**U.T. Economics Summer 2013 Math Camp**

**Date**: Thursday, August 15 through Monday, August 19

**Topics**: Expected utility and information structures, dynamic constrained optimization, properties of value functions.
**Readings**: CSZ 3.7, MWG 6.A-C

## Information structure problems

A. This is a problem on normalizations in expected utility theory.

1. An expected utility decision maker has beliefs $(\beta, 1-\beta)$ on $\{\omega_1, \omega_2\}$, and they have two possible actions to take, $a_1$ and $a_2$. Their utilities are given by

| | $\omega_1$ | $\omega_2$ |
|---|---|---|
| $a_1$ | 9 | 1 |
| $a_2$ | 2 | 3 |

Find, as a function of $\beta$, the decision makers optimal action. Show that it is the same function for the following two utility functions.

| | $\omega_1$ | $\omega_2$ |
|---|---|---|
| $a_1$ | 7 | 0 |
| $a_2$ | 0 | 2 |

| | $\omega_1$ | $\omega_2$ |
|---|---|---|
| $a_1$ | 3.5 | 0 |
| $a_2$ | 0 | 1 |

2. Show that the solution to $\max_{a \in A} \int u(a,y)\, d\beta(y)$ is the same as the solution to $\max_{a \in A} \int [r \cdot u(a,y) + g(y)]\, d\beta(y)$ for any $r > 0$ and any function $g(\cdot)$. [The point is that the last two sets of utilities games from these kinds of calculations.]

B. The timeline: first the prosecuting attorney commits to an information structure; second, the signal is observed by the judge/jury; third, the judge/jury makes their decision.

The Judge/Jury has a prior probability $p$ of the accused person being $G$uilty and probability $1-p$ of being $I$nnocent. Judge/Jury can either convict or acquity, and, after normalization their utility function is given by

| | $I$nnocent | $G$uilty |
|---|---|---|
| Convict | 0 | $z$ |
| Acquit | 1 | 0 |

with $z > 0$.

The prosecuting attorney (DA) receives expected utility is 1 if the accused is convicted, 0 else. The DA commits to an information structure, that is, they commit to a distribution, $Q$, over the space of beliefs, $\Delta(\Omega) = [0,1]$, such that $\int_{[0,1]} \beta\, dQ(\beta) = p$. The DA's problem is

$$(1) \qquad \max_{Q \in \Delta(\Delta(\Omega))} Q([\tfrac{1}{1+z}, 1]) \text{ subject to } \int_{\Delta(\Omega)} \beta\, dQ(\beta) = p.$$

1. Show judge/jury's optimal action is to convict if their beliefs, $\beta$, satisfy $\beta z \geq (1-\beta)$, i.e. $\beta \geq \frac{1}{1+z}$.
2. Show that the solution to the DA's problem is a $Q$ that puts mass $1-\gamma$ on 0 and $\gamma$ on $\frac{1}{1+z}$.
3. Show that the accused is convicted $p(1+z) > p$ of the time.

C. The police arrest a man and accuse him of a crime. Given the police department's record, there is a prior probability $\rho$, $0 < \rho < 1$, that the man is guilty, $\omega = g$, and a $(1-\rho)$ probability that the man is innocent, $\omega = i$. The man will be tried in front of a jury of $M$ people. These $M$ people will cast random, stochastically independent votes, $V_m = G$ for guilty and $V_m = I$ for innocent, $m = 1, \ldots, M$ with probabilities

$$P(V_m = G|\omega = i) = p, \quad P(V_m = G|\omega = g) = q, \quad 0 < p < \frac{1}{2} < q < 1.$$

Suppose that social utility depends on the innocence or guilt of the defendant, $\omega = i, g$, and the jury's decision, $V = I, G$, and

$$\underbrace{0 = u(V = G|\omega = i)}_{\text{worst mistake}} < \underbrace{u(V = I|\omega = g) = r}_{\text{mistake}} < \underbrace{u(V = G|\omega = g) = u(V = I|\omega = i) = 1}_{\text{correct decision}}.$$

1. Consider the unanimity rule for the jury, "Convict only if all jurors return a guilty vote," i.e. $V = G$ if $V_1 = V_2 = \cdots = V_M = G$, and $V = I$ otherwise. What are

$$P(V = G|\omega = g),\ P(V = I|\omega = g),\ P(V = G|\omega = i), \text{ and } P(V = I|\omega = i)?$$

2. If juries are costless, set up the problem for finding the optimal $M$ and characterize its dependence on $\rho$.
3. Repeat the previous, but now assume that the cost of a jury is an increasing function of the jury size and characterize the dependence of the optimal $M$ on the cost.

## Dynamic problems

D. Suppose that the growth curve in the fishery model of CSZ §3.7 is twice continuously differentiable. Your job is to compare the optimal growth paths and optimal steady states for the three utility functions $u(x_t) = \log(x_t)$, $v(x_t) = 2\sqrt{x_t}$, and $w(x_t) = x_t$.
   1. Give the three corresponding Euler equations.
   2. Give the optimal steady states as a function of the discount factor.
   3. Starting from the same $x_0$ below the steady states you just found, which utility function involves faster growth of the fish stock? [For the utility function $w(\cdot)$, you need to remember that the Euler equations were derived under the assumption that the solutions were strictly positive, and that may not be true.]
   4. Suppose now that $x_0$ is above the steady states and repeat the previous.

