Some Notes on the Opportunity Cost of Capital

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This is also called the time value of money. If you tie up money/capital, e.g. by putting it into new equipment, then that is money that you are not investing at whatever other rate of return you could be getting. In other words, the opportunity cost of money/capital is what it could be earning you if you used it someplace else, in the best possible someplace else.

The essential idea is that 1\$ of capital now returns $(1 + r)$\$ in one period if r is the per period **rate of return (ror)**. To put it another way, receiving 1\$ one period in the future is only worth $\frac{1}{(1+r)}$\$ right now. We now start looking at implications of this using the concept of the **net present value** of a flow of money. This is the key concept for valuing projects that will have costs and benefits spread over time, and projects that do not meet that description are rare indeed.

We will see both the discrete and the continuous version of the formulas, and you should be familiar with the uses of both.

Discrete Discounting

We will call the time periods $t = 0, 1, 2, \ldots$ as 'years,' but one could work with weeks, months, days, decades, or whatever period makes sense for the situation at hand. These are discrete periods in time of equal length, and the discrete time discounting that we use involves sums of the form

$$\sum_{t=0}^{\infty} \rho^t B_t \text{ or } \sum_{t=0}^{\infty} \left(\frac{1}{(1+r)}\right)^t B_t \tag{.1}$$

where $\rho := \frac{1}{(1+r)}$ is called the **discount factor**, the factor by which we discount the value of benefits or costs to be accrued in the future. The sum in equation (.1) is called the **net present value (npv)** of the sequence of benefits, $B_0, B_1, B_2, \ldots$. The B_t can be positive or negative, depending e.g. on whether the benefits of the project is larger or smaller than the costs in period t.

The logic comes from noting that investing a quantity x at $t = 0$ returns $(1+r)x$ at $t = 1$, $(1 + r)^2x$ at $t = 2$, $(1 + r)^3x$ at $t = 3$, and so on and on, with the general answer being $(1 + r)^t x$ at time t. Here $r > 0$, and we think of $100 \cdot r$ as the 'interest rate.' This means that receiving an amount y at some future $t = 1, 2, \ldots$ is only worth $x = y/(1 + r)^t$ at $t = 0$. Setting $\rho = 1/(1 + r)$ gives one of the rationales for studying sums of the form $\sum_{t=0}^{\infty} \rho^t B_t$.

Note that as well as depending on the sequence $B_0, B_1, B_2, \ldots$, the npv also depends on the discount factor: the closer ρ is to 1, that is, the smaller is r, the rate of return on capital, the more weight is given to the B_t's in the far future; the closer ρ is to 0, that is, the larger is r, the rate of return on capital, the more weight is given to the B_t in the near future.

Geometric Sums

With luck, you have seen geometric sums before, but even if you haven't, their basics are quite simple. The first observation is that $\sum_{t=0}^{\infty} \rho^t = \frac{1}{1-\rho}$. To see why, note that $\sum_{t=0}^{T} \rho^t = (1 + \rho + \rho^2 + \cdots + \rho^T)$, which implies that

$$(.2) \qquad (1-\rho) \cdot \left(\sum_{t=0}^{T} \rho^t\right) = \\ (1 + \rho + \rho^2 + \cdots + \rho^T) + \\ (-\rho - \rho^2 - \cdots - \rho^T - \rho^{T+1}) = (1 - \rho^{T+1}).$$

If $\rho < 1$, that is, if the rate of return is positive, then when T is large, ρ^{T+1} is approximately equal to 0. Putting this together,

$$(.3) \qquad (1-\rho)\left(\sum_{t=0}^{\infty} \rho^t\right) = 1, \text{ rearranging, } \sum_{t=0}^{\infty} \rho^t = \tfrac{1}{1-\rho}.$$

The formula in equation (.2) has more implications and uses. For example, note that

$$(.4) \qquad \sum_{t=4}^{9} \rho^t = \rho^4 + \rho^5 + \rho^6 + \rho^7 + \rho^8 + \rho^9 \\ = \rho^4(1 + \rho^1 + \rho^2 + \rho^3 + \rho^4 + \rho^5) \\ = \rho^4 \cdot \frac{(1-\rho^6)}{(1-\rho)}$$

because $6 = (9 - 4) + 1$. Personally, my impulse is to re-derive such formulas as needed rather than try to memorize them, this also helps me remember what I am trying to do. In any case, during any exams, the necessary formulas, and some unnecessary ones as well, will be provided.

