An Introduction to Mathematical Analysis in Economics:
Some Advanced Math from an Elementary Point of View

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Contents

Dedications 11

Preface 13
  Road Map 13
  Uses of the book 15
  List of notation 16

Chapter 1. Logic 17
  1.1. Examples of Implication 17
  1.2. Sets, Subsets, and Implication 18
  1.3. The Formation of Statements and Their Truth Values 19
    1.3.a. Ands/Ors/Not as Intersections/Unions/Complements 19
    1.3.b. Implies/Equivalence as Subset/Equality 20
    1.3.c. Proofs, a First Look 21
    1.3.d. Quantifiers 23
  1.4. Taxonomy of Proofs 25
  1.5. Bibliography for Chapter 1 27

Chapter 2. Set Theory 29
  2.1. Some Simple Questions 29
  2.2. Notation and Other Basics 30
  2.3. Products, Relations, Correspondences, and Functions 33
  2.4. Optimal Choice for Finite Sets 36
  2.5. Direct and Inverse Images, Compositions 41
    2.5.a. Direct Images 41
    2.5.b. Inverse Relations 42
    2.5.c. Injections, Surjections, and Bijections 43
    2.5.d. Compositions of Functions 44
  2.6. Weak and Partial Orders, Lattices 45
  2.7. Monotonic Changes in Optima: Supermodularity, and Lattices 48
    2.7.a. The implicit function approach 49
    2.7.b. The simple supermodularity approach 50
    2.7.c. Monotone comparative statics 51
  2.8. Tarski’s Lattice Fixed Point Theorem and Stable Matchings 52
  2.9. Equivalence relations 57
  2.10. Finite and Infinite Sets 58
  2.11. The Axiom of Choice and Some Equivalent Results 64
  2.12. Revealed Preference and Rationalizability 65
Chapter 3. The Space of Real Numbers

3.1. Why We Want More than the Rationals
3.2. Basic Properties of Rationals
3.3. Distance, Cauchy Sequences, and the Real Numbers
3.4. The Completeness of the Real Numbers
3.5. Newton’s Bisection Method
3.6. Supremum and Infimum
3.7. Summability
3.7.a. The Fishery Model
3.7.b. Summability
3.7.c. Optimality, Euler Equations, and Steady States
3.7.d. Monotone Sequences
3.8. Patience, Liminf, and Limsup
3.8.a. Liminf and Limsup
3.8.b. Patience
3.8.c. Discount Factors Close to One

Chapter 4. Metric Spaces

4.1. The Basics Definitions for Metric Spaces
4.2. Discrete Spaces
4.3. $\mathbb{R}^\ell$, the $\ell$-fold Product of $\mathbb{R}$
4.3.a. The Normed Vector Space Structure of $\mathbb{R}^\ell$
4.3.b. Dot Products and the Cauchy-Schwarz Inequality
4.3.c. The $p$-Norms, $p \neq 1, 2, \infty$
4.3.d. Completeness
4.3.e. Closure and Convergence
4.3.f. Separability
4.3.g. Compactness in $\mathbb{R}^\ell$
4.3.g.1. Context Matters
4.3.g.2. The Definition Redux
4.3.g.3. Boundedness and Total Boundedness
4.3.g.4. The Finite Intersection Property
4.3.g.5. Characterizations of Compactness
4.3.g.6. Some Applications
4.3.h. Continuous Functions on $\mathbb{R}^\ell$
4.3.i. Uniform Continuity
4.3.j. Connectedness
4.3.k. Banach’s Contraction Mapping Theorem
4.3.l. Finite State Markov Chains
4.3.m. Finite State Markovian Dynamic Programming
4.3.m.1. The Stage-Coach Problem
4.3.m.2. Generalizing the Stage-Coach Problem 132
4.3.m.3. Adding Stochastic Structure and an Infinite Horizon 133
4.3.n. Perturbed Markov Chains 135
4.4. The Space of Compact Sets and the Theorem of the Maximum 138
4.4.a. Examples 138
4.4.b. The Metric Space of Compact Sets 141
4.4.c. *Completeness, Separability, and Compactness 142
4.4.d. Upper and Lower Hemicontinuity 144
4.4.e. The Theorem of the Maximum 146
4.4.f. Applications 147
4.4.g. A Warning 148
4.5. $C_b(M)$, the Continuous, Bounded Functions 148
4.5.a. Examples: Isometries, Cdf’s and Stochastic Processes 149
4.5.b. Completeness 153
4.5.c. Dimensionality, Finite and Infinite 154
4.5.d. The Levy Distance on Cdf’s 155
4.5.e. Compactness in $C(M)$ when $M$ is Compact 156
4.6. $D(\mathbb{R})$, the Space of Cumulative Distribution Functions 160
4.6.a. Tightness in $D(\mathbb{R})$ 160
4.6.b. Weak Convergence of Cumulative Distribution Functions 160
4.6.c. The Denseness of Finitely Supported Probabilities 162
4.7. Approximation in $C(M)$ when $M$ is Compact 162
4.7.a. The Lattice Version of Stone-Weierstrass 163
4.7.b. The Algebraic Version of Stone-Weierstrass 165
4.7.c. Applications of Stone-Weierstrass 166
4.7.d. Separability of $C(M)$, $M$ Compact 169
4.8. Regression Analysis as Approximation Theory 169
4.8.a. Orthogonal Projections 170
4.8.b. Linear Regression 172
4.8.c. Nonlinear Regression 172
4.8.c.1. The Span-of-a-Finite-Set Case 172
4.8.c.2. Degrees of Complexity 173
4.8.c.3. The Span-of-a-Compact-Set Case 174
4.8.d. Testing for Neglected Nonlinearity 174
4.9. Countable Product Spaces and Sequence Spaces 175
4.9.a. Examples 175
4.9.b. The Product Metric on $\times_{a \in A} M_a$ 179
4.9.c. Cylinder Sets 180
4.9.d. Compactness, Completeness, and Separability in $\times_{a \in A} M_a$ 181
4.9.e. The Universal Separable Metric Space 182
4.9.f. Deterministic Dynamic Programming 183
4.10. Defining Functions Implicitly and by Extension 183
4.10.a. The Implicit Function Theorem 183
4.10.b. The Extension Theorem for Dense Sets 188
### CONTENTS

