U.T. Economics Summer 2011 Math Camp

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Topics: Statements as sets, notation, indicator functions, quantifiers
Readings: CSZ, Ch. 1, 2.1 - 3.

A. Express the following arguments in terms of subsets, in terms of statements, and in terms of indicator functions (using the notation of CSZ Chapter 1), and argue that the conclusions, i.e. the part after the word "therefore," must be true if the first two statements, the premises, are true.
   1. All humans are mortal, all Texans are human, therefore all Texans are mortal.
   2. All informative things are useful things, some websites are not useful things, therefore some websites are not informative.
   3. Reptiles have no fur, snakes are reptiles, therefore there are no furry snakes.

B. Repeat the previous, but now you must decide whether or not the conclusions follow from the premises, proving your answer.
   1. Some cars are grey, some grey things are televisions, therefore all cars are televisions.
   2. Some apes have no tails, apes are mammals, therefore some mammals are tailless.
   3. No fish are cats (catfish nothwithstanding), and no cats can fly, therefore all fish can fly.
   4. No fish are cats, and no cats can fly, therefore some fish can fly.

C. [With apologies to Lewis Carroll] We let X, the "universe of discourse," be the set of people. We define $B \subset X$ as the set of Babies, $I \subset X$ as the set of Illogical people, D the set of Despised people, and M the set of people who can Manage a crocodile.
   1. Express the following three statements as subset relations.
      (a) All babies are illogical.
      (b) Nobody is despised who can manage a crocodile.
      (c) Illogical persons are despised.
   2. Express the same three statements as implications.
   3. Prove that if the three statements are true, then no baby can manage a crocodile.

D. [With yet more apologies to Lewis Carroll] The universal set, X, for this problem is the set of things possibly met with while voyaging at sea, and the "log" below is a book full of observations. Consider the following four statements:
   (a) None of the unnoticed things, met with at sea, are mermaids.
   (b) Things entered in the log, as met with at sea, are sure to be worth remembering.

(c) I have never met with anything worth remembering, when on a voyage.
(d) Things met with at sea, that are noticed, are sure to be recorded in the log.

1. For $x \in X$, let "$\mathbb{N}(x)$" be the statement "x is noticed," and let "$\mathbb{M}(x)$" be the statement "x is a mermaid." Give statement (a) and its contrapositive in terms of $\mathbb{N}$ and $\mathbb{M}$. Also, give statement (a) in its set form.
2. Give the statement form, the contrapositive, and the set form for statement (b) above.
3. Give the statement form, the contrapositive, and the set form for statement (c) above.
4. Give the statement form, the contrapositive, and the set form for statement (d) above.
5. From statements (a)-(d) above, derive the conclusion that "I have never met a mermaid at sea."

E. Express the following arguments in terms of sets and indicator functions, decide whether or not the conclusions follow from the premises, and prove your answer.
1. All professors are wise people, Max Stinchcombe is a professor, therefore there exists a wise person.
2. Every wise person is a professor, Max Stinchcombe is a professor, therefore Max Stinchcombe is a wise person.
3. There is at least one professor at U.T. who is wise, Max Stinchcombe is a professor, therefore Max Stinchcombe is wise.
4. There is at least one professor at U.T. who is wise, Max Stinchcombe is wise, therefore Max Stinchcombe is professor.
5. All creatures from other planets are friendly, all martians are from another planet, therefore there are friendly martians.
6. All infinite subsets of the set $\{1, 2\}$ contain nineteen elements.

F. A **sequence in** $\mathbb{R}$ is a list $x = (x_1, x_2, x_3, \ldots)$ such that each $x_n \in \mathbb{R}$. The set of all sequences is denoted $\mathbb{R}^{\mathbb{N}}$. Within the set of all sequences we have the following distinguished sets:
- $\ell_\infty = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists B \in \mathbb{R})(\forall n \in \mathbb{N})[\,|x_n| \leq B\,]\}$.
- $c = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists r \in \mathbb{R})(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[\,|x_n - r| \leq \epsilon\,]\}$.
- $c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : (\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[\,|x_n| \leq \epsilon\,]\}$.
- $\ell_2 = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists B \in \mathbb{R})(\forall N \in \mathbb{N})[\,\sum_{n=1}^{N} |x_n|^2 \leq B\,]\}$.
- $\ell_1 = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists B \in \mathbb{R})(\forall N \in \mathbb{N})[\,\sum_{n=1}^{N} |x_n| \leq B\,]\}$.

You are now going to show that $\ell_1 \subsetneq \ell_2 \subsetneq c_0 \subsetneq c \subsetneq \ell_\infty$.
1. Show that $(\forall x \in \ell_1)[x \in \ell_2]$ and that $(\exists x \in \ell_2)[x \notin \ell_1]$.
2. Show that $(\forall x \in \ell_2)[x \in c_0]$ and that $(\exists x \in c_0)[x \notin \ell_2]$.
3. Show that $(\forall x \in c_0)[x \in c]$ and that $(\exists x \in c)[x \notin c_0]$.
4. Show that $(\forall x \in \ell_c)[x \in \ell_\infty]$ and that $(\exists x \in \ell_\infty)[x \notin c]$.