E. [Non-renewable resources as renewable resources with a 0 growth rate]: There is a total stock of $X$ of a resource, and the problem is to choose a consumption path, $c_0, c_1, c_2, \ldots$, $c_t \geq 0$, so as to maximize $\sum_t \rho^t u(c_t)$ subject to the constraint that $\sum_t c_t \leq X$. The Lagrangean for this problem is $L(c; \lambda) = \sum_t \rho^t u(c_t) + \lambda(X - \sum_t c_t)$ where $c = (c_0, c_1, c_2, \ldots)$ represents the whole infinite length vector. We suppose throughout that $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

   1. Explain why the constraint must be binding.
   2. Write out the Kuhn-Tucker conditions assuming that at the optimum, each $c_t^* > 0$.
   3. Show that the Kuhn-Tucker conditions that you just found deliver the Euler equation for a 'renewable' resource with the growth curve $F(x) \equiv 0$.

   For the rest of this problem, we suppose that $u(c) = 2\sqrt{c}$.

   4. Find the optimal consumption path as a function of $X$ and $\rho$.
   5. Give an intuitive explanation for why the $c_0^*$ that you just found should be smaller when $\rho$ is larger.
   6. Show that once $t$ is large enough, $c_t^*$ is larger for larger $\rho$.
   7. Give an intuitive explanation for why, for large enough $t$, $c_t^*$ should be larger for larger $\rho$. [This answer should be fairly tightly related to the answer you gave for why $c_0^*$ is smaller for larger $\rho$, so don't worry if it is slightly repetitive.]

F. [Efficiency and dynamics]: This problem is the beginning of the study of how differential patience/impatience interacts with efficient intertemporal allocation. There are two people with funny hair and entirely too much energy named Thing 1 and Thing 2. There are two time periods, $t = 0, 1$. 1's vector of consumption in the two periods is $(x_{1,0}, x_{1,1}) \geq (0, 0)$, and 2's vector of consumption in the two periods is $(x_{2,0}, x_{2,1}) \geq (0,0)$. 1's utility is $U_1(x_{1,0}, x_{1,1}) = x_{1,0} + \rho_1 x_{1,1}$, while 2's utility is $U_2(x_{2,0}, x_{2,1}) = u_2(x_{2,0}) + \rho_2 u_2(x_{2,1})$, where $u_2(x) = 2\sqrt{x}$. Both $\rho_1$ and $\rho_2$ are strictly positive and strictly less than 1.

There is a total amount $X$ of the consumption good available at $t = 0$. It can either be consumed or it can be invested at a rate of return $r > 0$. This means that saving $s$, i.e. consuming only $X - s$, right now allows for consumption of $(1+r)s$ at $t = 1$.

The first two problems involve finding the individually optimal allocations of consumption across time. They are good practice for the problem of finding the set of efficient allocations of consumption across time and people.

1. Consider Thing 2's problem when there is no Thing 1,

$$\max U_2(x_{2,0}, x_{2,1}) \text{ subject to } x_{2,0} \leq x,\ x_{2,1} \leq (1+r)[X - x_{2,0}],\ (x_{2,0}, x_{2,1}) \geq 0.$$

   a. Give the associated Lagrangean.
   b. Without solving the problem, explain how the optimal $x^*_{2,1}$ depends on $\rho_2$.
   c. Show that the Kuhn-Tucker conditions for this problem are the same as the Euler equation for the fishery growth problem.
   d. Solve for the optimal $(x^*_{2,0}, x^*_{2,1})$ as a function of $X$, $r$, and $\rho_2$.

2. Now consider Thing 1's problem when there is no Thing 2,

$$\max U_1(x_{1,0}, x_{1,1}) \text{ subject to } x_{1,0} \leq X,\ x_{1,1} \leq (1+r)[X - x_{1,0}],\ (x_{1,0}, x_{1,1}) \geq 0.$$

   a. Give the associated Lagrangean.
   b. Without solving the problem, explain how the optimal $x^*_{1,1}$ depends on $\rho_1$.
   c. Explain how the solution Kuhn-Tucker conditions for this problem depend on the relation between $\rho_1$ and $(1+r)$.
   d. Solve for the optimal $(x^*_{1,0}, x^*_{1,1})$ as a function of $X$, $r$, and $\rho_1$. [You will need to break up your answer into cases.]

We now turn to the problem of finding the dynamic efficient allocations for Thing 1 and Thing 2. To do this, we consider the solutions to problems of the form

$$\max\ [\alpha U_1(x_{1,0}, x_{1,1}) + (1-\alpha) U_2(x_{2,0}, x_{2,1})] \text{ subject to}$$
$$x_{1,0} + x_{2,0} \leq X,$$
$$x_{1,1} + x_{2,1} \leq (1+r)[X - (x_{1,0} + x_{2,1})],$$
$$(x_{2,0}, x_{2,1}) \geq (0,0),\ (x_{1,0}, x_{1,1}) \geq (0,0), \tag{2}$$

where $0 \leq \alpha \leq 1$.