4.10.c. The Extension Theorem for Closed Sets .................................................. 189  
4.11. The Metric Completion Theorem ................................................................. 190  
4.12. The Lebesgue Measure Space ....................................................................... 192  
4.12.a. The Space \((M, d_1)\) ................................................................................. 192  
4.12.b. The Basic Properties of \((M, d_1)\) ........................................................... 194  
4.12.c. A Special Subset of \((M, d_1)\) ................................................................. 196  
4.12.e. The Weak and the Strong Law of Large Numbers ................................. 198  
4.13. A First Look at Banach Spaces ................................................................... 200  
4.14. Bibliography for Chapter 4 ......................................................................... 203  
4.15. End of Chapter Problems ............................................................................ 204  

Chapter 5. Convex Analysis, Fixed Points, and Equilibria in \(\mathbb{R}^\ell\) ................................. 209  
5.1. The Basic Geometry of Convexity .................................................................. 209  
5.1.a. Examples of Convex Sets ........................................................................... 209  
5.1.b. Half-Spaces ............................................................................................... 210  
5.1.c. Vector Subspaces ....................................................................................... 211  
5.1.d. Interiors and Boundaries .......................................................................... 212  
5.1.e. Sums of Sets ............................................................................................... 213  
5.2. The Dual Space of \(\mathbb{R}^\ell\) .......................................................................... 214  
5.3. Separation Theorems in \(\mathbb{R}^\ell\) ................................................................. 215  
5.3.a. Degrees of Separation .............................................................................. 215  
5.3.b. Separating Points from Closed Convex Sets in \(\mathbb{R}^\ell\) ......................... 216  
5.3.c. Convex Hulls .............................................................................................. 217  
5.3.d. Neoclassical Duality ................................................................................... 218  
5.3.e. Support Hyperplanes and Support Functions ............................................. 221  
5.3.f. The Two Fundamental Theorems of Welfare Economics ............................. 222  
5.4. Concave and Convex Functions ..................................................................... 223  
5.4.a. Convex Preferences and Quasi-Concave Utility Functions ......................... 224  
5.4.b. Maximization and Derivative Conditions .................................................. 224  
5.4.c. Convexity, Minkowski’s Inequality, and \(p\)-norms .................................... 226  
5.4.d. The Continuity of Concave Functions ....................................................... 227  
5.5. The Kuhn-Tucker Theorem ........................................................................... 227  
5.5.a. Examples .................................................................................................... 228  
5.5.b. Saddle Points, or Why Does it Work? ....................................................... 236  
5.5.c. The Implicit Function Theorem ................................................................... 238  
5.5.d. The Envelope Theorem ............................................................................ 239  
5.6. Fixed Points ................................................................................................... 240  
5.6.a. Some Applications of Brouwer’s Theorem ................................................. 240  
5.6.b. Some Applications of Kakutani’s Theorem ................................................ 242  
5.6.c. Fixed points of functions .......................................................................... 244  
5.6.d. Fixed points of correspondences ................................................................. 246  
5.7. Existence of General Equilibrium with Preferences that Depend on Others’ Consumption ................................................................................................................. 249  
5.8. Perfect, Proper, and Stable Equilibria ............................................................. 249
5.9. Appendix - Some Proofs Looking for a Home 250
5.10. Differentiability and Concavity 260
5.10.a. The two results 260
5.10.b. The one dimensional case, \( f : \mathbb{R}^1 \to \mathbb{R} \) 261
5.10.c. The multi-dimensional case, \( f : \mathbb{R}^n \to \mathbb{R} \) 261
5.10.d. A fair amount of matrix algebra background 261
5.10.e. The Alternating Signs Determinant Test for Concavity 264
5.11. End of Chapter Problems 265