Up Front Costs, Backloaded Benefits

From the pieces we can evaluate the net present value of a project that requires and investment of $C > 0$ for periods $t = 0, 1, \ldots, T-1$, and then returns a profit $B > 0$ for periods $T, T+1, T+2, \ldots, T+T'$. This has net present value

$$(.5) \qquad npv(\rho) = \left(\tfrac{1}{(1-\rho)}\right)\left[-C(1-\rho^T) + B\rho^T(1-\rho^{T'+1})\right].$$

Mathematically, the easiest of the interesting cases has $T' = \infty$, that is, the project delivers a stream of profits, $B > 0$, that lasts for the foreseeable future. In this case,

$$(.6) \qquad npv(\rho) = \left(\tfrac{1}{(1-\rho)}\right)\left[B\rho^T - C(1-\rho^T)\right].$$

There are two things to notice about this equation.

First, this equation cross 0 from below exactly once, at point we'll denote $\rho^\dagger$. Let r_{IRR} satisfy $\frac{1}{(1+r)} = \rho^\dagger$. The "$IRR$" stands for **internal rate of return**, this is the rate of return at which the project breaks even. Here are the rules that come from doing this calculation.

R.1 If the opportunity cost of capital, r, is greater than r_{IRR}, then it is *not* worth investing in the project.
R.2 If the opportunity cost of capital, r, is less than r_{IRR}, then it *is* worth investing in the project.

The second thing to notice about equation (.6) is that if ρ is close enough to 1, that is, if r, the opportunity cost of capital, is low enough, then the net present value is positive. Here is one way to think about this result: if there aren't many productive alternatives around, it becomes worthwhile to invest in projects that only pay off in the far future.

There is another way to measure how "good" a project is, the **payback period**. Suppose that $B_0, B_1, B_2, \ldots$ is the expected stream of net benefits from a project, positive or negative. The payback period is the first T at which the running sum, $\sum_{t=0}^{T} \rho^t B_t$, gets and stays positive. At that point, everything that's been put into the project has been paid back (with interest), and the project has become a steady source of future profit.

It is an empirical observation that very few firms take on projects with payback periods any longer than 3 to 5 years, and that is an old figure that is probably an overestimate of present behavior. There are very few R&D projects that pay back their expenses over so short a time period. For example, Apple spends less than %3 of its yearly profits on R&D and has a product cycle requiring that the new products arrive frequently, guaranteeing that they are, mostly, small steps rather than large innovations — the really large innovations, microchips, the internet, touch screens, these all take a more serious investment of time and resources. Apple relies instead on the ruthlessly good design of products and interfaces, but uses technologies with origins almost exclusively in government funded research.

The conditions for innovation require what is perhaps best thought of as an industrial 'commons,' an idea with roots at least as old as the Enlightenment.

Commons Problems with a Dynamic Component

The term "commons" refers, historically in England, to the common grounds for a village, the area where everyone could run their sheep. The more sheep that are run on an area, the less productive it is. However, if I keep my sheep off to let it recover, all that will happen is that your sheep will benefit. As with many simple stories, this contains a very important truth, and Elinor Ostrom's work has examined the many varied and ingenious ways that people have devised to solve or circumvent this problem. However, it is a problem and it does need a solution. Here we are interested in common resources that pay off over long periods of time.

The first systematic analysis known to me came in the late 1600's, in the *Oisivités* of Louis XIV's defense minister, Sébastien Le Prestre de Vauban, who noted the following.

- Forests were systematically over-exploited in France, they are a public access resource, a commons.
- After replanting, forests start being productive in slightly less than 100 years but don't become fully productive for 200 years.
- Further, no private enterprise could conceivably have so long a time-horizon, essentially for discounting reasons.

