U.T. Economics Summer 2011 Math Camp

Date: Friday, August 12 - Sunday, August 14

Topics: More on convexity, concavity, and the K-T conditions
Readings: CSZ 5.3-8, MWG M.C-D, M.J-K

This is a set of problems meant to consolidate your knowledge of constrained optimization and the K-T conditions.

A. [Efficiency and dynamics]: This problem is the beginning of the study of how differential patience/impatience interacts with efficient intertemporal allocation. There are two people with funny hair and entirely too much energy named Thing 1 and Thing 2. There are two time periods, $t = 0, 1$. 1's vector of consumption in the two periods is $(x_{1,0}, x_{1,1}) \geq (0, 0)$, and 2's vector of consumption in the two periods is $(x_{2,0}, x_{2,1}) \geq (0, 0)$. 1's utility is $U_1(x_{1,0}, x_{1,1}) = x_{1,0} + \rho_1 x_{1,1}$, while 2's utility is $U_2(x_{2,0}, x_{2,1}) = u_2(x_{2,0}) + \rho_2 u_2(x_{2,1})$, where $u_2(x) = 2\sqrt{x}$. Both ρ_1 and ρ_2 are strictly positive and strictly less than 1.

There is a total amount X of the consumption good available at $t = 0$. It can either be consumed or it can be invested at a rate of return $r > 0$. This means that saving s, i.e. consuming only $X - s$, right now allows for consumption of $(1 + r)s$ at $t = 1$.

The first two problems involve finding the individually optimal allocations of consumption across time. They are good practice for the problem of finding the set of efficient allocations of consumption across time and people.

1. Consider Thing 2's problem when there is no Thing 1,

$$\max U_2(x_{2,0}, x_{2,1}) \text{ subject to } x_{2,0} \leq x,\ x_{2,1} \leq (1 + r)[X - x_{2,0}],\ (x_{2,0}, x_{2,1}) \geq 0.$$

a. Give the associated Lagrangean.
b. Without solving the problem, explain how the optimal $x^*_{2,1}$ depends on ρ_2.
c. Show that the Kuhn-Tucker conditions for this problem are the same as the Euler equation for the fishery growth problem.
d. Solve for the optimal $(x^*_{2,0}, x^*_{2,1})$ as a function of X, r, and ρ_2.

2. Now consider Thing 1's problem when there is no Thing 2,

$$\max U_1(x_{1,0}, x_{1,1}) \text{ subject to } x_{1,0} \leq X,\ x_{1,1} \leq (1 + r)[X - x_{1,0}],\ (x_{1,0}, x_{1,1}) \geq 0.$$

a. Give the associated Lagrangean.
b. Without solving the problem, explain how the optimal $x^*_{1,1}$ depends on ρ_1.
c. Explain how the solution Kuhn-Tucker conditions for this problem depend on the relation between ρ_1 and $(1 + r)$.

d. Solve for the optimal $(x^*_{1,0}, x^*_{1,1})$ as a function of X, r, and ρ_1. [You will need to break up your answer into cases.]

We now turn to the problem of finding the dynamic efficient allocations for Thing 1 and Thing 2. To do this, we consider the solutions to problems of the form

$$\begin{aligned}(1)\qquad \max\ [\alpha U_1(x_{1,0}, x_{1,1}) + (1-\alpha)U_2(x_{2,0}, x_{2,1})]\ \text{ subject to}& \\ x_{1,0} + x_{2,0} \leq X,& \\ x_{1,1} + x_{2,1} \leq (1+r)[X - (x_{1,0} + x_{2,1})],& \\ (x_{2,0}, x_{2,1}) \geq (0,0),\ (x_{1,0}, x_{1,1}) \geq (0,0),&\end{aligned}$$

where $0 \leq \alpha \leq 1$.

3. When $\alpha = 0$, this is the first problem you solved above, when $\alpha = 1$, it is the second problem.
   a. Explain how the solution $(x^*_{1,0}, x^*_{1,1}), (x^*_{2,0}, x^*_{2,1})$ depends on α. [You can do this without solving the problem in (1).]
   b. Show that for $0 < \alpha < 1$, any solution to the problem in (1) is efficient.
   c. Give the Lagrangean and the Kuhn-Tucker conditions for the problem in (1).
   d. Argue that for $0 < \alpha < 1$, the Kuhn-Tucker conditions can never be satisfied at $x_{2,0} = 0$ or $x_{2,1} = 0$. Interpret this in terms of marginal utilities.
   e. Show that for small enough strictly positive α, the solution to the problem in (1) involves $(x^*_{1,0}, x^*_{1,1}) = (0,0)$.

