Assignment #3 for **Mathematics for Economists**
Fall 2016

Due date: Mon. Oct. 24.

Readings: CSZ, Ch. 4.10 - 12, Ch. 5.3

Problems

A. **From Chapter 4.10**: 4.10.3.
B. **From Chapter 4.10**: 4.10.6.
C. **From Chapter 4.10**: 4.10.14.
D. **From Chapter 4.10**: 4.10.21.
E. **From Chapter 4.11**: 4.11.8 and 4.11.10
F. **From Chapter 4.11**: 4.11.12.
G. A function $f : M \to \mathbb{R}$, (M, d) a metric space, is **upper semi-continuous (usc)** iff $f^{-1}([r, \infty))$ is closed for every $r \in \mathbb{R}$. The closed sets $[r, \infty)$ open upwards, which is one reason to call these functions *upper* semi-continuous.
1. $f : \mathbb{R} \to \mathbb{R}$ is **right-continuous** if $(\forall x \in \mathbb{R})[f(x) = \lim_{\epsilon \downarrow 0} f(x + \epsilon)]$. Show that any non-decreasing, right-continuous f is usc. Give an example of a discontinuous, non-decreasing, right-continuous f.
2. The sequence characterization of upper semi-continuity: Show that f is usc iff $(\forall x \in M)(\forall x_n \to x)[f(x) \geq \limsup_n f(x_n)]$.
3. The ϵ-δ characterization of upper semi-continuity: show that f is usc iff $(\forall x \in M)(\forall \epsilon > 0)(\exists \delta > 0)[\,[d(x, x') < \delta] \Rightarrow [f(x) + \epsilon > f(x')]\,]$.
4. Show that the set of usc functions is a cone, that is, if f, g are usc and $\alpha, \beta \geq 0$, then $\alpha f + \beta g$ is usc.
5. If $f : M \to \mathbb{R}$ and there is a sequence of continuous functions, f_n, with the property that for all $x \in M$, $f_n(x) \downarrow f(x)$, then f is usc.
6. $f : M \to \mathbb{R}_+$ is usc iff the correspondence $x \mapsto [0, f(x)]$ is upper hemi-continuous.
7. $K \subset \mathbb{R}^\ell$ is compact iff for all usc $f : K \to \mathbb{R}$, $(\exists x^* \in K)[f(x^*) \geq f(X)]$.
8. Show that the class of usc functions on $\mathbb{R}^\ell$ is not **separable**, that is, for any countable dense subset $D \subset \mathbb{R}^\ell$, there are two usc functions f, g such that $f_{|D} = g_{|D}$ but $f \neq g$.

H. For metric spaces (M, d) and (M', d'), a function $f : M \to M'$ is **Cauchy continuous** if for any d-Cauchy sequence x_n in M, $f(x_n)$ is a d'-Cauchy sequence in M'.
1. We know that every Lipschitz continuous function is uniformly continuous but that the reverse is not true. Show that if f is uniformly continuous, then it is Cauchy continuous.
2. The usual metric on $\mathbb{R}$ is $d(x, y) = |x - y|$. An alternative metric is $d'(x, y) = |F(x) - F(y)|$ where $F(r) = \frac{e^r}{1+e^r}$. Let $(M, d) = (\mathbb{R}, d)$, $(M', d') = (\mathbb{R}, d')$ and let $f(x) = x$. Show that f is Cauchy continuous but not uniformly continuous.
3. Show that if f is Cauchy continuous, then it is continuous.
4. Give a continuous function that is not Cauchy continuous.

I. For a metric space (M, d), let $C_b(M; \mathbb{R})$ denote the set of bounded, continuous functions, that is, the set of functions with $\|f\|_\infty := \sup_{x \in M} |f(x)| < \infty$.
   1. For $f, g \in C_b(M; \mathbb{R})$, show that $\|f\|_\infty + \|g\|_\infty \geq \|f + g\|_\infty$. Give an example with strict inequality.
   2. Show that $d_\infty(f, g) := \|f - g\|_\infty$ is a metric on $C_b(M; \mathbb{R})$.
   3. Show that $(C_b(M; \mathbb{R}), d_\infty)$ is a complete metric space.

J. Let X and Θ be metric spaces. Suppose that
   - $u : X \times \Theta \to \mathbb{R}$ is a bounded, jointly continuous function,
   - $\theta \mapsto A(\theta) \subset X$ a non-empty valued, compact-valued continuous correspondence, and
   - $(x, \theta) \mapsto P(x, \theta) \in \Delta(\Theta)$ satisfies $[(x_n, \theta_n) \to (x, \theta)] \Rightarrow (\forall f \in C_b(\theta; \mathbb{R}))[\int f \, dP(x_n, \theta_n) \to \int f \, dP(x, \theta)]$.

   Finally, for $v_1 \in C_b(\Theta; \mathbb{R})$ and fixed $\beta \in (0, 1)$, define $v_2 = T(v_1)$ by
$$v_2(\theta) = \max_{x \in A(\theta)} [u(x, \theta) + \beta \int v_1 \, dP(x, \theta)].$$
   1. For $v_1 \in C_b(\theta; \mathbb{R})$, show that $v_2 = T(v_1) \in C_b(\Theta; \mathbb{R})$.
   2. Show that for $v, w \in C_b(\Theta; \mathbb{R})$, $d_\infty(T(v), T(w)) \leq \beta d_\infty(v, w)$.

K. Read Chapter 5.3 in CSZ.