

## U.T. Economics Summer 2011 Math Camp

**Date:** Thursday and Friday, August 18 and 19

**Topics:** Discrete time search theory based on value function iterations, the policy improvement algorithm, and the uniqueness of the solution to the Bellman/Pontryagin equation.

**Readings:** CSZ 4.11

### 1. MODEL: RANDOM JOB OFFERS

At times  $t = 0, 1, 2, \dots$ , the DM receives a random wage offer  $W_t$ . the collection  $\{W_t : t \geq 0\}$  is iid. If the DM accepts the offer  $W_t = w$ , they are out of the labor market and continue to receive  $w$  until the crack of doom. If the DM rejects the offer, they receive  $c$  for this period, and will have another chance in the next period. Utility in each period is given by consumption and consumption is either  $c$  or the accept  $w$ . [We can add a savings problem on top of this, and you will need to learn how to do that at some point in the future.] The utility of a consumption stream  $x_t$ ,  $t = 0, 1, 2, \dots$  is  $U(x_0, x_1, x_2, \dots) = \sum_{t \geq 0} \beta^t u(x_t)$ ,  $0 < \beta < 1$ . Given a present offer of  $w$ , i.e.  $W_0 = w$ , following the optimal policy leads to the value function  $V(w)$ . To be a bit more specific:

- accepting the offer  $W_0 = w$  at  $t = 0$  leads to utility  $\sum_{t \geq 0} \beta^t u(w) = u(w) \frac{1}{1-\beta}$ ;
- rejecting the first offer and accepting the offer  $W_1 = w$  at  $t = 1$  leads to utility  $u(c) + \sum_{t \geq 1} \beta^t u(w) = u(c) + u(w) \frac{\beta}{1-\beta}$ ;
- etc.

### 2. VALUE FUNCTION ANALYSIS

If we suppose that starting tomorrow, at  $t = 1$ , the DM will start behaving optimally, then today the problem they face today, holding the offer  $W_0 = w$ , is

$$(1) \quad \max_{rej, acc} \left\{ \underbrace{u(c) + \beta E V(W_1)}_{rej}, \underbrace{u(w)(1-\beta)^{-1}}_{acc} \right\},$$

where the first term is the utility of rejecting the present offer, and the second term is the utility of accepting it. Behaving optimally today plus behaving optimally thereafter is behaving optimally at all points in time, hence the **Bellman equation** (or **Pontryagin equation**) for the **value function**,

$$(2) \quad V(w) = \max \left\{ u(c) + \beta E V(W_1), u(w) \frac{1}{1-\beta} \right\}.$$

Note that this is an equation in the function  $V(\cdot)$  that should be satisfied for each of infinitely many possible values of  $w$ . Such things are called **functional equations**. We will (later) be able to show that the solution to this functional equation is unique.

A **policy function** is a function from the present state,  $w$ , to the optimal action, reject or accept. Given the true value function, the *optimal* policy function rejects at

$w$  when  $u(c) + \beta E V(W_1) \leq (<) u(w) \frac{1}{1-\beta}$ , and accepts otherwise. From this, one can directly read off the dependence of the optimal cutoff wage on  $c$ .

### 3. ITERATING ON THE VALUE FUNCTION

In this simple case, we might not proceed in the iterative fashion about to be described because there might be more direct methods, but we can see how it works out. Suppose that we are looking at the problem at  $t = 0$  and we know that no utility will be delivered at  $t \geq 1$  (think “Eat, drink, and be merry, for tomorrow we die”). Then the value of where we end up is  $V_0(w_1) \equiv 0$ . The value function for this one period problem is

$$(3) \quad V_1(w) = \max \left\{ \underbrace{u(c) + \beta E V_0(W_1)}_{rej}, \underbrace{u(w) + \beta E V_0(w)}_{acc} \right\}.$$

Thus,  $V_1(w) = \max\{u(c), u(w)\}$ . The associated policy function is  $a_0(w)$  =reject if  $w < c$ , accept else.