B. [A proof problem] Suppose that $C = \mathbb{R}^\ell_+$ and that $f : C \to \mathbb{R}$ is concave and differentiable. Show that $[\,[\mathbf{x}^* \in C] \wedge (\forall \mathbf{x} \in C)[D_x f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \leq 0]\,] \Leftrightarrow [\,[D_x f(\mathbf{x}^*) \leq 0] \wedge [\mathbf{x}^* \geq 0] \wedge [\mathbf{x}^* \cdot D_x f(\mathbf{x}^*) = 0]\,]$.

C. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = (1 + x_2)^3 x_1^2 + x_2^2$ has only one point $\mathbf{x}^*$ with $D_x f(\mathbf{x}^*) = 0$, but that f has neither a global maximum or a global minimum. [Just a reminder that FOC are necessary but not sufficient.]

D. [Practice with K-T conditions at the boundaries] Solve the problem $V(\mathbf{p}, w) = \max \vec{1} \cdot \mathbf{x}$ s.t. $\mathbf{px} \leq w$, $\mathbf{x} \geq 0$ where $\mathbf{p} \gg 0$ and $\vec{1}$ is the length-ℓ vector of 1's.

E. For $\mathbf{x} \in \mathbb{R}^3$, find $V(b) = \max\,(100 - \mathbf{x} \cdot \mathbf{x})$ subject to
1. $\mathbf{p} \cdot \mathbf{x} \leq b$, $\mathbf{x} \geq 0$ where $\mathbf{p} = (1,2,1)^T$.
2. $\mathbf{p} \cdot \mathbf{x} \geq b$, $\mathbf{x} \geq 0$ where $\mathbf{p} = (1,2,1)^T$.

In each case, verify that $V'(b) = \lambda^*(b)$.

F. Let $V(b_1, b_2) = \max \mathbf{y} \cdot \mathbf{x}$ s.t. $\mathbf{x} \cdot \mathbf{x} \leq b_1$ and $\mathbf{px} \leq b_2$ where $\mathbf{y} = (1,4,1)^T$ and $\mathbf{p} = (1,2,3)^T$. Verify that $\partial V / \partial b_i = \lambda^*_i$. [For different values of b_1 and b_2, different constraints are binding.].

G. Let $V(\mathbf{p}, w) = \max \frac{1}{2}\log(1 + x_1) + \frac{1}{4}\log(1 + x_2)$ subject to $\mathbf{px} \leq w$, $\mathbf{x} \geq 0$. Verify that $\partial V(\mathbf{p}, w)/\partial w) = \lambda^*$. [Corner solutions matter here.]

H. Solve the problem $\max\,(\frac{1}{2}x_1 - x_2)$ s.t. $x_1 + e^{-x_1} + (x_3)^2 \leq x_2$, $x_1 \geq 0$. [Here, the objective function could be regarded as depending on x_1, x_2 and x_3 even though x_3 has no effect on f.]
I. Solve $\max\,(1 - \mathbf{x}\cdot\mathbf{x})$ s.t. $\mathbf{x} \geq (2,3)^T$ by direct geometry and by examining the K-T conditions.

Useful notation and definitions

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\ell$, $[\mathbf{x}, \mathbf{y}] := \{\alpha\mathbf{x} + (1-\alpha)\mathbf{y} : \alpha \in [0,1]\}$ denotes the line segment joining $\mathbf{x}$ and $\mathbf{y}$. The **convex hull** of a set S is defined as $\bigcap\{C : S \subset C,\ C \text{ is a convex set}\}$. This is the smallest convex set containing S.

Definition 1. *$\mathbf{v} \in S$ is an **extreme point of** $S \subset \mathbb{R}^\ell$ if for all $[\mathbf{x}, \mathbf{y}] \subset S$, $\mathbf{v} \in [\mathbf{x}, \mathbf{y}]$ implies that $\mathbf{v} = \mathbf{x}$ or $\mathbf{v} = \mathbf{y}$. $\text{extr}(S)$ denotes the set of extreme points of S.*

Some examples: $(0,1) \subset \mathbb{R}^1$ has no extreme points, and $[0,1] \subset \mathbb{R}^1$ has the extreme points 0 and 1. $\mathbb{R}^\ell_+$ has the single extreme point, 0. For $w > 0$, $\mathbf{p} \gg 0$, $\{\mathbf{x} \in \mathbb{R}^\ell_+ : \mathbf{p}\cdot\mathbf{x} \leq w\}$ has the extreme points $\{0, \{\frac{w}{\mathbf{p}_i}\mathbf{e}_i\} : i = 1, \ldots, \ell\}$. Extreme points exist when C is convex and compact (and we will define compactness later.)

