U.T. Economics Summer 2011 Math Camp
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Topics: increasing differences, comparative statics.
Readings: CSZ, Ch. 2.8

A. Let $X = T = \mathbb{R}$ and $f(x,t) = x - \frac{1}{2}(x-t)^2$.
   1. Find $x^*(t)$ and verify directly that $dx^*/dt > 0$.
   2. Find f_x, f_{xx}, and f_{xt}, and verify that you should have expected $dx^*/dt > 0$.
   3. Draw a three-dimensional figure (and this is a skill worth developing), draw f and verify from your picture that $f_{xt} > 0$ and that it is this fact that makes $dx^*/dt > 0$.
   4. To practice with what goes wrong with derivative analysis when there are corner solutions, repeat this problem with $X = \mathbb{R}_+$, $T = \mathbb{R}$, and $g(x,t) = x - \frac{1}{2}(x+t)^2$.

B. [Monopolists and supply] Suppose that the inverse demand curve for a good produced by a monopolist is $x(p)$, so that consumer surplus is $CS(p) = \int_p^\infty x(r)\,dr$ when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.

   The monopolist's profit when he produces x is $\pi(x) = x \cdot p(x) - c(x)$, where $c(x)$ is the cost of producing x. The maximization problem for the monopolist and for society is

$$\max_{x \geq 0} \; [\pi(x) \;+\; 0 \cdot CS(p(x))], \text{ and}$$
$$\max_{x \geq 0} \; [\pi(x) \;+\; 1 \cdot CS(p(x))].$$

   Set $f(x,t) = \pi(x) + tCS(p(x))$, where $X = \mathbb{R}_+$ and $t \in T = \{0,1\}$. Show that $f(x,t)$ has increasing differences, and give the externalities intuition for why the monopolist under-produces and over-charges relative to the social optimum.

C. [Partially non-excludable goods] One class of goods that tend to be underprovided are the non-excludable ones, that is, the goods for which it is not possible to prevent people who have not paid for it from having access to it. Examples include clean air, clean water, the sight of a beautiful building, the use of roads that do not have toll booths, fish in the public lake, lighthouses, national defense, widespread vaccination. An excludable good is one for which it is possible to prevent people who have not paid for it from having access to it. Examples include food, private beach front property (with effective gates/fences). There are degrees of non-excludability between 0 and 1: only some of the hikers in the national parks have permits; only some of the fishers have licenses; private beach front property may be legally accesible from the water if not from land.

Suppose that producing $x \geq 0$ of a good costs $C(x)$. Let $W(x)$ denote the total societal willingness to pay for x, and suppose that $W(\cdot)$ is a non-decreasing function. Let $t \in [0, 1]$ denote the proportion of the total willingness to pay that can be collected. The collectible surplus maximization problem is

$$P(t): \quad \max_{x \geq 0}[t \cdot W(x) - C(x)].$$

1. Let $x^*(t)$ denote the solution to $P(t)$. How does $x^*(\cdot)$ depend on t? Explain your answer both with mathematics and intuitively.
2. Show that $x^*(t)$ is always at least weakly lower than the level that would maximize social surplus.

D. The amount of a pollutant that can be emitted by a firm is regulated to be no more than $t \geq 0$. The cost function for a monopolist producing x is $c(x, t)$ with $c_t < 0$ and $c_{xt} < 0$. These derivative conditions mean that increases in the allowed emission level lower costs and lower marginal costs, so that the firm will always choose t. For a given t, the monopolist's maximization problem is therefore

$$\max_{x \geq 0} f(x, t) = xp(x) - c(x, t),$$

where $p(x)$ is the demand function. Show that increases in t lead the monopolist to produce more. Explain the result both mathematically and economically.

E. [Faustmann rotation versus optimal single rotation] For $t \geq 0$ let $Q(t)$ denote a benefit function with $Q(t) > 0$ for $t > 0$. This problem asks you to go through several steps to compare the solutions to the problems

$$\max_{t \geq 0} Q(t)e^{-rt} \text{ and } \max_{t \geq 0} Q(t)\left[e^{-rt} + e^{-r2t} + e^{-r3t} + \cdots\right].$$

The first problem is the optimal single-rotation problem, the second is the Faustmann optimal rotation problem.

1. Argue that solving the optimal single-rotation problem is equivalent to solving the problem
$$\max_{t \geq 0} \left[\log(Q(t)) - rt\right].$$
2. Argue that solving the Faustmann optimal rotation problem is equivalent to solving the problem
$$\max_{t \geq 0} \left[\log(Q(t)) - rt - \log\left(1 - e^{-rt}\right)\right].$$
3. Show that the solution to the optimal Faustmann rotation is smaller (shorter) than the solution to the optimal single-rotation problem.
4. Give the intuition for the previous result.

F. [Private amenities from a 'forest'] A private woodlot owner receives an amenity flow of $A(t)$ while the trees are growing on her lot and the lot was replanted at $t = 0$. We assume that $A(t) > 0$ for $t > 0$, and that she is interested in both the amenity flow and the present value of net revenue from a single rotation. That is, we assume that she cuts down her lot of trees at T_A defined by

$$T_A = \text{argmax}_{t \geq 0}\, Q(t)e^{-rt} + \int_0^t A(s)e^{-rs}\, ds.$$

1. Show that the function $g(t) = \int_0^t A(s)e^{-rs}\, ds$ is increasing by giving its derivative for $t > 0$.
2. Show that T_A is larger than, or at least as large as, T_S, the solution to the problem $\max_{t \geq 0} Q(t)e^{-rt}$. Explain this both mathematically and intuitively.
3. Formulate and solve the same problem for the case of the optimal rotational forest in perpetuity (the Faustmann rotation) and show that adding the amenity flows into the optimization problem makes the optimal rotation longer.

G. [Public and private amenities from a 'forest'] Suppose in the previous problem that there is also a public amenity flow of $B(t) > 0$ for $t > 0$ that is received from the forest. Let T_B denote the solution to the problem $\max_{t \geq 0} Q(t)e^{-rt} + \int_0^t [A(s) + B(s)]e^{-rs}\, ds$. Show that T_B is larger than, or at last as large as, T_A from the previous problem, both in the single and in the Faustmann rotation case.

H. Let $Q(t)$ be the value of cutting down a forest that has grown from 0 to t, and, for the first two problems below, assume that after the trees are cut down, the owner will sell it for an agreed on price, v. Let $r > 0$ be the continuously compounded rate of interest, and suppose that Q has the usual properties: $Q(t) > 0$ and $Q'(t) > 0$ for all $t > 0$, and $Q''(t) > 0$ for an interval $(0, T)$ while $Q''(t) < 0$ for (T, ∞).

   1. How does the solution to the problem $\max_{t \geq 0}(Q(t) + v)e^{-rt}$ depend on r? Show the result mathematically and explain the intuition. [The math part is easier if you take logarithms first.]
   2. How does the solution to the problem $\max_{t \geq 0}(Q(t) + v)e^{-rt}$ depend on v? Show the result mathematically and explain the intuition. [The phrase "opportunity cost" should figure prominently in your explanation.]