

U.T. Economics Summer 2011 Math Camp

Date: Wednesday, August 17

Topics: Two flavors of search theory: continuous time, based on hazard rate analysis; and discrete time based on value function iterations and the policy improvement algorithm.

Readings: Dynamic optimization readings

When pdf's exist, **hazard rates exist**. We will be using hazard rates to talk about the arrival time of success, so, following the convention in queueing theory, we will use W as the random **waiting time**. The **hazard rate at t** is $h_W(t) = \frac{f_W(t)}{1-F_W(t)}$, which can be understood as $\frac{1}{dt}P(t < W < t + dt | W > t)$ for small (infinitesimal) $dt > 0$.

A random variable, $W \geq 0$, is **incomplete** if it has a mass point at ∞ , corresponding to $\lim_{t \rightarrow \infty} F_W(t) < 1$. For a possibly incomplete W with density on $[0, \infty)$, the following summarizes the relation between the **density**, $f_W(t)$, the **cumulative distribution function (cdf)**, $F_W(t)$, the **reverse cdf**, $G_W(t)$, the **hazard rate**, $h_W(t)$, the **cumulative hazard**, $H_W(t)$, and the **mass at infinity**, q_W , for $t \geq 0$:

$$(1) \quad F_W(t) = \int_0^t f_W(x) dx; \quad G_W(t) = 1 - F_W(t); \quad h_W(t) = \frac{f_W(t)}{G_W(t)};$$
$$H_W(t) = \int_0^t h_W(x) dx, \quad G_W(t) = e^{-H_W(t)}; \quad \text{and } q_W = e^{-H_W(\infty)}.$$

If $H_W(t) = \int_0^t h(x) dx \uparrow \infty$ as $t \uparrow \infty$, then $q_W = 0$ so that $W < \infty$ with probability 1, so that $F_W(t) \uparrow 1$ and $G_W(t) \downarrow 0$ as $t \uparrow \infty$.

From $G_W(t) = e^{-H_W(t)}$ one sees that any non-negative h can serve as the hazard rate for some waiting time, W , and W is incomplete iff h is integrable. The following are well-known examples.

- i. An incomplete negative exponential has cdf $(1 - q_W)(1 - e^{-\lambda t})$ and everywhere decreasing hazard rate $\lambda \frac{1}{(q_W/(1-q_W))e^{\lambda t} + 1}$. If $q_W = 0$, then the hazard rate is constant and the waiting time is memoryless.
- ii. A Weibull distribution is of the form $W = X^\gamma$, $\gamma > 0$, where X has a negative exponential distribution with parameter λ . The cdf of an incomplete Weibull is $(1 - q_W)(1 - e^{-\lambda t^{1/\gamma}})$, and the hazard rate is $h_W(t) = \frac{\lambda t^{\frac{1-\gamma}{\gamma}}}{(q_W/(1-q_W))e^{\lambda t^{1/\gamma}} + 1}$. If $\gamma > 1$, the $h_W(0+) = \infty$ and the hazard rate is everywhere decreasing to 0. If $\gamma < 1$, then $h_W(0) = 0$ and the hazard rate is first increasing then decreasing.

At a flow cost of $c > 0$, one can keep searching for a source of higher profits (a low cost source of a crucial input, a process breakthrough, a new product). If found, expected profits of π^* result. If one abandons the search, the alternative yields expected profits of π_* , $\pi^* > \pi_* > 0$. Let W denote the waiting time till the source is found. Since one optimally searches in the more likely locations or ideas first, the arrival rate of W is decreasing over time. The choice problem is at what time, t_1 , does one stop searching and accept the lower π_* ?

The expected benefits of waiting an extra instant dt at t_1 are $(\pi^* - \pi_*)h_W(t_1)$, the expected costs are $(c + r\pi_*)$ because $r\pi_*$ is the perpetual annuity flow value of π_* . At an interior optimum, $0 < t_1^* < \infty$, the necessary first order conditions are $(\pi^* - \pi_*)h_W(t_1) = (c + r\pi_*)$. In order for this to be a local maximum rather than a local minimum, i.e. in order that the benefits of waiting are positive before t_1 and negative after t_1 , the hazard rate, $h_W(t)$, must be decreasing, $h'_W(t_1) < 0$.