From these observations, Vauban concluded that the only institutions that could, and should, undertake such projects were the government and the church. His calculations involved summming the **un-discounted benefits, delayed and large, and costs, early and small**, on the assumption that society would be around for at least the next 200 years to enjoy the net benefits.

The American System of Manufactures

At the end of the American Civil war, the U.S. government decided that it needed rifles with interchangeable parts. This required a huge change in technological competencies, the development of tools to make tools. The American system was also known, in the early days, as *armory practice*, it evolved in the government funded armories, spread from the armories in Springfield and Harper's Ferry to private companies, Colt in particular.

The private companies formed near where there were people with the needed competencies, you don't set up a firm to using specialized machinery too far from people who know how it works well enough to fix it. To put it another way, there was an industrial commons created and sustained by government expenditures, and in using this commons, private companies were able to flourish. The parallel between Apple's history and Colt's, separated by almost a century and a half, is striking.

The system was generalized to many other products, if you know how to do one thing really well, then you probably know how to do similar things pretty well (Singer sewing machines for example). The system itself, the idea of creating tools to create tools, then spread through Europe within a couple of decades at the beginning of the second Industrial Revolution, and has since taken over the world. This is but one example of a very general pattern, most of the really large innovations in the technologies now so widespread in electronics, computing, optics, agriculture, medicine, came from funding organized by and through the government. Venture capitalists, start-up firms, private firms in general, cannot, because of the opportunity cost of capital, have the patience to invest in these things. Vauban's core insights, that these long term projects are good for society and that only institutions with a very long time horizon can undertake them, these still resonate today.

Other commons examples

The essential observation is that markets generally underprovide goods when the benefits spill over to other people. If the maximizer pays attention to the problem

$$\max_{x} \ [B_1(x) - C(x)] \tag{.7}$$

but society would pay attention to

$$\max_{x} \ [(B_1(x) + B_2(x)) - C(x)] \tag{.8}$$

we know what will happen if $B_2(\cdot)$ is an increasing function.

Arthur Andersen

The consulting firm, Arthur Andersen, used to hire bright youngsters straight from college, work them incredibly hard, pay them not very much, but provide them with really valuable experience, a form of apprenticeship. When an employee has valuable skills, they can take those skills out on the market and earn a good salary. The Andersen apprenticeships earned the company money, they would not have behaved this way without it being profitable.

The benefits to their employees after they left, the $B_2(\cdot)$ above, were not particularly part of the firm's calculation in deciding how many people to train and how much to teach them, the x. Now, that's not quite correct inasmuch as the future benefits to employees who leave with valuable experience meant that the firm could continue to pay ambitious, hard-working young college graduates less than they were actually worth.

However, it has never been seriously suggested that firms take over the business of teaching the literacy and numeracy that underpins modern society. When the $B_2(x)$ includes these kinds of trainings, its value far exceeds what any firm can aspire to. Further, because the training takes decades to complete, no firm can afford it given the opportunity cost of capital arguments, not even if they could recoup the expenses with some form of slavery.

Microchips

Venture capitalists and technology firms were very late, and reluctant entrants into the business of making microchips. The original demand for such a product came from NASA (the National Aeronautics and Space Administration), who wanted very light, very small computing power that would survive some really awful conditions. NASA not only paid for the research into the feasibility of such a creation, it provided a guaranteed market for the first chips, paying $1,000 apiece for chips that soon cost only $20 to $30 apiece. The benefits of going through this learning by doing process have given us the computer age.

Internet

Think DARPA.

Material sciences

A silly example is velcro, more substantive historically, think transistors, silicon chips, touch-sensitive glass screens for modern computers. Underlying all of these, and the many examples not mentioned, is the post-graduate education system, producing science Ph. D.'s in a system heavily subsidized by the government. Again, this an investment that will not payoff for so long that the internal rate of return (IRR) is not sustainable for a commercial firm, and the benefits are not recoverable unless one starts down a system starting with indentured servants and ending in slavery.

Agricultural sciences

Hybrid crops were mostly, and the original techniques for genetic modification of crops were nearly completely, funded by governments interested in the long-term welfare of society.

