Some Examples of Optimal Timing Algebra
Monday Sept 21, 2015

Suppose that $B(x) = \frac{x^2}{x^2+1}$ and consider the problem

$$\max_{x \geq 0} B(x)e^{-0.06 \cdot x}.$$

The following graph shows $B(x)$ on top and $B(x)e^{-0.06 \cdot x}$ below it. The maximum happens at $x \simeq 3.114760$, and yields a value of 0.7452.

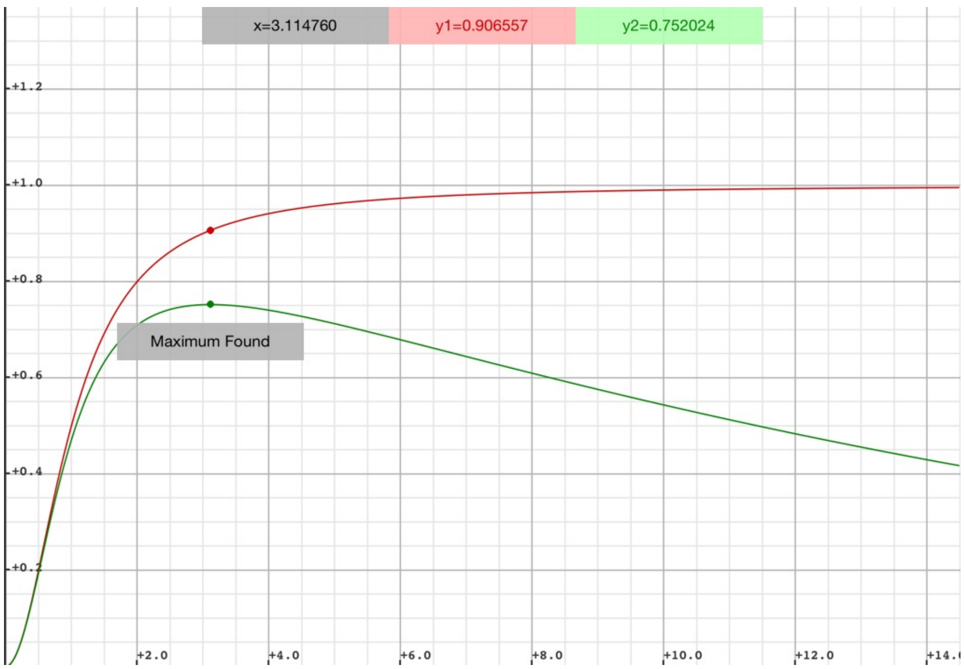

Figure 1. $B(x)$ and $B(x)e^{-0.06 \cdot x}$

To solve this more directly, take logarithms, then set the derivative equal to 0, yielding the FOCs $\frac{B'(x)}{B(x)} = 0.06$, that is,

$$\frac{B'(x)}{B(x)} = \frac{2}{x(x^2+1)} = \frac{6}{100}.$$

Getting out your calculator, you can check that

$$\frac{2}{3.11476(3.11476^2+1)} \simeq \frac{6}{100}.$$

Let us repeat the previous with the interest rates 0.01 (very patient) and 0.25 (rather impatient).

The following graph shows $B(x)$ on top and $B(x)e^{-0.01x}$ below it, the maximum happens at $x = 5.791038 > 3.11476$ as monotone comparative statics already told us.