Now back up one period from the end of the world, suppose that we will receive utility in periods  $t = 0$  and  $t = 1$  but no utility for  $t \geq 2$ , and that at  $t = 1$  we will follow the optimal policy for being in a situation where there will be no future utility,

$$(4) \quad V_2(w) = \max \left\{ \underbrace{u(c) + \beta E V_1(W_1)}_{rej}, \underbrace{(1 + \beta)u(w)}_{acc} \right\},$$

and the policy function,  $a_1(w)$  is the argmax for this problem. The next equation is

$$(5) \quad V_3(w) = \max \left\{ \underbrace{u(c) + \beta E V_2(W_1)}_{rej}, \underbrace{(1 + \beta + \beta^2)u(w)}_{acc} \right\},$$

and we just keep on going. The claim is that the limit function,  $V(w) = \lim_n V_n(w)$  is the unique function satisfying eqn. (2).

### 4. POLICY FUNCTION ITERATIONS

Again, in this simple case, we might not proceed in the iterative fashion about to be described in favor of more direct methods, but we can see how it works out. Propose a policy, say  $a_0(w)$  =reject if  $w < w_o$ , accept else. There is some expected value to following this policy. Find it, and call it  $V_0(\cdot)$ . Now consider the problem in which we commit to using  $a_0(\cdot)$  for  $t \geq 2$ , but are considering doing the best we can subject to this limitation. The problem is

$$(6) \quad V_1(w) = \max \left\{ u(c) + \beta E V_0(W_1), u(w) \frac{1}{1-\beta} \right\}.$$

Call the argmax from the above policy  $a_1(\cdot)$ , there is some expected value to following this policy, find it, call it  $V_1$  and the new problem is

$$(7) \quad V_2(w) = \max \left\{ u(c) + \beta E V_1(W_1), u(w) \frac{1}{1-\beta} \right\}.$$

This delivers a sequence of policy functions, each one of which is better than the previous one (a pretty easy argument).

## 5. DIRECT CALCULATION METHODS

Suppose that we consider a policy reject if  $W \leq w_0$ , accept otherwise. Let  $T$  be the random time until the first  $W_t > w_0$ . If  $T = \tau$ , the sequence of realized  $W_t$ 's satisfies  $W_0, W_1, \dots, W_{\tau-1} \leq w_0, W_\tau > w_0$ , and the associated utility is

$$(8) \quad \sum_{t=0}^{\tau-1} \beta^t u(c) + \sum_{t \geq \tau} \beta^t u(W_\tau).$$

The probability that  $\tau = 0$  is  $(1 - F(w_0))$ ,  $P(\tau = 0) = F(w_0)^0(1 - F(w_0))$ ,  $P(\tau = 1) = F(w_0)^1(1 - F(w_0))$ ,  $P(\tau = 2) = F(w_0)^2(1 - F(w_0))$ , etc. In other words,  $\tau$  has a geometric distribution. From these, we can calculate the utility associated with following this policy from  $t = 0$  onwards,

$$(9) \quad EU(w_0) = E \left[ \sum_{t=0}^{\tau-1} \beta^t u(c) + \sum_{t \geq \tau} \beta^t u(W_\tau) \right] = \\ \frac{1}{1 - \beta F(w_0)} \left[ u(c) \beta F(w_0) + E(u(W)|W > w_0) \frac{1 - F(w_0)}{1 - \beta} \right],$$

where  $W$  is a random variable independent of everything in sight and having the same distribution as all of the  $W_t$ . The resulting direct calculation problem is  $\max_{w_0} EU(w_0)$ .

## 6. PROBLEMS

- A. From the observation that there is a unique solution to  $V(w) = \max\{u(c) + \beta EV(W_1), u(w) \frac{1}{1-\beta}\}$  and that the optimal policy function solves it, give the dependence of the optimal cutoff on  $c$ .
- B. [Discrete time discounting] Suppose that the net benefits for a project are  $B_0, B_1, \dots$  where: net benefits per period are bounded, i.e. there exists a (perhaps quite large) number  $\bar{B}$  such that for all  $t$ ,  $|B_t| \leq \bar{B}$ ; there exists a time period  $T$  such that  $B_t < 0$  for  $t = 1, \dots, T$  and  $B_t > 0$  for  $t = T + 1, T + 2, \dots$ ; and for a sufficiently large  $T'$ ,  $\sum_{t=0}^{T'} B_t > 0$ , i.e. that if we do not discount, then the net benefits outweigh the net costs.