Continuous Discounting

Continuous discounting makes the calculation of optimal timing of decisions and other continuous timing decisions much easier. These involve expressions involving the terms e^{rt} and/or e^{-rt} where $e \simeq 2.718281828459\ldots$ is the basis of the natural logarithms. The crucial competence to develop is the ability to recognize when you need which of the various formulas below, but the development of these formulas will, of necessity, make reference to facts you are not likely to have seen without a calculus class.

From your calculus class

You should have been exposed to the following,

$$\lim_{n\to\infty} \left(1 + \tfrac{x}{n}\right)^n = e^x. \tag{.9}$$

Taking $x = rt$ delivers e^{rt} as the limit in the above expression, but we do not here need the formal definition of a limit, just the idea that e^x is a very good approximation to $\left(1 + \frac{x}{n}\right)^n$ when n is moderately large.

If the interest rate is 6% per year, it is 6/4% per fiscal quarter, 6/12% per month, 6/52% per week, and 6/365%. If you calculate

- the quarterly rates of return four times over the course of the year where you re-invest all of your quarterly earnings, you make $(1 + \frac{0.06}{4})^4$ on each dollar invested at the beginning of the year,
- the monthly rates of return twelve times over the course of the year where you re-invest all of your monthly earnings, you make $(1 + \frac{0.06}{12})^{12}$ on each dollar invested at the beginning of the year,
- the weekly rates of return 52 times over the course of the year where you re-invest all of your weekly earnings, you make $(1 + \frac{0.06}{52})^{52}$ on each dollar invested at the beginning of the year,
- the daily rates of return 365 times over the course of the year where you re-invest all of your daily earnings, you make $(1 + \frac{0.06}{365})^{365}$ on each dollar invested at the beginning of the year.

It turns out that 365 is close enough to ∞, that is, 365 is large enough that the difference between $(1 + \frac{0.06}{365})^{365}$ and $e^{0.06}$ doesn't matter very much. If the difference is still to big, figure out how many seconds there are in a year and work with that. Once we get close enough to that limit out at ∞, we call it **continuous compounding** or **continuous time growth**, and most modern calculators have the button allowing you to calculate e^{rt} or $1/e^{rt} = e^{-rt}$ for any relevant r and t.

In continuous time, investing a quantity x at $t = 0$ returns $x \cdot e^{rt}$ at time $t > 0$. This means that receiving an amount y at $t > 0$ is only worth $y/e^{rt} = ye^{-rt}$ at $t = 0$. It is this continuity of time that makes these calculations so useful for optimal timing problems. This is not a deep or subtle observation, rather it is the observation that if one is summing $\sum_{t=0}^{T} \rho^t B_t$ and optimizing over T or considering a choice of a t that optimizes $\Pi(t)\rho^t$, having the time intervals be discrete, say years, means that the mathematical formulation of the problem can only return answers in years, whereas the true best timing might be "next September," at $T = 3/4$ roughly.

The appearance of the integral

If you have a function of time t, say $h(t) > 0$ for $a \le t < b$, then the area between the curve and the time axis and bounded between the times a and b is denoted

$$\int_a^b h(t)\,dt \text{ or } \int_a^b h(x)\,dx. \tag{.10}$$

There are two extra subtleties, one involving negative areas and the other involving time flows that stretch out into the indefinite future.

S.1 When $h(t) < 0$, we count the area between the curve and the time axis as being negative. For example, if $h(t) = -9$ for $3 \leq t < 9$ and $h(t) = 10$ for $9 \leq t < 23$, $\int_3^{23} h(t)\,dt$ is equal to $-9 \cdot (9-3) + 10 \cdot (23-9)$, that is $\int_3^{23} h(t)\,dt = -54 + 140 = 86$.

S.2 When we have benefits, $h(t)$, starting now, at $t = 0$, and extending off into the indefinite future, we will write $\int_0^\infty h(t)\,dt$ for their value. You should worry, for just a couple of seconds, about the problem that the area under a curve that stretches off forever is going to be infinite. This is one problem that discounting solves.

