Assignment #1 for **Mathematics for Economists**
Fall 2016

Due date: Wed. Sept. 21.

Readings: CSZ, Ch. 2.10, Ch. 3.1-11.

Problems

A. Show that there are uncountably many complete and transitive preferences relations on $\mathbb{N}$. [Looking at CSZ 2.10, and especially the Cantor-Bernstein Theorem might help if you are stuck.]

B. **From Chapter 3.3**: 3.3.2 (p. 75), 3.3.10 (p. 77).

C. **From Chapter 3.3**: 3.3.14, 16, and 17 (p. 78).

D. **From Chapter 3.3**: 3.3.22 (p. 80), and 3.3.31 (p. 82).

E. **From Chapter 3.4**: 3.4.2 (p. 83), 3.4.7 and 12 (p. 84).

F. **From Chapter 3.5**: 3.5.3 (p. 88).

G. **From Chapter 3.6**: 3.6.4 (p. 91), 3.6.6, parts 1 and 2 (p. 92).

H. **From Chapter 3.7**: 3.7.12 (p. 96), 3.7.16 and 17 (p. 98).

I. Suppose that the growth curve in the fishery model of §3.7 is twice continuously differentiable. Your job is to compare the optimal growth paths and optimal steady states for the three utility functions $u(x_t) = \log(x_t)$, $v(x_t) = 2\sqrt{x_t}$, and $w(x_t) = x_t$.
   1. Give the three corresponding Euler equations.
   2. Give the optimal steady states as a function of the discount factor.
   3. Starting from the same x_0 below the steady states you just found, which utility function involves faster growth of the fish stock? [For the utility function $w(\cdot)$, you need to remember that the Euler equations were derived under the assumption that the solutions were strictly positive, and that may not be true.]
   4. Suppose now that x_0 is above the steady states and repeat the previous.

J. Write down the infima and suprema of the following sets. In each case say whether the infima and suprema are minimum and maximum elements.
   1. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.
   2. $\{\frac{3n}{4n+1} : n \in \mathbb{N}\}$.
   3. $\mathbb{Q} \cap (0, 1)$.
   4. $\{q \in \mathbb{Q} : q^2 < 7\}$.

K. Show the following.
   1. $\lim_{n\to\infty} \frac{2n^2+19n+3}{3n^2-35n+10^6} = \frac{2}{3}$.
   2. $\lim_{n\to\infty} \frac{2n^2+\sin n}{n+19} = \infty$.
   3. $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

L. Problems about long-run averages and discounting. For this problem, sequences start at $t = 0$, that is, a sequence is $(x_0, x_1, x_2, \ldots)$.
   1. Suppose that $x_t \to x$, show that $\frac{1}{T+1}\sum_{t\leq T} x_t \to x$.
   2. Consider the sequence

$$(x_t)_{t=0}^{\infty} = (\underbrace{0, 0}_{2^1}, \underbrace{1, 1, 1, 1}_{2^2}, \underbrace{0, 0, 0, 0, 0, 0, 0, 0}_{2^3}, \ldots).$$

Show that $\liminf_T \frac{1}{T+1}\sum_{t\leq T} x_t = \frac{1}{3}$ and $\limsup_T \frac{1}{T+1}\sum_{t\leq T} x_t = \frac{2}{3}$.

3. For $0 < \beta < 1$, express the density on $\{0, 1, 2, \ldots\}$ given by $p_t = (1-\beta)\beta^t$ as a convex combination of uniform densities on $\{0, 1, \ldots, t\}$, $t = 0, 1, 2, \ldots$.
4. Show that if $\lim_{T\uparrow\infty} \frac{1}{T+1}\sum_{t\leq T} x_t = r$, then $\lim_{\beta\uparrow 1}(1-\beta)\sum_{t=0}^{\infty} x_t\beta^t = r$.

M. Optimizing quadratic functions on $\mathbb{R}^k$. Throughout this problem: V is a negative definite, symmetric $k \times k$ matrix; $\mathbf{b}$ and $\mathbf{a}$ are a non-zero vectors in $\mathbb{R}^k$; M is an $\ell \times k$ matrix with of full row rank with $\ell < k$; and $U(\mathbf{x}) := \frac{1}{2}\mathbf{x}'V\mathbf{x} - \mathbf{a}'\mathbf{x}$.

1. A $k \times k$ matrix V is not invertible iff $V\mathbf{x} = 0$ for some non-zero $\mathbf{x} \in \mathbb{R}^k$. Using this fact, show the following: if V is negative definite, then it is invertible.
2. Give the gradient of $U(\mathbf{x})$.
3. Give the solution to $\max_{\mathbf{x}\in\mathbb{R}^k} U(\mathbf{x})$.
4. Let $\mathbf{x}^\circ$ be the solution from the previous problem and suppose that $d(\mathbf{x}^\circ, L) > 0$ where $L = \{\mathbf{x} : M\mathbf{x} = \mathbf{b}\}$. Give the solution to
$$\max_{\mathbf{x}\in\mathbb{R}^k} U(\mathbf{x}) \ \text{ s.t. } \ M\mathbf{x} = \mathbf{b}.$$
5. Let $V(\mathbf{b}) = \max_{\mathbf{x}\in\mathbb{R}^k} U(\mathbf{x})$ s.t. $M\mathbf{x} = \mathbf{b}$. From matrix algebra calculations, give the gradient of V when $d(\mathbf{x}^\circ, L) > 0$.