Theorem 1 (Krein-Milman for $\mathbb{R}^\ell$)**.** *A compact convex $C \subset \mathbb{R}^\ell$ is equal to the convex hull of its extreme points. In particular, $\text{extr}(C) \neq \varnothing$ for compact convex C.*

For the following, you can take as given that the set $C = \{\mathbf{x} \in \mathbb{R}^\ell : \mathbf{x}\cdot\mathbf{x} \leq 1\}$ is compact and convex, and that $\text{extr}(C) = \{\mathbf{x} \in \mathbb{R}^\ell : \mathbf{x}\cdot\mathbf{x} = 1\}$. The geometry of this should be pretty clear.

We usually maximize a concave function over a convex set. Suppose instead that we are maximizing f and that $f : C \to \mathbb{R}$ is a convex function where C is a compact and convex set.

Lemma 1. *The solutions to the previous problem have non-empty intersection with $\text{extr}(C)$.*

Problems

J. Find $V(a,b) = \max x^2$ subject to $a \leq x$, $x \leq b$, $a < b$, $a, b \in \mathbb{R}$, and examine its differentiablity.
K. Find $V(r) = \max\,(x - r)^2$ s.t. $-1 \leq x \leq +1$ and examine its differentiability at $r = 0$.
L. For $\mathbf{x} \in \mathbb{R}^2$, using direct geometry, solve the problem $\max \mathbf{x}\mathbf{x}$ s.t. $\mathbf{x} \in [0,1] \times [0,1]$, and then check the K-T conditions at the solution. In what direction does the gradient point? How does the separation of the feasible set from the better-than set work?

M. [Simple eigenvalues and eigenvectors] Let M be the 2×2 matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and solve the problem $\max \mathbf{x}^T M\mathbf{x}$ s.t. $\mathbf{x}\mathbf{x} \leq 1$. The objective function is convex, and you know the set of extreme points of the constraint set. The K-T conditions should lead you to $M\mathbf{x} = \lambda\mathbf{x}$, equivalently, $(M - \lambda I)\mathbf{x} = 0$. The equation $\det(M - \lambda I) = 0$ has two solutions, at both of them, the gradient of the objective function points outwards, at one of them you are at a solution. [Another case of the K-T conditions not being sufficient.]

N. [More general eigenvalues and eigenvectors] Let M be a positive definite $\ell \times \ell$ matrix. Solve the problem $\max \mathbf{x}^T M\mathbf{x}$ s.t. $\mathbf{x}\mathbf{x} \leq 1$. Letting $\mathbf{x}_1^*$ be a solution to that problem, solve the problem $\max \mathbf{x}^T M\mathbf{x}$ s.t. $\mathbf{x}\mathbf{x} \leq 1$, $\mathbf{x}\mathbf{x}_1^* = 0$. Given the first two solutions, $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$, solve the problem $\max \mathbf{x}^T M\mathbf{x}$ s.t. $\mathbf{x}\mathbf{x} \leq 1$, $\mathbf{x}\mathbf{x}_1^* = 0$, $\mathbf{x}\mathbf{x}_2^* = 0$. Continue until you have ℓ solutions.

Suppose that we are at a solution $\mathbf{x}^*$ to $\max f(\mathbf{x})$ s.t. $g(\mathbf{x}) \leq \mathbf{b}$. Let $J = \{j \in \{1, \ldots, m\} : g_j(\mathbf{x}^*) = b_j\}$. If the gradients $\{D_x g_j(\mathbf{x}^*) : j \in B\}$ are not linearly independent, then it will be impossible to find a unique set of $\lambda_j^* \geq 0$, $j \in J$ such that $D_x f(\mathbf{x}^*) = \sum_{j \in J} \lambda_j^* D_x g(\mathbf{x}^*)$. This screws up the relation between the derivative of $V(\cdot)$ and λ. The linear independence of the constraints binding at a solution is called **constraint qualification**. Violations of constraint qualification cause problems.

One more problem

O. Consider $V(b_1, b_2) = \max (x_1 \cdot x_2)^{\frac{1}{2}} + x_1 + x_2$ s.t. $(x_1^2 + x_2^2) \leq b_1$, $x_1 + x_2 \leq b_2$, $x_1, x_2 \geq 0$. At the point $(b_1, b_2) = (2, 1)$, what can be said about $\partial V(2, 1)/\partial b_i$? What other pairs of (b_1, b_2) share this distasteful problem?