U.T. Economics Summer 2013 Math Camp

Date: Friday, August 9 and Monday August 10

Topics: Increasing differences, supermodularity
Readings: CSZ, Ch. 2.8

A. Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$ so that $CS(p) = \int_p^\infty x(r)\,dr$ is the consumer surplus when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. (From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.) The monopolist's profit when they produce x is $\pi(x) = x \cdot p(x) - c(x)$ where $c(x)$ is the cost of producing x. The maximization problem for the monopolist is

$$\max_{x \geq 0} \; \pi(x) + 0 \cdot CS(p(x)). \tag{1}$$

Society's surplus maximization problem is

$$\max_{x \geq 0} \; \pi(x) + 1 \cdot CS(p(x)). \tag{2}$$

Set $f(x, \theta) = \pi(x) + \theta CS(p(x))$, $\theta \in \{0, 1\}$, and verify that $f(x, \theta)$ is supermodular. What does this mean about a monopolist's output relative to the social optimum?

B. [Background question for a tragedy of the commons model] Suppose that two countries, i and j, put out fishing fleets of size $q_i, q_j \geq 0$, measured in standardized "boats." When $q = (q_i + q_j)$ is the total size of the fleet on the water, the per boat take is $v(q)$ where $v(0) > 0$, $v'(q) < 0$, and $v''(q) < 0$. The utility of country i is

$$u_i(q_i, q_j) = q_i v(q_i + q_j) - c_i(q_i)$$

where $c_i(\cdot)$ is an increasing cost function. Question: is $q_i^*(q_j)$ increasing or decreasing in q_j?

C. [Laws of unintended consequences] Suppose that there is a one-dimensional policy, to be set at a level $t \geq 0$. For examples, this could be a frequency of vehicle inspection, a tax level, a maximal amount of pollution that any given vehicle can emit per mile. Sometimes only parts of the benefits, $B(t)$, or parts of the costs, $C(t)$, are included. We want to see what happens to the optimal t when they are all included. We suppose for the first parts of the problem that both the social benefits and social costs increasing in t.

1. Carefully compare the properties of the *sets* of optimal t's for the problems

$$\max_{t \geq 0} [B(t) - C(t)] \;\text{ and }\; \max_{t \geq 0} [(B(t) + B_2(t)) - C(t)]$$

where $B_2(\cdot)$ is another increasing benefit function.

2. Carefully compare the properties of the *sets* of optimal t's for the problems

$$\max_{t \geq 0} [B(t) - C(t)] \;\text{ and }\; \max_{t \geq 0} [B(t) - (C(t) + C_2(t))]$$

where $C_2(\cdot)$ is another increasing cost function.

3. Carefully compare the properties of the *sets* of optimal t's for the problems

$$\max_{t\geq 0}\,[B(t)-C(t)] \text{ and } \max_{t\geq 0}\,[(B(t)+B_2(t))-(C(t)+C_2(t))]$$

when

a. the net benefits $B_2(\cdot)-C_2(\cdot)$, are increasing, and

b. the net benefits $B_2(\cdot)-C_2(\cdot)$, are decreasing.

4. Explain how one or more of these comparisons apply to vehicle inspection rates that are designed to decrease accidents to an optimal rate, but also benefit the environment.
5. Explain how one or more of these comparisons apply to limits on pollution per vehicle that are designed to increase air quality to some optimal level but have the effect of making the cheaper (older and more polluting) cars unavailable.
6. Explain how one or more of these comparisons apply to royalties charged to private logging firms on public lands when illegal logging is an option.
7. Explain how one or more of these comparisons apply to the following slippery slope: a system of cameras installed to deter crime can be linked to facial recognition software allowing governments to track all citizens' movements.

D. [Faustmann versus single rotation] For $t \geq 0$ let $Q(t)$ denote a benefit function with $Q(t) > 0$ for $t > 0$. This problem asks you to go through several steps to compare the solutions to the problems

$$\max_{t\geq 0} Q(t)e^{-rt} \text{ and } \max_{t\geq 0} Q(t)\left[e^{-rt}+e^{-r2t}+e^{-r3t}+\cdots\right].$$

When $Q(t)$ is the benefit of a forest left to grow from time 0 until time t, the first problem is called the optimal single-rotation problem, while the second is the Faustmann optimal rotation problem.