To more formally analyze the problem of maximizing expected profits by choice of how long to search without success, note the following:

- a. if $1_{[0,t_1)}(W) = 1$, i.e. if $W < t_1$, then one incurs the search cost $\int_0^W (-c)e^{-rt} dt$ and receives the discounted profits of π^*e^{-rW} ; and
- b. if $1_{[t_1,\infty)}(W) = 1$, then one incurs the search cost $\int_0^{t_1} (-c)e^{-rt} dt$ and receives the discounted profits of $\pi_*e^{-rt_1}$.

Thus the problem is

$$(2) \quad \max_{t_1 \geq 0} \mathbb{E} \left[1_{[0,t_1)}(W) \left(\int_0^W (-c)e^{-rt} dt + \pi^*e^{-rW} \right) + 1_{[t_1,\infty)}(W) \left(\int_0^{t_1} (-c)e^{-rt} dt + \pi_*e^{-rt_1} \right) \right].$$

The first term in rounded brackets in (2) is equal to $-c\frac{1}{r}(1 - e^{-rW}) + \pi^*e^{-rW}$, and the second term is $-c\frac{1}{r}(1 - e^{-rt_1}) + \pi_*e^{-rt_1}$. Rewriting, the expectation in (2) is

$$(3) \quad \psi(t_1) = \int_0^{t_1} \left(-c\frac{1}{r}(1 - e^{-rw}) + \pi^*e^{-rw} \right) f_W(w) + \left(-c\frac{1}{r}(1 - e^{-rt_1}) + \pi_*e^{-rt_1} \right) G_W(t_1).$$

Taking derivatives with respect to t_1 , cancelling and rearranging yields

$$(4) \quad \psi'(t_1) = e^{-rt_1} [(\pi^* - \pi_*) f_W(t_1) + (c + r\pi_*) G_W(t_1)].$$

Multiplying and dividing by $G_W(t_1)$, $\psi'(t_1)$ can be re-expressed as

$$(5) \quad \psi'(t_1) = (e^{-rt_1} G_W(t_1)) [(\pi^* - \pi_*) h_W(t_1) + (c + r\pi_*)].$$

$\psi'(t_1) = 0$ only at those t_1^o for which $h_W(t_1^o) = \frac{c+r\pi_*}{\pi^*-\pi_*}$. At any such t_1^o ,

$$(6) \quad \psi''(t_1^o) = (e^{-rt_1} G_W(t_1))' [0] + (\pi^* - \pi_*) h'_W(t_1^o)$$

which can only be strictly negative if $h'_W(t_1^o) < 0$. Since interior t_1^* only happen when (5) is satisfied and $h'_W(t_1^*) < 0$, the comparative statics of t_1^* are immediate: it is decreasing in c , r , and π_* , and increasing in π^* .

Some examples may clarify the optimal waiting search times.

- i. If W is an incomplete negative exponential, then $h_W(0) = \lambda(1 - q_W)$, so that $t_1^* = 0$ if $\lambda(1 - q_W) < \frac{c+r\pi_*}{\pi^*-\pi_*}$ because search is never expected to be productive enough to be warranted. If $\lambda(1 - q_W) > \frac{c+r\pi_*}{\pi^*-\pi_*}$ then $0 < t_1^* < \infty$.
- ii. If W is a complete negative exponential, i.e. if $q_W = 0$ in the previous example, then $t_1^* = 0$ if $\lambda < \frac{c+r\pi_*}{\pi^*-\pi_*}$, $t_1^* = \infty$ if $\lambda > \frac{c+r\pi_*}{\pi^*-\pi_*}$, and $t_1^* = [0, \infty]$ if $\lambda = \frac{c+r\pi_*}{\pi^*-\pi_*}$.
- iii. If W has an incomplete Weibull distribution with $\gamma > 1$, then $0 < t_1^* < \infty$ because the initial searching is always productive enough in expected terms. If $\gamma < 1$

and the hazard rate is everywhere less than or equal to $\frac{c+r\pi_*}{\pi^*-\pi_*}$, then $t_1^* = 0$. If the hazard rate crosses $\frac{c+r\pi_*}{\pi^*-\pi_*}$, then either t_1^* is 0 or is the second (crossing) point where $h_W(t) = \frac{c+r\pi_*}{\pi^*-\pi_*}$.

