U.T. Economics Summer 2011 Math Camp

Date: Thursday, August 11

Topics: Dynamic programming, expected utility theory
Readings: CSZ 4.11, 3.7, 6.2.d-e, MWG 6.A-C

A. CSZ 4.11.12 (p. 160).

B. Suppose that the growth curve in the fishery model of §3.7 is twice continuously differentiable. Your job is to compare the optimal growth paths and optimal steady states for the three utility functions $u(x_t) = \log(x_t)$, $v(x_t) = 2\sqrt{x_t}$, and $w(x_t) = x_t$.
   1. Give the three corresponding Euler equations.
   2. Give the optimal steady states as a function of the discount factor.
   3. Starting from the same x_0 below the steady states you just found, which utility function involves faster growth of the fish stock? [For the utility function $w(\cdot)$, you need to remember that the Euler equations were derived under the assumption that the solutions were strictly positive, and that may not be true.]
   4. Suppose now that x_0 is above the steady states and repeat the previous.

C. [Non-renewable resources as renewable resources with a 0 growth rate]: There is a total stock of X of a resource, and the problem is to choose a consumption path, $c_0, c_1, c_2, \ldots,\ c_t \geq 0$, so as to maximize $\sum_t \rho^t u(c_t)$ subject to the constraint that $\sum_t c_t \leq X$. The Lagrangean for this problem is $L(c; \lambda) = \sum_t \rho^t u(c_t) + \lambda(X - \sum_t c_t)$ where $c = (c_0, c_1, c_2, \ldots)$ represents the whole infinite length vector. We suppose throughout that $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

   1. Explain why the constraint must be binding.
   2. Write out the Kuhn-Tucker conditions assuming that at the optimum, each $c_t^* > 0$.
   3. Show that the Kuhn-Tucker conditions that you just found deliver the Euler equation for a 'renewable' resource with the growth curve $F(x) \equiv 0$.

   For the rest of this problem, we suppose that $u(c) = 2\sqrt{c}$.

   4. Find the optimal consumption path as a function of X and ρ.
   5. Give an intuitive explanation for why the c_0^* that you just found should be smaller when ρ is larger.
   6. Show that once t is large enough, c_t^* is larger for larger ρ.
   7. Give an intuitive explanation for why, for large enough t, c_t^* should be larger for larger ρ. [This answer should be fairly tightly related to the answer you gave for why c_0^* is smaller for larger ρ, so don't worry if it is slightly repetitive.]

D. The police arrest a man and accuse him of a crime. Given the police department's record, there is a prior probability ρ, $0 < \rho < 1$, that the man is $g$uilty, $\omega = g$, and a $(1-\rho)$ probability that the man is $i$nnocent, $\omega = i$. The man will be tried in front of a jury of M people. These M people will cast random, stochastically independent votes, $V_m = G$ for guilty and $V_m = I$ for innocent, $m = 1, \ldots, M$ with probabilities

$$P(V_m = G|\omega = i) = p, \quad P(V_m = G|\omega = g) = q, \quad 0 < p < \frac{1}{2} < q < 1.$$

Suppose that social utility depends on the innocence or guilt of the defendant, $\omega = i, g$, and the jury's decision, $V = I, G$, and

$$\underbrace{0 = u(V = G|\omega = i)}_{\text{worst mistake}} < \underbrace{u(V = I|\omega = g) = r}_{\text{mistake}} < \underbrace{u(V = G|\omega = g) = u(V = I|\omega = i) = 1}_{\text{correct decision}}.$$

1. Consider the unanimity rule for the jury, "Convict only if all jurors return a guilty vote," i.e. $V = G$ if $V_1 = V_2 = \cdots = V_M = G$, and $V = I$ otherwise. What are

   $$P(V = G|\omega = g), \; P(V = I|\omega = g), \; P(V = G|\omega = i), \text{ and } P(V = I|\omega = i)?$$

2. If juries are costless, set up the problem for finding the optimal M and characterize its dependence on ρ.
3. Repeat the previous, but now assume that the cost of a jury is an increasing function of the jury size and characterize the dependence of the optimal M on the cost.