Assignment #10 for **Mathematics for Economists**
Economics 392M.8, Fall 2013

Due date: Mon. Nov. 18.

Readings: CSZ, Ch. 5.1-8.

This week's basic results are the separating hyperplane theorem and the continuity of concave/convex functions on the relative interior of their domain. We will spend most of our time on applications of convexity: neoclassical duality theory for the consumer; Blackwell's ranking of the informativeness of statistics/information structures; Nash bargaining theory; Farkas's Theorem of the Alternative and its implications for the equivalence of arbitrage free pricing and the existence of a martingale measure.

A. For a compact $K \subset \mathbb{R}^\ell$, define $\mu_K : \mathbb{R}^\ell \to \mathbb{R}$ by $\mu_K(\boldsymbol{y}) = \max_{\boldsymbol{x} \in K} \boldsymbol{y}\boldsymbol{x}$. For the following sets K, give the support function, and check that at any point $\boldsymbol{y}^\circ$ where $\mu_K(\cdot)$ is differentiable, $D_{\boldsymbol{y}}\mu_K(\boldsymbol{y}^\circ) = \text{argmax}_{\boldsymbol{x}\in K}\, \boldsymbol{y}^\circ \boldsymbol{x}$.
   1. $K = \{\boldsymbol{x}_1, \boldsymbol{x}_2\} \subset \mathbb{R}^2$, $\boldsymbol{x}_1 = (2, 3)'$, $\boldsymbol{x}_2 = (-1, 4)$.
   2. $K = \{\boldsymbol{x} : \|\boldsymbol{x}\| = 1\}$.
   3. $K = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{y}\| = 1\}$ where $\boldsymbol{y}$ is a point in $\mathbb{R}^\ell$.
   4. $K = \{\boldsymbol{x} \in \mathbb{R}^2_+ : \boldsymbol{p}\boldsymbol{x} \leq w\}$ where $\boldsymbol{p} \gg 0$ and $w > 0$.

B. **Basic neoclassical duality for the theory of the consumer**: Suppose that preferences $\succeq$ on $\mathbb{R}^2_+$ can be represented by the utility function $u(x_1, x_2) = x_1 + \sqrt{x_2}$.
   1. Find
      a. $x(\boldsymbol{p}, w)$,
      b. the income and price elasticities of consumption,
      c. the indirect utility function,
      d. the Hicksian demand function,
      e. the expenditure function,
      f. the Slutsky substitution matrix.
   2. Check directly that
      a. the indirect utility function is hd(0) in $(\boldsymbol{p}, w)$,
      b. the indirect utility function is strictly increasing in w,
      c. the indirect utility function is decreasing in $\boldsymbol{p}$, and give the region(s) in which it is strictly decreasing,
      d. the indirect utility function is quasi-convex,
      e. the Hicksian demand function is hd(0) in $\boldsymbol{p}$,
      f. the Hicksian demand function is strictly increasing in u,
      g. the Hicksian demand function is the derivative of the expenditure function,
      h. the expenditure function is hd(1) in $\boldsymbol{p}$,
      i. the expenditure function is strictly increasing in u,
      j. the expenditure function is strictly increasing in $\boldsymbol{p}$,
      k. the expenditure function is concave in $\boldsymbol{p}$,
      l. the following equalities hold

$$h(\boldsymbol{p}, u) = x(\boldsymbol{p}, e(\boldsymbol{p}, u)), \quad \text{and} \quad x(\boldsymbol{p}, w) = h(\boldsymbol{p}, v(\boldsymbol{p}, w)),$$

m. Roy's identity, and

n. the symmetry of the Slutsky matrix.

C. **From Chapter 8.12**: 8.12.4.

D. [Farkas] Prove the following: if A is an $m \times n$ matrix, $\boldsymbol{b} \in \mathbb{R}^m$, and $F = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$, then either (i) $F \neq \emptyset$ or (ii) $(\exists \boldsymbol{y} \in \mathbb{R}^m)[\boldsymbol{y}'A \geq 0 \wedge \boldsymbol{y}'\boldsymbol{b} < 0]$, but not both.

E. Check whether the following systems have non-negative solutions.

1. $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (1, 1)'$ and

$$A = \begin{bmatrix} 4 & 1 & -5 \\ 1 & 0 & 2 \end{bmatrix}.$$

2. $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (2, 2, 2, 1)'$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$