Assignment #12 (last one!) for **Mathematics for Economists**
Economics 392M.8, Fall 2013

Due date: Fri. Dec. 6.

Readings: CSZ, Ch. 5.11-12.

We are going to spend the last three meetings on Farkas's Lemma and arbitrage, and then on some fixed point theorems and their uses.

Lemma (Farkas): *If A is an $m \times n$ matrix, $\boldsymbol{b} \in \mathbb{R}^m$, and $F = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$, then either*
(i) $F \neq \emptyset$, *or*
(ii) $(\exists \boldsymbol{y} \in \mathbb{R}^m)[\boldsymbol{y}'A \geq 0 \wedge \boldsymbol{y}'\boldsymbol{b} < 0]$,
but not both.

Sketch of proof. $F \subset \mathbb{R}^m$ is a closed convex cone. Either (i′) $\boldsymbol{b} \in F$ or (ii′) $\boldsymbol{b} \notin F$, but not both. If (i′), then $F \neq \emptyset$, if (ii′), then let $\boldsymbol{y} = (\boldsymbol{b} - \boldsymbol{z}^*)$ where $\boldsymbol{z}^*$ solves $\min_{\boldsymbol{z} \in F}(\boldsymbol{b}-\boldsymbol{z})'(\boldsymbol{b}-\boldsymbol{z})$, and use the same argument as used in the proof of the separation of closed convex sets from points not in the set. □

To get at arbitrage in markets, we have the notion of assets that return different amounts in different states, where states will be realized in the future. Specializing, suppose that we are at time $t = 0$, there are m assets, and at $t = 1$, one of n states will happen. Letting S be the set of states, we have $\#S = n$, and probability distributions over S are denoted by $\Delta(S) = \{\pi \in \mathbb{R}^S_+ : \sum_{s \in S} \pi_s = 1\}$.

Returns, measured in dollars returned per dollar of investment (where "dollar" is understood broadly) and that asset $i \in \{1, \ldots, m\}$ returns $a_{i,j}$ if state $j \in \{1, \ldots, n\}$ happens. Let A be the $n \times m$ matrix of returns.[1] If these are investments in limited liability corporations, then $A \geq 0$. We are interested in prices of the assets, $\boldsymbol{p} \in \mathbb{R}^m_{++}$ and properties implied by the no arbitrage condition. The typical setting has many more states than assets, $n > m$, e.g. there is a large number of investment vehicles, but a huge number of payoff relevant outcomes.

Suppose, and this is important, that the decision maker can hold any amount of any of the assets, positive or negative, so that a portfolio is a vector, $\boldsymbol{y} \in \mathbb{R}^m$. The vector of the state-contingent wealth that results from portfolio $\boldsymbol{y}$ is $\boldsymbol{y}'A \in \mathbb{R}^n$, and the cost to buy this portfolio is $\boldsymbol{y}'\boldsymbol{p}$. We say that prices satisfy the **no arbitrage condition** if for any portfolio, $\boldsymbol{y}$,

$$[\boldsymbol{y}'A \geq 0] \Rightarrow [\boldsymbol{y}'\boldsymbol{p} \geq 0].$$

By Farkas's Lemma, if prices satisfy the no arbitrage condition, then there exists $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \geq 0$, such that $A\boldsymbol{x} = \boldsymbol{p}$. Since $\boldsymbol{p} \gg 0$, we know that $\boldsymbol{x} \neq 0$. Normalize $\boldsymbol{x}$ into a probability $\pi \in \Delta(S)$ by $\pi_s = \boldsymbol{x}_s / (\sum_{s \in S} \boldsymbol{x}_s)$. Since scaling all prices by a positive constant affects nothing, we have

$$\boldsymbol{p} = A\pi.$$

[1] It is worth thinking through a probability space and random vector formulation of this.

We have now shown that if the prices allow no arbitrage, then it is as if the price of each asset is given by its expected payoff under a probability distribution. Such a probability distribution is called a **risk neutral probability distribution**. If π describes the probability distribution over the different states, then any risk neutral investor is absolutely indifferent between all possible portfolios because the price of any asset is exactly its expected value.

Problems

A. [Market completeness] Suppose, in contravention of all evidence and good sense, that $m \geq n$ and the rank of A is n, i.e. the n columns of A are linearly independent.
   1. The market is **complete** if for any state-dependent vector of payoffs, $\boldsymbol{z} \in \mathbb{R}^n$, there exists a portfolio, $\boldsymbol{y}$, that generates $\boldsymbol{z}$. Show that $m \geq n$ and the n columns of A being linearly independent implies market completeness.
   2. What about the converse of the previous statement? Does market completeness imply that $m \geq n$ and the columns of A are linearly independent?
   3. Show that market completeness implies that the risk neutral probability is unique.
   4. Show that market incompleteness implies that the set of risk neutral probabilities is a compact convex set.