1. Show that for  $\rho$  sufficiently close to but still strictly less than 1,  $\sum_{t=0}^{\infty} \rho^t B_t > 0$ .

One interpretation of discounting has to do with the ability to invest resources at  $t$  and receive a larger set of resources at  $t + 1$  or later. If this is true, receiving an amount  $x$  at  $t$  is worth less than receiving the same  $x$  at  $t + 1$  or later. This has to do with  $\rho = \frac{1}{1+r}$  where  $r$  is the interest rate. Another interpretation of discounting has to do with the uncertainty of future benefits.

Let  $\tau$  be a random variable taking on the values  $0, 1, \dots$ , and having the **geometric distribution**, i.e.  $P(\tau = t) = \delta^t(1 - \delta)$ .

2. Give the expectation of  $\tau$ ,  $E\tau$ , as a function of  $\delta$ .
3. Show that  $\tau$  is "memoryless," i.e. that  $P(\tau = t + s | \tau \geq t) = P(\tau = s)$  for  $s, t \geq 0$ .

4. Give  $E \sum_{t=0}^{\tau} B_t$ . ( $\sum_{t=0}^{\tau} B_t$  is called, for pretty straightforward reasons, a “random sum.” Random sums turn out to be very important in many contexts beyond discounting.)
- C. Let  $f(c, w_0) = \frac{1}{1-\beta F(w_0)} \left[ u(c)\beta F(w_0) + E(u(W)|W > w_0) \frac{1-F(w_0)}{1-\beta} \right]$  and consider the problem  $\max_{w_0} f(c, w_0)$ . How does the optimal  $w_0$  depend on  $c$ ? [We’ve seen this result before.]
- D. For each of the following cases, find  $E(W|W > w_0)$  and  $E(u(W)|W > w_0)$  as a function of  $w_0$ .
- (a)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = w + 2\sqrt{w}$ .
  - (b)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = \frac{w^{1-a}}{1-a}$ ,  $0 < a < 1$ .
  - (c)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = 1 - e^{-\gamma w}$ ,  $\gamma > 0$ .
  - (d)  $W \sim \text{neg. exp.}(\lambda)$ ,  $u(w) = 1 - e^{-\gamma w}$ ,  $\lambda, \gamma > 0$ . [An especially interesting case has  $\gamma = \lambda$ .]
- E. Set  $c = 90$  and  $\beta = 0.95$ . For each of the following cases, perform the first two iterations on the policy function.
- (a)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = w + 2\sqrt{w}$ .
  - (b)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = \frac{w^{1-a}}{1-a}$ ,  $0 < a < 1$ .
  - (c)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = 1 - e^{-\gamma w}$ ,  $\gamma > 0$ .
  - (d)  $W \sim \text{neg. exp.}(\frac{1}{150})$ ,  $u(w) = 1 - e^{-\gamma w}$ ,  $\gamma > 0$ .
- F. Set  $c = 90$  and  $\beta = 0.95$ . For each of the following cases, find the optimal job search policy if it is possible to give it analytically, otherwise specify how you would find it numerically.
- (a)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = w + \sqrt{w}$ .
  - (b)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = \frac{w^{1-a}}{1-a}$ ,  $0 < a < 1$ .
  - (c)  $W \sim \text{Unif}[80, 220]$ ,  $u(w) = 1 - e^{-\gamma w}$ ,  $\gamma > 0$ .
  - (d)  $W \sim \text{neg. exp.}(\frac{1}{150})$ ,  $u(w) = 1 - e^{-\gamma w}$ ,  $\gamma > 0$ .