U.T. Economics Summer 2011 Math Camp

Date: Monday, August 8

Topics: Dot products, directional derivatives, tangent planes.
Readings: CSZ 5.3-8, MWG M.A

Dot (or inner) product notation: Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\ell$, the **dot product** or **inner product** of $\mathbf{x}$ and $\mathbf{y}$ is defined as $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{\ell} x_i y_i$. In words, we multiply each component of $\mathbf{x}$ by the corresponding component of $\mathbf{y}$ and add the results. Often we write the dot product as $\mathbf{xy}$, and you will also see the following notations for the dot product: $\mathbf{x}^T\mathbf{y}$; $\mathbf{x}'\mathbf{y}$; and $\langle \mathbf{x}, \mathbf{y} \rangle$.

The **(Euclidean) length of a vector** $\mathbf{x}$ is defined as $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. More explicitly, $\|\mathbf{x}\| = \sqrt{\sum_i x_i^2}$, which is exactly the distance between 0 and $\mathbf{x}$ that you learned in geometry class all those years ago.

We will begin by finding some of the implications of the result

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\theta)$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Because of this result, we say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

For $\mathbf{x} \in \mathbb{R}^\ell$, $f : \mathbb{R}^\ell \to \mathbb{R}$, $g : \mathbb{R}^\ell \to \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$, we are interested in the problems

$$V(\mathbf{b}) = \max f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq \mathbf{b}, \text{ and} \tag{1}$$

$$V(\mathbf{b}) = \max f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq \mathbf{b},\ \mathbf{x} \geq 0. \tag{2}$$

We will study it using the associated **Lagrangean function**, $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda^T(\mathbf{b} - g(\mathbf{x}))$, $\lambda \in \mathbb{R}^m_+$.

A. Some calculations and graphical exercises.
1. If $\mathbf{x} = (1, 3, 7)'$ and $\mathbf{y} = (2, 9, 0)$, give $\mathbf{xy}$, $\|\mathbf{x}\|$, $\|\mathbf{y}\|$.
2. Draw the set of $\mathbf{x} \in \mathbb{R}^2$ such that $\|\mathbf{x}\| \leq 1$, $\|\mathbf{x}\| \leq 2$, and $\|\mathbf{x}\| \leq 17$.
3. For $\mathbf{p} = (1, 3)$, draw the set $\{\mathbf{x} \in \mathbb{R}^2_+ : \mathbf{px} \leq w\}$ for $w = 3$, $w = 5$, and $w = 37$.
4. If $\mathbf{x} = (1, 3, 7)'$, give two linearly independent $\mathbf{y}$ such that $\mathbf{xy} = 0$, and give the equation of the plane of points that are orthogonal to $\mathbf{x}$.

B. Graph the following affine functions as well as two or three representative level sets and their gradients (i.e. the direction of fastest increase of the functions).
1. $f(x_1, x_2) = 7 + 3x_1 + 4x_2$, equivalently $f(\mathbf{x}) = 7 + \mathbf{xp}$ where $\mathbf{p} = (3, 4)'$.
2. $g(x_1, x_2) = -3 + 5x_1 - 2x_2$, equivalently $g(\mathbf{x}) = -3 + \mathbf{xy}$ where $\mathbf{y} = (5, -2)'$.
3. $h(x_1, x_2) = 21 + (-3x_1 - 7x_2) = 21 - 3x_1 - 7x_2$, equivalently $h(\mathbf{x}) = 21 + \mathbf{xz}$ where $\mathbf{z} = (-3, -7)'$.

C. Give the affine function tangent to the following utility functions at the given points $\mathbf{x}^\circ$ and $\mathbf{y}^\circ$. Also give (and draw) the level sets in $\mathbb{R}^2_+$ through the $\mathbf{x}^\circ$ as well as the direction in which the function increases the fastest.

1. $u(x_1, x_2) = \log(x_1) + 3\log(x_2)$ at $\mathbf{x}^\circ = (7, 3)'$ and $\mathbf{y}^\circ = (5, 5)'$.
2. $v(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$ at $\mathbf{x}^\circ = (7, 12)'$ and $\mathbf{y}^\circ = (19, 5)'$.
3. $w(x_1, x_2) = x_1 + 2\sqrt{x_2}$ at $\mathbf{x}^\circ = (3, 16)'$ and $\mathbf{y}^\circ = (9, 25)'$.

D. From $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\theta)$, we know that if $\mathbf{x}$ and $\mathbf{y}$ are in the same quadrant, then $\mathbf{x} \cdot \mathbf{y} \geq 0$. The following depend on $\mathbf{x}$ and $\mathbf{y}$ being on the edges of the same quadrant, and they are more useful than their simplicity would suggest. Taken together, they go by the name of **complementary slackness**, and we will see them extensively in inequality constrained maximization.