1. Argue that solving the optimal single-rotation problem is equivalent to solving the problem $\max_{t\geq 0}[\log(Q(t))-rt]$.
2. Argue that solving the Faustmann optimal rotation problem is equivalent to solving the problem $\max_{t\geq 0}[\log(Q(t))-rt-\log(1-e^{-rt})]$.
3. Show that the solution to the optimal Faustmann rotation is smaller (shorter) than the solution to the optimal single-rotation problem. Explain why this *should* be true.
4. Give the dependence of the solutions on r.

E. The amount of a pollutant that can be emitted is regulated to be no more than $\theta \geq 0$. The cost function for a monopolist producing x is $c(x,\theta)$ with $c_\theta < 0$ and $c_{x\theta} < 0$. These derivative conditions means that increases is the allowed emission level lower costs and lower marginal costs, so that the firm will always choose θ.

For a given θ, the monopolist's maximization problem is therefore

$$\max_{x \geq 0} f(x, \theta) = xp(x) - c(x, \theta) \tag{3}$$

where $p(x)$ is the (inverse) demand function. Show that output decreases as $\theta \downarrow$.

Sometimes the set of available choices also shifts with θ. The next result concerns this possibility.

Theorem 1. *Suppose that X and Θ are non-empty subsets of $\mathbb{R}$, that $\Gamma(\theta) = [g(\theta), h(\theta)] \cap X$ where g and h are weakly increasing functions with $g \leq h \leq \infty$, that $f : X \times \Theta \to \mathbb{R}$ is supermodular (i.e. has increasing differences in x and θ). Then the smallest and the largest solutions to the problem $P(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta)$ are weakly increasing functions. Further, if f is strictly supermodular, then every selection from $\Psi(\theta) := \operatorname{argmax}_{x \in \Gamma(\theta)} f(x, \theta)$ is weakly increasing.*

F. This result is, more or less, available in the textbook. I would prefer it if you tried to figure out the arguments for the problems below before looking at the arguments in the book. However, don't spend more than 20 or 30 minutes on this, just enough time to see what, if anything, you find difficult. Once you know that, feel free to read the textbook on this topic. Then, try to give the proof without having the textbook open.
   1. Prove the part of the result before the word "Further".
   2. Prove the part of the result after the word "Further".

G. You start with an amount θ, choose an amount, c, to consume in the first period, and have $f(\theta - c)$ to consume in the second period, and your utility is $u(c) + \beta u(f(\theta - c))$ where $u' > 0$ and $u'' < 0$. We suppose that $r \mapsto f(r)$ is increasing.
   1. Consider the two-period consumption problem,

   $$P_c(\theta) = \max \quad u(c) + \beta u(f(\theta - c)) \text{ subject to } c \in [0, \theta]. \tag{4}$$

   Prove that, because $f(\cdot)$ is increasing, this problem is equivalent to the two-period savings problem,

   $$P_s(\theta) = \max \quad u(\theta - s) + \beta u(f(s)) \text{ subject to } s \in [0, \theta]. \tag{5}$$

   2. Prove that savings, $s^*(\theta)$, are weakly increasing in θ.
   3. Now define $V(y) = \max_{c_0, c_1, \ldots} \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $x_0 = y$, $s_t = x_t - c_t$, $x_{t+1} = f(s_t)$, and $c_t \in [0, x_t]$, $t = 0, 1, \ldots$. Assuming that the maximization problem for V has a solution, show that $V(\cdot)$ is increasing. From this, prove that the solution to the following infinite horizon savings problem is weakly increasing in θ,

   $$P(\theta) = \max \quad u(\theta - s) + \beta V(f(s)) \text{ subject to } s \in [0, \theta]. \tag{6}$$