- iv. If W is a complete Weibull with $\gamma > 1$, then $0 < t_1^* < \infty$, but if $\gamma < 1$, then either $t_1^* = 0$ or $t_1^* = \infty$.

Problems

A. [Continuous time discounting] When the continuously compounded interest rate is r , the worth (or value) of a flow of q /year over the next T years is $W(q, r, T) = \int_0^T qe^{-rt} dt$. The corresponding total flow is $F(q, T) = \int_0^T q dt = qT$. Since $e^{-rt} < 1$ for all $t > 0$, $W(q, r, T) < F(q, T)$. More generally, if the flow is $q(t)$, the net worth is $\int_0^T q(t)e^{-rt} dt$ and the total flow is $\int_0^T q(t) dt$.

1. If $T = 30$ and $W = \$250,000$ and the interest rate is 6%, i.e. $r = 0.06$, what are the corresponding yearly payments, q ? What is the difference between the total flow and W ? How do your answers change if $T = 15$?
2. Repeat the previous problem with $r = 0.04$, i.e. with a 1/3 reduction in the interest rate. Do the yearly payments go up by more or less than 1/3?
3. Show that if $W(q^\circ, r, T) = W^\circ$ and T is large, then $q^\circ \simeq rW^\circ$. To put this another way, if T is large, then flow payments corresponding to W° are approximately rW° . [Thus, if $r = 0.05$, then yearly payments corresponding to a value of, say, $W^\circ = \$100,000$, is $q^\circ = \$5,000$.]

B. Each of the following distributions is given in one way or another. In each case, give the cdf, the reverse cdf, the density, the hazard rate, and the expectation. Then, as a function of r, π^*, π_* , find the optimal t_1^* for the continuous time search problem described above.

1. The uniform distribution on the interval $[a, b]$, $a < b$, which has density

$$f(t) = \frac{1}{b-a} 1_{[a,b]}(t).$$

2. The negative exponential distribution with parameter $\lambda > 0$, which has cdf

$$F(t) = 1_{[0,\infty)}(t)[1 - e^{-\lambda t}].$$

3. The Weibull distribution with parameters $\lambda, \gamma > 0$, which is of the form $W = X^\gamma$ where X is a negative exponential with parameter λ . [Hint: for every $t \geq 0$, you know that $P(W \leq t) = P(X^\gamma \leq t) = P(X \leq t^{1/\gamma})$, and you have the cdf for the negative exponential in the previous problem.]

4. The Pareto(α, x) distribution on $[0, \infty)$, $\alpha, x > 0$,

$$F(t) = 1_{[0,\infty)}(t)[1 - \left(\frac{x}{t+x}\right)^\alpha].$$

C. [Mixture models] A mixture model is one using a distribution over distributions. In the theory of experiments, one can understand an information structure as giving a distribution over posterior distributions.

1. Suppose that our statistical model has five parameters, $\alpha \in [0, 1]$, $\mu_1, \mu_2 \in \mathbb{R}$, and $\sigma_1^2, \sigma_2^2 \in \mathbb{R}_{++}$. Specifically, that $g(x|\alpha, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \alpha f(x|\mu_1, \sigma_1^2) + (1 - \alpha)f(x|\mu_2, \sigma_2^2)$ where $f(x|\mu, \sigma^2)$ is the density of a Gaussian with mean μ and

variance σ^2 . Given iid data (x_1, \dots, x_n) , a **maximum likelihood estimator (MLE)** solves the problem

$$\max_{\alpha, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2} \prod_{i=1}^n g(x_i | \alpha, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2).$$

- (i) Set $\alpha = 0$, and solve for the MLE for μ_2, σ_2^2 .
 - (ii) Show that there is no MLE in the problem.
2. Suppose that the distribution of a waiting time W is negative exponential with parameter λ , but that λ is unknown, being uniformly distributed over an interval $[a, b] \subset \mathbb{R}_{++}$. Give the hazard rate for W and show that it is decreasing.