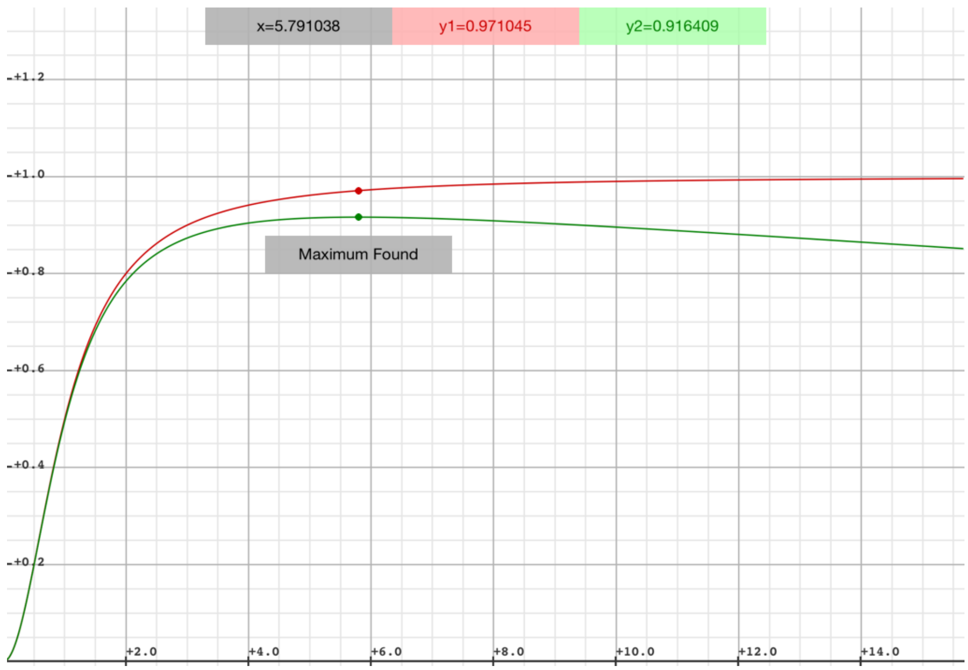

FIGURE 2. $B(x)$ and $B(x)e^{-0.01 \cdot x}$

Getting out your calculator, you can check the FOCs $B'(x)/B(x) = r$,

$$\frac{2}{5.791038(5.791038^2 + 1)} \simeq \frac{1}{100}.$$

The following graph shows $B(x)$ on top and $B(x)e^{-0.25\cdot x}$ below it, the maximum happens at $x = 1.833751 < 3.11476 < 5.791038$ as monotone comparative statics already told us.

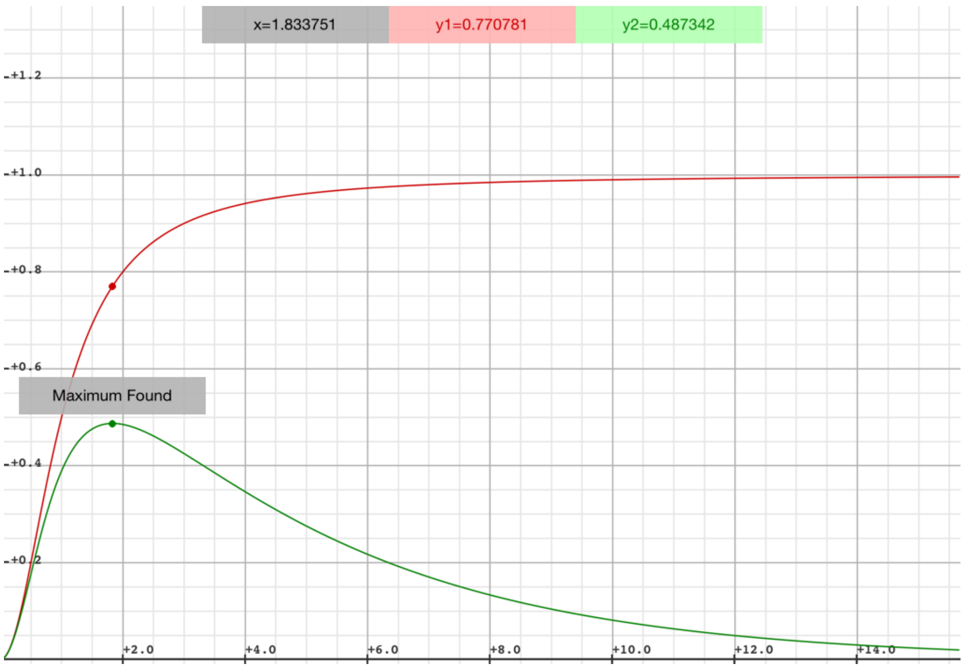

FIGURE 3. $B(x)$ and $B(x)e^{-0.25\cdot x}$

Getting out your calculator, you can check the FOCs $B'(x)/B(x) = r$,

$$\frac{2}{1.833751(1.833751^2+1)} \simeq \frac{25}{100}.$$

There are some additional lessons to take from these graphs and analyses.

1. For the lower values of r, $r = 0.01$ and $r = 0.06$, the graph of profits, $\pi(x) = B(x)e^{-rx}$ is pretty flat near the top. This means that optimal profits are not too sensitive to mistakes. This is good because we expect to make errors when estimating $B(x)$ and $B'(x)$.
2. For the higher value of r, $r = 0.25$, the graph of profits, $\pi(x) = B(x)e^{-rx}$ is pretty curved near the top. This means that optimal profits are much more sensitive to mistakes that arise from having to estimate $B(x)$ and $B'(x)$. However, the signals are now more informative, essentially because the flatness of $B(x)$ is overcome by the higher discount factor driving down the value.
3. The optimal profit is a decreasing function of r, that is $\pi^*(r) = \pi(x^*(r))$, is a decreasing function. This is because higher interest rates devalue future rewards and rewards will not accrue until the project is done.

If you feel more comfortable after having worked some numerical examples, some simple variants of the previous analyses can be done with

$$B(x) = \left(\frac{x}{x+1}\right)^5, \text{ or } B(x) = \left(\frac{x^3}{x^3+1}\right)^7,$$

or any of the variants that suggest themselves from this.