**U.T. Economics Summer 2011 Math Camp**

**Date**: Tuesday, August 9 and Wednesday August 10

**Topics**: Saddle points, complementary slackness, convexity and concavity and the sufficiency of the K-T derivative conditions (FOC)
**Readings**: CSZ 5.8, MWG M.C-D, M.J-K

Some notes on topics covered

A set $C \subset \mathbb{R}^\ell$ is **convex** if $(\forall \mathbf{x}, \mathbf{y} \in C)(\forall \alpha \in (0,1))[\,\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in C\,]$. For a convex $C \subset \mathbb{R}^\ell$, a function $f : C \to \mathbb{R}$ is **concave** if the subgraph of $f$ is a convex set, where the subgraph of $f$ is the set $\{(\mathbf{x}, y) \in C \times \mathbb{R} : y \leq f(\mathbf{x})\}$. Equivalently, $f : C \to \mathbb{R}$ is concave if $(\forall \mathbf{x}, \mathbf{y} \in C)(\forall \alpha \in (0,1))[f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y})]$. A function $f$ is **convex** if $-f$ is concave, which reverses the inequalities in the definitions.

If $C$ is a convex set and the function $f : C \to \mathbb{R}$ has derivatives, then at any $\mathbf{x}^\circ \in C$, we can go in a straight line toward any $\mathbf{x} \in C$, i.e. travel along the vector $(\mathbf{x} - \mathbf{x}\circ)$. The change in $f$ along a tangent plane at $\mathbf{x}^\circ$ is $D_x f(\mathbf{x}^\circ)(\mathbf{x} - \mathbf{x}^\circ$. A point $\mathbf{x}^\circ$ is a **local maximum for $f$ in $C$** if for $(\forall \mathbf{x} \in C)[D_x f(\mathbf{x}^\circ)(\mathbf{x} - \mathbf{x}^\circ) \leq 0]$. Two comments.

i. A local maximum for a concave function is a global maximum.
ii. A verbal short-hand for the condition is that $\mathbf{x}^\circ$ is a global maximum if $D_x f(\mathbf{x}^\circ) = 0$ or if $\mathbf{x}^\circ$ is at a boundary of $C$ and $D_x f(\mathbf{x}^\circ)$ points outwards.

For $\mathbf{x} \in \mathbb{R}^\ell$, $f : \mathbb{R}^\ell \to \mathbb{R}$, $g : \mathbb{R}^\ell \to \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$, we are interested in the problems

$$V(\mathbf{b}) = \max f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq \mathbf{b}, \text{ and} \tag{1}$$

$$V(\mathbf{b}) = \max f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq \mathbf{b},\ \mathbf{x} \geq 0. \tag{2}$$

We will study the solutions using the associated **Lagrangean function**,

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda \cdot (\mathbf{b} - g(\mathbf{x})),\ \ \lambda \in \mathbb{R}^m_+. \tag{3}$$

A point $(\mathbf{x}^*, \lambda^*)$ is a **saddle point** for (1) if

$$(\forall \mathbf{x})(\forall \lambda \geq 0)[L(\mathbf{x}, \lambda^*) \leq L(\mathbf{x}^*, \lambda^*) \leq L(\mathbf{x}^*, \lambda)]. \tag{4}$$

A point $(\mathbf{x}^*, \lambda^*)$ is a **saddle point** for (2) if

$$(\forall \mathbf{x} \geq 0)(\forall \lambda \geq 0)[L(\mathbf{x}, \lambda^*) \leq L(\mathbf{x}^*, \lambda^*) \leq L(\mathbf{x}^*, \lambda)]. \tag{5}$$

The basic result is that if $(\mathbf{x}^*, \lambda^*)$ is a saddle point, then $\mathbf{x}^*$ solves the maximization problem that gave rise to the Lagrangean function. Four observations are in order.

i. An early step in proving this involves proving **complementary slackness**, for any saddle point, $\lambda^* \cdot (\mathbf{b} - g(\mathbf{x}^*)) = 0$. This means that $V(b) \equiv L(\mathbf{x}^*, \lambda^*)$, a fact that we will use in the envelope theorem proof that $D_b V(b) = \lambda$.

ii. If $f(\cdot)$ is concave, $g(\cdot)$ is convex, and both are differentiable, then for any $\lambda^\circ \geq 0$, $D_x L(\mathbf{x}^*, \lambda^\circ) = 0$ is both necessary and sufficient for $\mathbf{x}^*$ to solve $\max L(\mathbf{x}, \lambda^\circ)$ in (4).