B. [Option pricing] At $t = 0$, the price of a stock is S, the price of a bond is B, and there are two states, call them u, in which case, at $t = 1$, the price of the stock is $S_1 = Su$, $u > 1$, and d, in which case, at $t = 1$, the price of the stock is $S_1 = Sd$, $0 < d < 1$, and that whatever happens to the price of the stock, the price of the bond at $t = 1$ is rB, $r > 1$. If you buy, at $t = 0$, one unit of an option on the stock with strike price κ, then at $t = 1$, if the price of the stock S_1 satisfies $S_1 > \kappa$, then you can "exercise the option," i.e. you can buy the stock at price κ and sell it (immediately) at S_1, which yields a profit of $(S_1 - \kappa)$ per unit of the option. On the other hand, if $S_1 \leq \kappa$, then the option is worthless (because if you exercise it you lose money). Combining, at $t = 1$, the value of the option with strike price κ is $\max(0, S_1 - \kappa)$. We will take two approaches to the question, "What is the value of such an option?" Unless $d < r < u$, holding the option makes no sense and will not be on the market.
   1. Suppose that $\pi = (\rho, (1 - \rho))$, i.e. the probability of the state u is ρ, and the probability of d is $(1 - \rho)$. Show that, using the bond's rate of increases as the discount factor, the expected discounted value of the κ-option is

      $$\rho \tfrac{1}{r} \max(0, Su - \kappa) + (1 - \rho) \tfrac{1}{r} \max(0, Sd - \kappa).$$

      Graph this as a function of ρ for fixed κ, and as a function of κ for fixed ρ.
   2. Suppose that the decision maker/investor has expected utility preferences given by the von Neumann-Morgenstern utility function $u(r) = -e^{-\gamma r}$, $\gamma > 0$, has a total wealth w to be allocated between the stock, the bond, and the option. Suppose that the prices are given by $\boldsymbol{p} = (p_S, p_B, p_0)'$. Find demand as a function of w, ρ, and $\boldsymbol{p}$.

3. [Black-Scholes] For the three assets, the stock, the bond, and the κ-option, the matrix A is given by

$$A = \begin{bmatrix} Su & Sd \\ Br & Br \\ \max(0, Su - \kappa) & \max(0, Sd - \kappa) \end{bmatrix}.$$

Suppose that there is no arbitrage, solve for the unique risk neutral probability measure, and give the expected value of the κ-option.

4. In the previous problem, the value of the option did not depend on probabilities given ahead of time. This is an implication of market completeness. When might one believe in this kind of implication?

C. [Portfolio demand in the presence of ambiguity] When there are two outcomes, $x < y$, and the probability of y is $p \in [a, b]$, $0 \leq a \leq b \leq 1$, the α-minmax EU preferences are represented by the utility function

$$U([a, b]) = \alpha \min_{p \in [a,b]} [(1-p)u(x) + pu(y)] + (1-\alpha) \max_{q \in [a,b]} [(1-q)u(x) + qu(y)].$$

The fact that $p \in [a, b]$ is a reflection of ambiguity, that is, of the investor not knowing the exact probability. When markets are incomplete, prices being arbitrage free means that there is a convex interval of probabilities when there are two outcomes.

1. Show that $U([a, b]) = \frac{1}{2}[au(x) + bu(y)] - \frac{1}{2}v[bu(y) - au(x)]$ for a v that depends on α. Give v, explain why $v > 0$ can be interpreted as ambiguity aversion, and give the conditions on α that yield $v > 0$.
2. Suppose that only the stock and the bond are available and that payoffs are given as above. Find the optimal portfolio demand as a function of w, $[a, b]$, w, and $\boldsymbol{p} = (p_S, p_B)'$.
3. For given $[a, b]$, find the certainty equivalent of the return to investing z dollars in the option.

D. In the textbook, the argument for the existence of an ergodic distribution for a finite Markov chain used Brouwer's fixed point theorem. This is a bit of overkill — Brouwer's fixed point theorem is a deep result, almost too deep to be using for a bit a matrix algebra. Derive the existence of $\pi \in \Delta(S)$ such that $P'\pi = \pi$ from Farkas's Lemma (here S is finite and P is a transition matrix). [Hint: Let $\mathbf{1}$ be an S-vector of 1's, consider the system of equations $A\pi = \boldsymbol{b}$ where A is $(S+1) \times S$ matrix with $P' - I$ being the top $S \times S$ part, and e' the bottom row, and $\boldsymbol{b} = (\mathbf{0}', 1)'$ where $\mathbf{0}$ is an S-vector of 0's.].

E. From Ch. 5.11: 5.11.4 and 5.11.7.

F. From Ch. 5.11: 5.11.10 and 5.11.17.

G. From Ch. 5.12: 5.12.18.

H. From Ch. 5.12: 5.12.19.

I. From Ch. 5.12: 5.12.23, 5.12.27 and 5.12.34.

J. From Ch. 5.12: 5.12.33.