3. When $\alpha = 0$, this is the first problem you solved above, when $\alpha = 1$, it is the second problem.
   a. Explain how the solution $(x^*_{1,0}, x^*_{1,1}), (x^*_{2,0}, x^*_{2,1})$ depends on $\alpha$. [You can do this without solving the problem in (2).]
   b. Show that for $0 < \alpha < 1$, any solution to the problem in (2) is efficient.

c. Give the Lagrangean and the Kuhn-Tucker conditions for the problem in (2).

d. Argue that for $0 < \alpha < 1$, the Kuhn-Tucker conditions can never be satisfied at $x_{2,0} = 0$ or $x_{2,1} = 0$. Interpret this in terms of marginal utilities.

e. Show that for small enough strictly positive $\alpha$, the solution to the problem in (2) involves $(x^*_{1,0}, x^*_{1,1}) = (0, 0)$.

## Value function properties and KT problems

G. Suppose that $u : \mathbb{R}^1_+ \to \mathbb{R}$ is concave and strictly increasing. For $0 < \beta < 1$ and $r > 0$, consider the problem

$$V(w) = \max_{c_1,c_2} [u(c_1) + \beta u(c_2)] \text{ subject to } c_1 \in [0, w],\ c_2 \in [0, (1 + r)(w - c_1)].$$

Show that $V(\cdot)$ is concave and strictly increasing.

H. Suppose that $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ is concave and strictly increasing. For fixed $\mathbf{p}$, show that the function $V(w) := \max u(\mathbf{x})$ subject to $\mathbf{px} \leq w$, $\mathbf{x} \geq 0$ is concave and strictly increasing.

## KT problems and further value function properties

I. [More about KT conditions] Suppose that $C = \mathbb{R}^\ell_+$ and that $f : C \to \mathbb{R}$ is concave and differentiable. Show that $[\,[\mathbf{x}^* \in C] \wedge (\forall \mathbf{x} \in C)[D_x f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \leq 0]\,] \Leftrightarrow [\,[D_x f(\mathbf{x}^*) \leq 0] \wedge [\mathbf{x}^* \geq 0] \wedge [\mathbf{x}^* \cdot D_x f(\mathbf{x}^*) = 0]\,]$.

J. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = (1 + x_2)^3 x_1^2 + x_2^2$ has only one point $\mathbf{x}^*$ with $D_x f(\mathbf{x}^*) = 0$, but that $f$ has neither a global maximum or a global minimum. [Just a reminder that FOC are necessary but not sufficient.]

K. [Practice with K-T conditions at the boundaries] Solve the problem $V(\mathbf{p}, w) = \max \vec{1} \cdot \mathbf{x}$ s.t. $\mathbf{px} \leq w$, $\mathbf{x} \geq 0$ where $\mathbf{p} \gg 0$ and $\vec{1}$ is the length-$\ell$ vector of 1's.

L. For $\mathbf{x} \in \mathbb{R}^3$, find $V(b) = \max(100 - \mathbf{x} \cdot \mathbf{x})$ subject to the following constraints.
1. $\mathbf{p} \cdot \mathbf{x} \leq b$, $\mathbf{x} \geq 0$ where $\mathbf{p} = (1, 2, 1)^T$.
2. $\mathbf{p} \cdot \mathbf{x} \geq b$, $\mathbf{x} \geq 0$ where $\mathbf{p} = (1, 2, 1)^T$.

In each case, verify that $V'(b) = \lambda^*(b)$.

M. Let $V(b_1, b_2) = \max \mathbf{y} \cdot \mathbf{x}$ s.t. $\mathbf{x} \cdot \mathbf{x} \leq b_1$ and $\mathbf{px} \leq b_2$ where $\mathbf{y} = (1, 4, 1)^T$ and $\mathbf{p} = (1, 2, 3)^T$. Verify that $\partial V/\partial b_i = \lambda_i^*$. [For different values of $b_1$ and $b_2$, different constraints are binding.].

N. Let $V(\mathbf{p}, w) = \max \frac{1}{2}\log(1 + x_1) + \frac{1}{4}\log(1 + x_2)$ subject to $\mathbf{px} \leq w$, $\mathbf{x} \geq 0$. Verify that $\partial V(\mathbf{p}, w)/\partial w) = \lambda^*$. [Corner solutions matter here.]

O. Solve the problem $\max(\frac{1}{2}x_1 - x_2)$ s.t. $x_1 + e^{-x_1} + (x_3)^2 \leq x_2$, $x_1 \geq 0$. [Here, the objective function could be regarded as depending on $x_1, x_2$ and $x_3$ even though $x_3$ has no effect on $f$.]

P. Solve $\max(1 - \mathbf{x} \cdot \mathbf{x})$ s.t. $\mathbf{x} \geq (2, 3)^T$ by direct geometry and by examining the K-T conditions.