Another pair of observations: the areas under curves add up; and if we multiply a function by a constant, the area under the curve is multiplied the same constant.

O.1 The area under the curve $h(t)$ between a and b plus the area under the curve $g(t)$ between b and c is

$$\int_a^b h(t)\,dt + \int_b^c g(t)\,dt. \tag{.11}$$

O.2 The area under the curve $2 \cdot h(t)$ between a and b is

$$\int_a^b 2 \cdot h(t)\,dt = 2 \cdot \int_a^b h(t)\,dt. \tag{.12}$$

A Small Detour

The expressions in equation (.10) are called "integrals," historically, the symbol "$\int$" comes from the letter capital "S," standing for "summation." A slightly more subtle aspect of these problems is that we change summations into integrals. To understand why this happens, let us think of the B_t in $\sum_{t=0}^\infty \rho^t B_t$ as being a flow, the benefits B_t are the benefits **per year** (or per day/week/month/etc. as appropriate). This means that $\sum_{t=0}^\infty \rho^t B_t$ can be rewritten as

$$\textstyle\int_0^1 \rho^1 B_1\,dt + \int_1^2 \rho^2 B_2\,dt + \int_2^3 \rho^2 B_2\,dt + \cdots. \tag{.13}$$

In continuous time, the flow of benefits can vary with time, that is, it need not be the step functions of (.13), B_1 for all $0 \leq t < 1$, B_2 for all $1 \leq t < 2$, etc. Further, the discount factor in continuous time is not the step function ρ^1 for all $0 \leq t < 1$, ρ^2 for all $1 \leq t < 2$, etc. Rather, it is e^{-rt} at $t > 0$. Putting these together, the continuous time net present value of a flow $q(t)$ is

$$\textstyle\int_0^T q(t)e^{-rt}\,dt \text{ or } \int_0^\infty q(t)e^{-rt}\,dt. \tag{.14}$$

Useful formulas

For simplicity, we will mostly study flows of net benefits that start negative, say at $-C$, for a time period from 0 to T, and then turn positive, say at B, from T on into the indefinite future. The **continuously compounded net present**

value (npv) of such a stream of benefits is

$$npv(r) = \int_0^T (-C)e^{-rt}\,dt + \int_T^\infty Be^{-rt}\,dt. \tag{.15}$$

"Mostly" does not mean "always," we will also study that case that we can forsee the end of the benefits, i.e. we will care about expressions of the form

$$npv(r) = \int_0^T (-C)e^{-rt}\,dt + \int_T^\tau Be^{-rt}\,dt \tag{.16}$$

where $\tau > T$ is the time at which the benefits B stop accruing.

Here are the useful formulas, these contain the basics of integrating continuously discounted piece-wise constant flows, and give the comparisons with discrete time discounting (recall that $\rho = \frac{1}{(1+r)}$ in these comparisons).

F.1 $\int_0^\infty e^{-rt}\,dt = \frac{1}{r}$, which you should compare to $\sum_{t=0}^\infty \rho^t = \frac{1}{(1-\rho)}$;

F.2 $\int_0^T e^{-rt}\,dt = \frac{1}{r}(1 - e^{-rT})$, which you should compare to $\sum_{t=0}^T \rho^t = \frac{1}{1-\rho}(1 - \rho^{T+1})$;

F.3 $\int_a^b e^{-rt} = \frac{1}{r}(e^{-ra} - e^{-rb}) = \frac{1}{r}e^{-ra}(1 - e^{-r(b-a)})$ which you should compare to $\sum_{t=\tau}^T \rho^t = \frac{1}{1-\rho}\rho^\tau(1 - \rho^{(T-\tau)+1})$.

Going back to the expression in (.15), we have $npv(r) = \frac{1}{r}(-C)(1 + e^{-rT}) + \frac{1}{r}Be^{rT}$. As in the discrete case, the IRR is the r that solves the equation $npv(r) = 0$. You should verify that the IRR solves $e^{-rT} = \frac{C}{B+C}$. Taking logarithms allows you to solve for the IRR. You should go through what different values of B, C and T do to the IRR, and ask yourself why your answers should make sense.