Assignment #2 for **Mathematics for Economists**
Fall 2016

Due date: Wed. Oct. 5.

Readings: CSZ, Ch. 4.1 - 9, Ch. 5.1

Problems

A. **From Chapter 4.3**: 4.3.6 (p. 116), 4.3.11 (p. 118), and 4.3.15 (p. 120).

B. **From Chapter 4.4**: 4.4.3 (p. 120) 4.4.6 (p. 121), 4.4.12, and 13 (p. 123)

C. **From Chapter 4.5**: 4.5.2 (p. 124), 4.5.8, 9, 10, and 11 (p. 126)

D. **From Chapter 4.6**: 4.6.3 (p. 129)

E. **From Chapter 4.7**: 4.7.14 (p. 132), 4.7.3, 6 (p. 130)

F. **From Chapter 4.8**: 4.8.9 (p. 140), 4.8.10, changing "six" to "three" (p. 140)i, and 4.8.13, pick two of the four (p. 141)

G. **From Chapter 4.9**: 4.9.2, (p. 143)

H. **From Chapter 5.1**: 5.1.14, 5.1.15 (p. 177), 5.1.18, 5.1.19 (p. 178), 5.1.28, and 29 (p. 180).

I. For $i = 1, \ldots, I$, let $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex, increasing cost function with $c_i(x_i)$ representing the cost of achieving pollution reduction of level $x_i \geq 0$ for source i. We assume throughout that $c_i(0) = 0$, that the sources are in the same geographical region, and that 'perfect' mixing occurs, that is, if source i reduces by x_i°, then the total regional reduction is $\sum_i x_i^\circ$ and every place in the region receives the same benefit from this.

   Let $C(R) = \min_{x_1,\ldots,x_I} \sum_i c_i(x_i)$ s.t. $\sum_i x_i \geq R$.

   1. Suppose that each $c_i(\cdot)$ is twice continuous differentiable, satisfies $c_i'(0) = 0$ and $c_i'(x_i) > 0$ for all $x_i > 0$. Give the FOcs for $C(R)$, show that $C(\cdot)$ is convex, and characterize its derivative.
   2. Now suppose that each $c_i(\cdot)$ is continuously differentiable and convex but may not satisfy the $c_i'(0) = 0$ condition. Give the Kuhn-Tucker conditions for $C(R)$, show that $C(\cdot)$ is convex, and characterize its derivative.
   3. Now suppose that each $c_i(\cdot)$ is increasing and convex. Characterize $C(R)$ as far as possible.
   4. Suppose now that each firm had to pay a price p per unit of the pollutant they emit. Give, in as much generality as you can manage, properties of the price $p(R)$ that results in a reduction of size R.

J. [About the Neyman-Pearson Lemma] Suppose that $\mathbf{X} = (X_1, \ldots, X_n)$ is a sequence of 0's and 1's that has the Bernoulli density $f(\mathbf{x}|\theta)$, $\theta \in \Theta = \{\theta_0, \theta_1\}$. There is a trade-off between α, the probability of a Type I error, and β, the probability of a Type II error. Let us suppose that we dislike both types of errors, and in particular that we are trying to devise a test, characterized by its rejection region, $\mathbf{X}_r$, to minimize

$$a \cdot \alpha(\mathbf{X}_r) + b \cdot \beta(\mathbf{X}_r),$$

where $a, b > 0$, $\alpha(\mathbf{X}_r) = P(\mathbf{X} \in \mathbf{X}_r|\theta_0)$, and $\beta(\mathbf{X}_r) = P(\mathbf{X} \notin \mathbf{X}_r|\theta_1)$. The idea is that the ratio of a to b specifies our trade-off between the two types of errors: the higher a is relative to b, the lower we want α to be relative to β.

Let $\mathbf{X}_{a,b} = \{\mathbf{x} : af(\mathbf{x}|\theta_0) < bf(\mathbf{x}|\theta_1)\} = \left\{\mathbf{x} : \frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)} > \frac{a}{b}\right\}$.

This decision rule is based on the **likelihood ratio**, and likelihood ratio tests appear regularly in statistics, often as part of calculating a uniformly most powerful test.

1. Show that a test of the form $\mathbf{X}_{a,b}$ solves the minimization problem given above. [Hint: Let $\phi(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbf{X}_r$ and $\phi(x) = 0$ otherwise. Note that $a \cdot \alpha(\mathbf{X}_r) + b \cdot \beta(\mathbf{X}_r) = a \int \phi(\mathbf{x}) f(\mathbf{x}|\theta_0)\, d\mathbf{x} + b \int (1 - \phi(\mathbf{x})) f(\mathbf{x}|\theta_1)\, d\mathbf{x}$ and that this is in turn equal to $b + \int \phi(\mathbf{x})[af(\mathbf{x}|\theta_0) - bf(\mathbf{x}|\theta_1)]\, d\mathbf{x}$. The idea is to minimize the last term in this expression by the choice of $\phi(\mathbf{x})$. Which $\mathbf{x}$'s should have $\phi(\mathbf{x}) = 1$?]
2. As a function of a and b, find $\mathbf{X}_{a,b}$ when $(X_1, \ldots, X_n)$ is iid Bern(θ), $\theta \in \Theta = \{\theta_0, \theta_1\} \subset (0, 1)$.

K. [Prices as coordination devices] Suppose that an organization has 4 subdivisions, each subdivision has 3 possible projects, projects $k = 1, \ldots, 12$. Project k, if run at proportion α, $0 \leq \alpha \leq 1$, gives benefit αB_k and costs αC_k of a scarce resource. The B_k and the C_k are as given in the following table.

Division	Project	B_k	C_k	B_k/C_k
I	1	600	100	6
	2	1,400	200	7
	3	1,000	200	5
II	4	500	50	10
	5	750	250	3
	6	1,000	200	5
III	7	900	100	9
	8	3,500	500	7
	9	1,600	400	4
IV	10	800	100	8
	11	1,000	250	4
	12	1,200	400	3

The company has a total of $1,200$ of the scarce resource.

1. Suppose first that each division is allocated 300 of the $1,200$ in resources, that is, if each of the four is allocated $\frac{1}{4}$ of the total. What is(are) the optimal plan(s) for each division? What are the resulting total profits?
2. Suppose now that the company allocates the $1,200$ optimally among the 12 projects. What is(are) the optimal plan(s) for the firm? What are the resulting total profits?
3. A simple rule for the center to announce to implement the best, coordinated solution, **without the central organizing office needing to know about the benefits and costs of the different projects**, is "Fund any project with $B/C > p$, talk to us about projects with $B/C = p$, and forget projects with $B/C < p$." What is the p that works?

4. Draw the parallel(s) between this solution and the solution to the Neyman-Pearson problem just above.
5. **Some commentary**. An alternative formulation of the previous rule is "Value the resource at a price p and pick projects to maximize profits, talk to us about projects that break even." One number, the price p to be paid for the resource, plus the simple and decentralizable rule, "maximize profits," achieves coordination on the scale of the firm. When the scarce resource is produced by another division within the firm, the price to the divisions using that scarce resource is called a **transfer price**. The general question is how to figure out the best transfer price without needing to know all of the information from the various divisions within the firm. For a lovely statement of the general ideas behind prices as transmitters of information, see esp. Sections III-VI in the paper "The Use of Knowledge in Society," by F. A. Hayek, *American Economic Review*, 35(4), 1945, 519-530.