Concavity and continuity, **Mathematics for Economists**
Economics 392M.8, Fall 2013

This alternative proof that a concave function is continuous on the relative interior of its domain first shows that it is bounded on small open sets, then from boundedness and concavity, derives continuity.

Theorem 1. *If $f : C \to \mathbb{R}$ is concave, $C \subset \mathbb{R}^\ell$ convex with non-empty interior, then f is continuous on int(C).*

Three lemmas deliver the proof. Throughout, we maintain the assumptions that $f : C \to \mathbb{R}$ is concave, that C is convex, and that it has non-empty interior. We first show that a concave f is bounded below on small balls around any point in the interior.

Lemma 1. *For all $\boldsymbol{x}_\circ \in$ int(C), there exists $\epsilon > 0$ and $r \in \mathbb{R}$ such that $f(B_\epsilon(\boldsymbol{x}_\circ)) \geq r$.*

Proof. Pick $\epsilon' > 0$ such that $B_{2\epsilon'}(\boldsymbol{x}_\circ) \subset C$. Consider the $2 \cdot \ell$ points $S = \{\boldsymbol{x}_0 \pm \epsilon' \boldsymbol{e}_i : i \in \{1, \ldots, \ell\}\}$ where $\boldsymbol{e}_i$ is the unit vector in the i'th direction. By concavity, any $\boldsymbol{y} \in \mathbf{co}\,(S)$ is a convex combination of the $\boldsymbol{x} \in S$, therefore $f(\boldsymbol{y}) \geq \sum \alpha_{\boldsymbol{x}} \boldsymbol{x} \geq \min\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\}$. Now pick $\epsilon > 0$ such that $B_\epsilon(\boldsymbol{x}_\circ) \subset \mathbf{co}\,(S)$. □

We now show that f is bounded on small balls around any point in the interior.

Lemma 2. *For $B_\epsilon(\boldsymbol{x}_\circ) \subset C$, if $f(B_\epsilon(\boldsymbol{x}_\circ)) \geq m$, then $|f(B_\epsilon(\boldsymbol{x}_\circ))| \leq |m| + 2f(\boldsymbol{x}_\circ)$.*

Proof. Translate $B_\epsilon(\boldsymbol{x}_\circ)$ by subtracting $\boldsymbol{x}_\circ$, i.e. suppose that $\boldsymbol{x}_\circ = 0$. For any $\boldsymbol{y} \in B_\epsilon(0)$, $f(0) \geq \frac{1}{2}f(\boldsymbol{y}) + \frac{1}{2}f(-\boldsymbol{y})$, so that

$$\tfrac{1}{2}f(\boldsymbol{y}) \leq f(0) - \tfrac{1}{2}f(-\boldsymbol{y}), \text{ or } f(\boldsymbol{y}) \leq 2f(0) - f(-\boldsymbol{y}). \tag{1}$$

Now $2f(0) - f(-\boldsymbol{y}) \leq 2f(0) - m \leq 2|f(0)| + |m|$ and $m \leq f(\boldsymbol{y})$, we have $m \leq f(\boldsymbol{y}) \leq 2|f(0)| + |m|$, so in particular, $|f(\boldsymbol{y})| \leq 2|f(0)| + |m|$. □

We now show that f is Lipschitz continuous on small neighborhoods.

Lemma 3. *For $B_r(\boldsymbol{x}_\circ) \subset C$ and B its closure, if $|f(B)| \leq M$, then for any $\epsilon \in (0, r)$, f is $\left(\frac{2M}{\epsilon}\right)$-Lipschitz on $B_{(r-\epsilon)}(\boldsymbol{x}_\circ)$.*

Proof. Pick $\boldsymbol{x} \neq \boldsymbol{y} \in B_{(r-\epsilon)}(\boldsymbol{x}_\circ)$. We want to show that $|f(\boldsymbol{y}) - f(\boldsymbol{x})| \leq \frac{2M}{\epsilon}\|\boldsymbol{y} - \boldsymbol{x}\|$.

Consider the point $\boldsymbol{z} := \boldsymbol{y} + \frac{\epsilon}{\|\boldsymbol{y}-\boldsymbol{x}\|}(\boldsymbol{y} - \boldsymbol{x})$ that belongs to B, and note that $\boldsymbol{y}$ is between $\boldsymbol{x}$ and $\boldsymbol{z}$, specifically, $\boldsymbol{y} = \frac{\epsilon}{\epsilon+\|\boldsymbol{y}-\boldsymbol{x}\|}\boldsymbol{x} + \frac{\|\boldsymbol{y}-\boldsymbol{y}\|}{\epsilon+\|\boldsymbol{y}-\boldsymbol{x}\|}\boldsymbol{z}$. By concavity, $f(\boldsymbol{y}) \geq \frac{\epsilon}{\epsilon+\|\boldsymbol{y}-\boldsymbol{x}\|}f(\boldsymbol{x}) + \frac{\|\boldsymbol{y}-\boldsymbol{y}\|}{\epsilon+\|\boldsymbol{y}-\boldsymbol{x}\|}f(\boldsymbol{z})$, hence $(\epsilon + \|\boldsymbol{y} - \boldsymbol{x}\|)f(\boldsymbol{y}) \geq \epsilon f(\boldsymbol{x}) + \|\boldsymbol{y} - \boldsymbol{y}\|f(\boldsymbol{z})$, or

$$\epsilon(f(\boldsymbol{x}) - f(\boldsymbol{y})) \leq \|\boldsymbol{y} - \boldsymbol{x}\|(f(\boldsymbol{y}) - f(\boldsymbol{z})). \tag{2}$$

Now $(f(\boldsymbol{y}) - f(\boldsymbol{z})) \leq 2M$, hence $(f(\boldsymbol{x}) - f(\boldsymbol{y})) \leq \frac{2M}{\epsilon}\|\boldsymbol{y} - \boldsymbol{x}\|$. Interchanging the names/roles of $\boldsymbol{x}$ and $\boldsymbol{y}$, $(f(\boldsymbol{y}) - f(\boldsymbol{x})) \leq \frac{2M}{\epsilon}\|\boldsymbol{y} - \boldsymbol{x}\|$. Combining, $|f(\boldsymbol{y}) - f(\boldsymbol{x})| \leq \frac{2M}{\epsilon}\|\boldsymbol{y} - \boldsymbol{x}\|$. □