iii. If $f(\cdot)$ is concave, $g(\cdot)$ is convex, and both are differentiable, then for any $\lambda^\circ \geq 0$, $D_x L(\mathbf{x}^*, \lambda^\circ) \leq 0$, $\mathbf{x}^* \geq 0$, and $\mathbf{x}^* \cdot D_x L(\mathbf{x}^*) = 0$ is both necessary and sufficient for $\mathbf{x}^*$ to solve

$$\max_{\mathbf{x}} L(\mathbf{x}, \lambda^\circ)$$

in (5).

iv. If $\mathbf{x}^\circ$ satisfies $0 \leq \mathbf{b} - g(\mathbf{x}^\circ)$, then $D_\lambda L(\mathbf{x}^\circ, \lambda^*) \geq 0$, $\lambda^* \geq 0$, and $\lambda^* \cdot D_\lambda L(\mathbf{x}^\circ, \lambda^*)$ is necessary and sufficient for $\lambda^*$ to solve

$$\min_{\lambda \geq 0} L(\mathbf{x}^\circ, \lambda)$$

in both (4) and (5).

## Problems

A. Read CSZ Ch. 5.8 and work most of the problems in it.

B. Find $V(w) = \max f(\mathbf{x})$ subject to $\mathbf{px} \leq w$ and $\frac{dV}{dw}$ for $\mathbf{p} \in \mathbb{R}^2_{++}$ and the following functions $f$. In each case, verify the relation between the Lagrange multiplier, $\lambda$ and $\frac{dV}{dw}$ and check that you have found a saddle point for the Lagrangean.
   1. $f(x_1, x_2) = \log(x_1) + 3\log(x_2)$.
   2. $f(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$. $\mathbf{x}^\circ = (7, 12)'$ and $\mathbf{y}^\circ = (19, 5)'$.
   3. $f(x_1, x_2) = x_1 + 2\sqrt{x_2}$.

C. The problems just given also have a $\mathbf{p}$ in them, i.e. $V(\mathbf{p}, w) = \max f(\mathbf{x})$ subject to $\mathbf{px} \leq w$. Another differentiability result for $V(\cdot, \cdot)$ is $D_{\mathbf{p}} V(\mathbf{p}, w) = \mathbf{x}^*(\mathbf{p}, w)$. Verify that this holds true for the three utility functions given in the previous problem, being careful with the third one, where there will be points at which $V$ is not differentiable.

D. If $u(x_1, x_2)$ is not differentiable, then writing down the FOC, i.e. the K-T conditions means that you have mis-understood basic aspects of the arguments. However, even for non-differentiable $u(\cdot, \cdot)$, $V(\mathbf{p}, w)$ can be very well-behaved. We will study the non-differentiable, concave function $u(x_1, x_2) = \min\{rx_1, x_2\}$ where $r > 0$. Along the line $x_2 = rx_1$, this function has a “fold” or a “kink,” and has no derivative. Let $V(\mathbf{p}, w) = \max u(x_1, x_2)$ subject to $x_1, x_2 \geq 0$, $p_1 x_1 + p_2 x_2 \leq w$.
   (a) Find $V(\mathbf{p}, w)$.
   (b) Find $\frac{\partial V}{\partial w}$ and interpret it.
   (c) Find $D_{\mathbf{p}} V$ and interpret it.

E. A set can either be **convex** or fail to be convex. A function $f: C \to \mathbb{R}$, $C \subset \mathbb{R}^\ell$ a convex set, can be either **concave**, or **convex**, both, or neither. For the following classes of sets and functions, check for these geometric properties.
1. $f: \mathbb{R}^\ell \to \mathbb{R}$ defined by $f(\mathbf{x}) = a + \mathbf{y}\mathbf{x}$.
2. $H^{\leq}_{\mathbf{y}}(r) = \{\mathbf{x} \in \mathbb{R}^\ell : \mathbf{x}\mathbf{y} \leq r\}$.
3. $H^{\leq}_{\mathbf{y}_1}(r_1) \cap H^{\leq}_{\mathbf{y}_2}(r_2)$.
4. $F: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a + bx + cx^2$.
5. $f(x) = e^x$, $g(x) = -e^x$, $h(x) = e^{-x}$, and $j(x) = -e^{-x}$.
6. $F: \mathbb{R}_+ \to \mathbb{R}$ defined by $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$.
7. $F: \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \frac{1}{1+e^{\lambda x}}$, $\lambda > 0$.
8. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(t) = (t\mathbf{x} + (1-t)\mathbf{y})^T M (t\mathbf{x} + (1-t)\mathbf{y})$ where $M$ is a symmetric, positive definite matrix.
9. $F: \mathbb{R}^\ell \to \mathbb{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T M \mathbf{x}$ where $M$ is a symmetric, positive definite matrix.
10. $C = \{\mathbf{x} \in \mathbb{R}^\ell : \mathbf{x}^T M \mathbf{x} \leq r\}$, $M$ symmetric and positive definite.

