U.T. Economics Summer 2013 Math Camp

Date: Monday, August 12

Topics: Dot products, directional derivatives, tangent planes.
Readings: CSZ 5.3-8, MWG Math appendix A

Dot (or inner) product notation: Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\ell$, the **dot product** or **inner product** of $\mathbf{x}$ and $\mathbf{y}$ is defined as $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{\ell} x_i y_i$. In words, we multiply each component of $\mathbf{x}$ by the corresponding component of $\mathbf{y}$ and add the results. Often we write the dot product as $\mathbf{xy}$, and you will also see the following notations for the dot product: $\mathbf{x}^T\mathbf{y}$; $\mathbf{x}'\mathbf{y}$; and $\langle \mathbf{x}, \mathbf{y} \rangle$.

The **(Euclidean) length of a vector** $\mathbf{x}$ is defined as $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. More explicitly, $\|\mathbf{x}\| = \sqrt{\sum_i x_i^2}$, which is exactly the distance between 0 and $\mathbf{x}$ that you learned in geometry class all those years ago.

We will begin by finding some of the implications of the result

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\theta)$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Because of this result, we say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

For $\mathbf{x} \in \mathbb{R}^\ell$, $f : \mathbb{R}^\ell \to \mathbb{R}$, $g : \mathbb{R}^\ell \to \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$, we are interested in the problems

$$V(\mathbf{b}) = \max f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq \mathbf{b}, \text{ and} \tag{1}$$

$$V(\mathbf{b}) = \max f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq \mathbf{b},\ \mathbf{x} \geq 0. \tag{2}$$

We will study it using the associated **Lagrangean function**, $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda^T(\mathbf{b} - g(\mathbf{x}))$, $\lambda \in \mathbb{R}^m_+$.

A. Some calculations and graphical exercises.
1. If $\mathbf{x} = (1, 3, 7)'$ and $\mathbf{y} = (2, 9, 0)$, give $\mathbf{xy}$, $\|\mathbf{x}\|$, $\|\mathbf{y}\|$.
2. Draw the set of $\mathbf{x} \in \mathbb{R}^2$ such that $\|\mathbf{x}\| \leq 1$, $\|\mathbf{x}\| \leq 2$, and $\|\mathbf{x}\| \leq 17$.
3. For $\mathbf{p} = (1, 3)$, draw the set $\{\mathbf{x} \in \mathbb{R}^2_+ : \mathbf{px} \leq w\}$ for $w = 3$, $w = 5$, and $w = 37$.
4. If $\mathbf{x} = (1, 3, 7)'$, give two linearly independent $\mathbf{y}$ such that $\mathbf{xy} = 0$, and give the equation of the plane of points that are orthogonal to $\mathbf{x}$.

B. Graph the following affine functions as well as two or three representative level sets and their gradients (i.e. the direction of fastest increase of the functions).
1. $f(x_1, x_2) = 7 + 3x_1 + 4x_2$, equivalently $f(\mathbf{x}) = 7 + \mathbf{xp}$ where $\mathbf{p} = (3, 4)'$.
2. $g(x_1, x_2) = -3 + 5x_1 - 2x_2$, equivalently $g(\mathbf{x}) = -3 + \mathbf{xy}$ where $\mathbf{y} = (5, -2)'$.
3. $h(x_1, x_2) = 21 + (-3x_1 - 7x_2) = 21 - 3x_1 - 7x_2$, equivalently $h(\mathbf{x}) = 21 + \mathbf{xz}$ where $\mathbf{z} = (-3, -7)'$.

C. Give the affine function tangent to the following utility functions at the given points $\mathbf{x}^\circ$ and $\mathbf{y}^\circ$. Also give (and draw) the level sets in $\mathbb{R}^2_+$ through the $\mathbf{x}^\circ$ as well as the direction in which the function increases the fastest.
   1. $u(x_1, x_2) = \log(x_1) + 3\log(x_2)$ at $\mathbf{x}^\circ = (7, 3)'$ and $\mathbf{y}^\circ = (5, 5)'$.
   2. $v(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$ at $\mathbf{x}^\circ = (7, 12)'$ and $\mathbf{y}^\circ = (19, 5)'$.
   3. $w(x_1, x_2) = x_1 + 2\sqrt{x_2}$ at $\mathbf{x}^\circ = (3, 16)'$ and $\mathbf{y}^\circ = (9, 25)'$.

D. From $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\theta)$, we know that if $\mathbf{x}$ and $\mathbf{y}$ are in the same quadrant, then $\mathbf{x} \cdot \mathbf{y} \geq 0$. The following depend on $\mathbf{x}$ and $\mathbf{y}$ being on the edges of the same quadrant, and they are more useful than their simplicity would suggest. Taken together, they go by the name of **complementary slackness**, and we will see them extensively in inequality constrained maximization.
   1. If $\mathbf{x}, \mathbf{y} \geq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then for $i = 1, \ldots, \ell$, $[\mathbf{x}_i > 0] \Rightarrow [\mathbf{y}_i = 0]$ and $[\mathbf{y}_i > 0] \Rightarrow [\mathbf{x}_i = 0]$.
   2. If $\mathbf{x} \geq 0$, $\mathbf{y} \leq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then for $i = 1, \ldots, \ell$, $[\mathbf{x}_i > 0] \Rightarrow [\mathbf{y}_i = 0]$ and $[\mathbf{y}_i > 0] \Rightarrow [\mathbf{x}_i = 0]$.
   3. For $\mathbf{y} \geq 0$, consider the problem $\min \mathbf{xy}$ subject to $\mathbf{x} \geq 0$. Show that $\mathbf{x}^*$ solves this problem iff $\mathbf{x}^* \geq 0$ and $\mathbf{x}^*\mathbf{y} = 0$.

E. The following observations are also quite simple, but will turn out to be quite useful. The results are stated for $\lambda > 0$, but work as well for $\lambda < 0$.
   1. For any non-zero $\mathbf{x} \in \mathbb{R}^\ell$, the solution to the problem $\max\{\mathbf{x} \cdot \mathbf{u} : \|\mathbf{u}\| \leq 1\}$ is $\mathbf{u}^* = \lambda\mathbf{x}$ where $\lambda > 0$ is equal to $1/\|\mathbf{x}\|$.
   2. Suppose that $f(\mathbf{x}) = a + \lambda\mathbf{yx}$ and $g(\mathbf{x}) = b + \mathbf{yx}$, so that $D_x f(\mathbf{x}^\circ) = \lambda D_x g(\mathbf{x}^\circ)$ for any $\mathbf{x}^\circ$, where $\lambda > 0$ and $\mathbf{y} \neq 0$. Then a unit change from $\mathbf{x}^\circ$ leads to the ratio of the changes of f and g being equal to λ, i.e. for any $\mathbf{x}^\circ$ and $\mathbf{uy} \neq 0$, the ratio

$$\Delta f(\mathbf{x}^\circ)/\Delta g(x^\circ) = [f(\mathbf{x}^\circ + \mathbf{u}) - f(\mathbf{x}^\circ)]/[g(\mathbf{x}^\circ + \mathbf{u}) - g(\mathbf{x}^\circ)] = \lambda.$$