Chapter 6. Measure Theory and Probability 267
6.1. The Basics Definitions for Measure Theory 268
6.1.a. Measurable Sets 268
6.1.b. Measurable Functions 270
6.1.c. Countably Additive Probabilities 272
6.1.d. Integrals of Measurable Functions on Probability Spaces 272
6.1.e. Vector Spaces of Integrable Functions 275
6.1.f. Riemann and Lebesgue Integrals 276
6.2. Some Examples 277
6.2.a. Basic Statistics 277
6.2.b. Limit Theorems 279
6.2.c. Measure Space Exchange Economies and Games 280
6.3. A Simple Case of Parametric Estimation 283
6.3.a. The Classical Statistical Model 283
6.3.b. Independence 284
6.3.c. Distributions Induced by Random Variables 284
6.3.e. Selection Issues 287
6.3.f. Maximum Likelihood Estimators 288
6.3.g. Unbiased Estimators and Linear Estimators 289
6.3.h. Variance 290
6.3.i. Best Linear Unbiased Estimators 291
6.3.j. Consistent Estimators 293
6.3.k. Consistency and Convergence 294
6.3.l. Efficiency, BLUE’s and BUE’s 294
6.3.m. Cramer-Rao Lower Bound 294
6.3.n. Neyman-Pearson 294
6.4. Integrating to the Limit 294
6.4.a. Types of Convergence 295
6.4.b. Fields as Metric Spaces 295
6.4.c. Infinitely Often and Almost Always 295
6.4.d. Borel-Cantelli and a Strong Law of Large Numbers 296
6.4.e. The Set Theoretic Monotone Class Theorem 296
6.4.f. Independence, Tail \( \sigma \)-Fields, and Some 0-1 Laws 297
6.4.g. Herds 297
6.5. The Basic Extension Theorem 297
6.6. The Central Limit Theorem 299
6.6.a. With Independent Identically Distributed Terms 299
6.6.b. With Independent Terms 299
6.6.c. With Dependence 299
6.7. A Seriouser Introduction to Stochastic Processes 299
6.7.a. Some Implications of Tightness 300
6.7.b. Cumulative Distribution Functions and Probabilities on $\mathbb{R}$ 300
6.7.c. The Tightness of Measures 301
6.7.d. The Prohorov Metric 303
6.7.e. The Continuous Mapping Theorem 303
6.7.f. Discrete Time Stochastic Processes 303
6.7.g. Continuous Time Stochastic Processes 303
6.8. The $L^p(\Omega, \mathcal{F}, P)$ Spaces 303
6.9. Measure Space Exchange Economies 304
6.9.a. The Model 304
6.9.b. Joint Measurability 305
6.9.c. Core Equivalence 305
6.9.d. Lyapunov’s Convexity Theorem 306
6.9.e. Fair Division 306
6.9.f. The basics of $\sigma$-fields 308
6.9.g. Properties of the Domain of $P$ 310
6.9.h. Convex sets 311
6.9.i. A finite dimensional vector space: $\mathbb{R}^n$ 312
6.10. Projection in Hilbert Spaces 318
6.10.a. Conditional Expectation as a Special Kind of Projection 318
6.10.b. Doob’s Theorem for Sub-$\sigma$-fields and Nonparametric Regression 318
6.10.c. Modeling Information as Sub-$\sigma$-fields 319
6.11. Martingales and Conditional Expectation 320
6.11.a. In Finance 320
6.11.b. Hysteresis 320
6.11.c. Convergence 320
6.13. Bibliography for Chapter 3 320
6.15. Some Problems Looking for a Home 321

Chapter 7. Convex Analysis, Fixed Points, and Equilibria in Vector Spaces 323
7.1. The Basic Geometry of Convexity 323
7.1.a. Examples of Convex Sets 323
7.1.b. Half-Spaces 324
7.1.c. Vector Subspaces 324
7.1.d. Sums and Differences of Sets 326
7.2. The Dual Space of $V$ 326
7.2.a. Continuous Linear Functions on Hilbert Spaces 326
7.2.b. Strong Norms and Weak Metrics 327
CONTENTS

7.2.c. Discontinuous Linear Functions on Normed Spaces 328
7.2.d. Probabilities as Linear Functionals 329
7.3. Separation Theorems in $V$ 331
7.3.a. Degrees of Separation 332
7.3.b. Separating Points from Closed Convex Sets in $\mathbb{R}^\ell$ 332
7.3.c. Convex Hulls 333
7.3.d. Revisiting Probabilities as Linear Functions 335
7.3.e. Neoclassical Duality 335
7.4. Support Hyperplanes and Support Functions 338
7.4.a. The Two Fundamental Theorems of Welfare Economics 339
7.5. Concave and Convex Functions 341
7.5.a. Convex Preferences and Quasi-Concave Utility Functions 341
7.5.b. Maximization and Derivative Conditions 342
7.5.c. Convexity, Minkowski’s Inequality, and $p$-norms 343
7.5.d. The Continuity of Concave Functions 344
7.6. Fixed Points 344

Chapter 8. Topological Spaces 345
8.1. Continuous Functions and Homeomorphisms 347
8.2. Separation Axioms 348
8.3. Convergence and Completeness 350