Assignment #6 for **Mathematics for Economists**
Economics 392M.8, Fall 2013

Due date: Mon. Oct. 14.

Readings: Chapters 4.8, 4.9, 4.10, 5.2, and 5.3

Problems

We now turn to continuity, uniform continuity, and Lipschitz continuity, as well as some of the beginnings of convexity.

A. **From Chapter 4.8**: 4.8.7 and 4.8.9.

B. **From Chapter 4.8**: 4.8.10, any two of the four parts of 4.8.13, 4.8.16.

C. **From Chapter 4.9**: 4.9.2.

D. **From Chapter 4.10**: 4.10.4 and 4.10.5.

E. A function $f : M \to \mathbb{R}$, (M, d) a metric space, is **upper semi-continuous (usc)** iff $f^{-1}([r, \infty))$ is closed for every $r \in \mathbb{R}$. The closed sets $[r, \infty)$ open upwards, which is one reason to call these functions *upper* semi-continuous.

1. $f : \mathbb{R} \to \mathbb{R}$ is **right-continuous** if $(\forall x \in \mathbb{R})[f(x) = \lim_{\epsilon \downarrow 0} f(x + \epsilon)]$. Show that any non-decreasing, right-continuous f is usc. Give an example of a discontinuous, non-decreasing, right-continuous f.
2. The sequence characterization of upper semi-continuity: Show that f is usc iff $(\forall x \in M)(\forall x_n \to x)[f(x) \geq \limsup_n f(x_n)]$.
3. The ϵ-δ characterization of upper semi-continuity: show that f is usc iff $(\forall x \in M)(\forall \epsilon > 0)(\exists \delta > 0)[\,[d(x, x') < \delta] \Rightarrow [f(x) + \epsilon > f(x')]\,]$.
4. Show that the set of usc functions is a cone, that is, if f, g are usc and $\alpha, \beta \geq 0$, then $\alpha f + \beta g$ is usc.
5. If $f : M \to \mathbb{R}$ and there is a sequence of continuous functions, f_n, with the property that for all $x \in M$, $f_n(x) \downarrow f(x)$, then f is usc.
6. $f : M \to \mathbb{R}_+$ is usc iff the correspondence $x \mapsto [0, f(x)]$ is upper hemi-continuous, i.e. iff
$$(\forall x \in M)(\forall \epsilon > 0)(\exists \delta > 0)\,[\,[d(x, y) < \delta] \Rightarrow [[0, f(y)] \subset [0, f(x)]^{\epsilon}]\,]\,. \tag{1}$$
7. $K \subset \mathbb{R}^{\ell}$ is compact iff for all usc $f : K \to \mathbb{R}$, $(\exists x^* \in K)[f(x^*) \geq f(X)]$.
8. Show that the class of usc functions on $\mathbb{R}^{\ell}$ is **not** separable, that is, for any countable dense subset $D \subset \mathbb{R}^{\ell}$, there are two usc functions f, g such that $f_{|D} = g_{|D}$ but $f \neq g$.

F. Read Chapter 5.2.

G. Read Chapter 5.3.