F. Let $C = [0,1] \subset \mathbb{R}^1$ and let $f: C \to \mathbb{R}$ be a concave differentiable function. Write out the three cases that guarantee that $\mathbf{x}^\circ \in C$ is a **global maximum for $f$ in** $C$, i.e. for $(\forall \mathbf{x} \in C)[D_x f(\mathbf{x}^\circ)(\mathbf{x} - \mathbf{x}^\circ) \leq 0]$.

G. Let $C = [0,1] \times [0,1] \subset \mathbb{R}^2$ and let $f: C \to \mathbb{R}$ be a concave differentiable function. Write out the five cases that guarantee that $\mathbf{x}^\circ \in C$ is a **global maximum for $f$ in** $C$, i.e. for $(\forall \mathbf{x} \in C)[D_x f(\mathbf{x}^\circ)(\mathbf{x} - \mathbf{x}^\circ) \leq 0]$.

H. This problem is about optimal emission reduction. At present, each of $I$ different firms produces a quantity $q_i^\circ$ of $NO_x$ as a by-product of their operations, so that the total quantity produced is $Q^\circ = \sum_i q_i^\circ$. At a cost $c_i(a_i)$, $c_i' > 0$ and $c_i'' > 0$, firm $i$ can abate to $q_i^\circ - a_i$. Total abatement is $A = \sum_i a_i$. We wish to reduce emissions from $Q^\circ$ to $Q^*$, i.e. to abate by an amount $A^* = Q^\circ - Q^*$ in a cost efficient fashion, that is, to solve

$$\min_{a_1,\ldots,a_I} \sum_i c_i(a_i) \text{ subject to } \sum_i a_i \geq A^*,\ a_i \geq 0 \text{ for } i = 1, \ldots, I.$$

The first problems ask you to work this with specific functions, the second set of problems asks you to work in more generality. Specifically, for the first set of problems, suppose that $c_i(a_i) = \frac{1}{2}\beta_i(a_i)^2$ where $\beta_1 < \beta_2 < \cdots < \beta_I$.

1. Give the Lagrangean for the efficient cost of abatement problem, then take its derivatives and set them equal to 0.
2. Show that the solution is $a_i^* = A^* \frac{1}{\beta_i} \frac{1}{\sum_j 1/\beta_j}$.
3. Suppose now that the $NO_x$ levels, $a_i$, can be accurately measured and are taxed at a rate $\tau$. Let $a_i^\tau$ denote firm $i$'s profit maximizing amount of abatement when the tax is $\tau$. Show that $a_i^\tau = \tau \frac{1}{\beta_i}$.

4. Find the tax rate $\tau(A^*)$ that achieves total abatement $A^*$ in an efficient manner. Explain **why** $\tau(\cdot)$ should be an increasing function. [If you did the algebra correctly, you will have found an increasing function. I want you to explain why you should have expected the function to be increasing.]
5. Suppose that the $NO_x$ is perfectly mixed, that the social costs/damage of the output level $Q$ is given by $C(Q) = \frac{1}{2}\gamma Q^2$. Characterize the optimal tax rate, $\tau^*$, and the optimal abatement, $A^*$.

For the next set of problems, assume only that $c_i' > 0$ and $c_i'' > 0$.

6. Give the Lagrangean for this problem.
7. Assuming that the solution involves each $a_i^* > 0$, show that at the solution, $c_i'(a_i^*) = c_j'(a_j^*)$ for each $i, j$ pair.
8. Suppose now that $a_i$ can be accurately measured and the Pigovian tax for it is $\tau$. Let $a_i^\tau$ denote firm $i$'s profit maximizing amount of abatement when the tax is $\tau$. Show that profit maximization by the firms will lead to $c_i'(a_i^\tau) = c_j'(a_i^\tau)$ for each $i, j$ pair.
9. Let $\tau(A^*)$ be the tax rate that achieves total abatement $A^*$ in an efficient manner. Show that $\tau(\cdot)$ is increasing.
10. Suppose that the $NO_x$ is perfectly mixed, that the social costs/damage of the output level $Q$ is given by $C(Q)$, and that marginal social cost/damage is increasing. Characterize the optimal tax rate, $\tau$, and the optimal abatement $A^*$.
11. Explain how tradeable permits can achieve $A^*$ at the same total cost.