1. If $\mathbf{x}, \mathbf{y} \geq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then for $i = 1, \ldots, \ell$, $[\mathbf{x}_i > 0] \Rightarrow [\mathbf{y}_i = 0]$ and $[\mathbf{y}_i > 0] \Rightarrow [\mathbf{x}_i = 0]$.
2. If $\mathbf{x} \geq 0$, $\mathbf{y} \leq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then for $i = 1, \ldots, \ell$, $[\mathbf{x}_i > 0] \Rightarrow [\mathbf{y}_i = 0]$ and $[\mathbf{y}_i > 0] \Rightarrow [\mathbf{x}_i = 0]$.
3. For $\mathbf{y} \geq 0$, consider the problem $\min \mathbf{x}\mathbf{y}$ subject to $\mathbf{x} \geq 0$. Show that $\mathbf{x}^*$ solves this problem iff $\mathbf{x}^* \geq 0$ and $\mathbf{x}^*\mathbf{y} = 0$.

E. The following observations are also quite simple, but will turn out to be quite useful. The results are stated for $\lambda > 0$, but work as well for $\lambda < 0$.

1. For any non-zero $\mathbf{x} \in \mathbb{R}^\ell$, the solution to the problem $\max\{\mathbf{x} \cdot \mathbf{u} : \|\mathbf{u}\| \leq 1\}$ is $\mathbf{u}^* = \lambda\mathbf{x}$ where $\lambda > 0$ is equal to $1/\|\mathbf{x}\|$.
2. Suppose that $f(\mathbf{x}) = a + \lambda\mathbf{y}\mathbf{x}$ and $g(\mathbf{x}) = b + \mathbf{y}\mathbf{x}$, so that $D_x f(\mathbf{x}^\circ) = \lambda D_x g(\mathbf{x}^\circ)$ for any $\mathbf{x}^\circ$, where $\lambda > 0$ and $\mathbf{y} \neq 0$. Then a unit change from $\mathbf{x}^\circ$ leads to the ratio of the changes of f and g being equal to λ, i.e. for any $\mathbf{x}^\circ$ and $\mathbf{u}\mathbf{y} \neq 0$, the ratio

$$\Delta f(\mathbf{x}^\circ)/\Delta g(x^\circ) = [f(\mathbf{x}^\circ + \mathbf{u}) - f(\mathbf{x}^\circ)]/[g(\mathbf{x}^\circ + \mathbf{u}) - g(\mathbf{x}^\circ)] = \lambda.$$

F. [A special case of Samuelson's analysis of the optimal provision of public goods] There are two goods, a private consumption good, x, and a public good, G. The public good is produced according to the production function $G = f(z)$ where $f(z) = z$. In other words, one unit of the private consumption good can be turned into one unit of the public good. Person i's utility function is $u_i(x_i, G) = \log(x_i) + \beta_i \log(G)$, and each $\beta_i > 0$, and each person has a total of 10 units of the private good, some of which must be turned into the public good.

The first problems concerns the case where there is just one person. It will provide background for the later parts of the problem.

(1) Person i's problem is

$$\max_{x_i,z,G} u_i(x_i, G) \text{ subject to } x_i + z \leq 10 \text{ and } G \leq z.$$

Set up the Lagrangean for this problem.

(2) Show that the solution to the previous problem has $x_i^* = 10/(\beta_i + 1)$ and $z^* = 10\beta_i/(\beta_i + 1)$.

(3) Explain why higher values of β_i lead to higher optimal values of z^*'s and lower optimal values of x_i^*'s. Your answer should involve the equality of person i's MRS between the private and the public good and the technological rate of substitution between z and G.

The next set of problems concern the case where there are 2 people, $I = \{1, 2\}$, both having 10 units of the private good.

(4) The efficient allocations can be found by solving the problem

$$\max_{x_1,x_2,z,G} u_1(x_1, G) \text{ subject to } u_2(x_2, G) \geq u_2^\circ,\ x_1 + x_2 + z \leq 20, \text{ and } G \leq z.$$

Set up the Lagrangean for this problem.

(5) Show that any solution to the previous problem involves the sum of the two people's MRS's between the private and the public good being equal to the technological rate of substitution.

(6) Explain why higher values of β_1 will increase the optimal level of G.

(7) Show that it is possible to make both people better off than they were in the single person problems discussed above.

(8) Efficient allocations can also be found be solving problems of the form

$$\max_{x_1,x_2,z,G}[w_1 \cdot u_1(x_1, G) + w_2 u_2(x_2, G)] \text{ subject to } x_1 + x_2 + z \leq 20, \text{ and } G \leq z$$

for utility "weights" $w_1, w_2 > 0$. Show that the solution to these problems *also* involves the sum of the two people's MRS's between the private and the public good being equal to the technological rate of substitution.