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Nash equilibrium and generalized integration for infinite normal form games

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Abstract

Infinite normal form games that are mathematically simple have been treated [Harris, C.J., Stinchcombe, M.B., Zame, W.R., in press. Nearly compact and continuous normal form games: characterizations and equilibrium existence. Games Econ. Behav.]. Under study in this paper are the other infinite normal form games, a class that includes the normal forms of most extensive form games with infinite choice sets.

Finitistic equilibria are the limits of approximate equilibria taken along generalized sequences of finite subsets of the strategy spaces. Points must be added to the strategy spaces to represent these limits. There are direct, nonstandard analysis, and indirect, compactification and selection, representations of these points. The compactification and selection approach was introduced [Simon, L.K., Zame, W.R., 1990. Discontinuous games and endogenous sharing rules. Econometrica 58, 861–872]. It allows for profitable deviations and introduces spurious correlation between players' choices. Finitistic equilibria are selection equilibria without these drawbacks. Selection equilibria have drawbacks, but contain a set-valued theory of integration for non-measurable functions tightly linked to, and illuminated by, the integration of correspondences.

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1. Introduction

A normal form game (nfg), $\Gamma = (S_i, u_i)_{i \in I}$, is specified by a finite player set, I, strategy sets, S_i, $i \in I$, and bounded utility functions, $u_i : S \to \mathbb{R}$, $S := \times_{i \in I} S_i$. This paper develops a theory of Nash equilibrium for nfgs specified at this level of generality. There are no topological or measure theoretic assumptions.

Compact and continuous nfgs are the starting point for the study of infinite games. An nfg is compact and continuous if each S_i is compact and each u_i is jointly continuous. The companion piece to this paper developed the theory of nfgs that are nearly compact and continuous (ncc).

1.1. Games that are nearly compact and continuous

A game Γ is ncc if it is possible to densely imbed each S_i in a compact space, $\widehat{S_i}$, in such a fashion that all of the u_j have jointly continuous extensions to the product $\times_{i \in I} \widehat{S_i}$. A game Γ is *integrable* if each u_i is integrable with respect to all products of finitely additive probabilities. A game Γ is *uniformly finitely approximable* (ufa) if each S_i can be approximated by finite sets using the Fudenberg and Levine's (1983) "most utility difference it can make to anyone" pseudo-metric,

$$d_{\mathcal{U}_i}(s_i, t_i) = \max_{k \in I} \sup_{s \in S} \left| u_k(s \backslash s_i) - u_k(s \backslash t_i) \right|. \tag{1}$$

The companion piece to this paper, Harris et al. (in press), showed that the three conditions, integrability, being ufa, and being ncc, are equivalent. This paper studies nfgs that fail to be integrable, ncc, or ufa, a class that includes the normal forms of most extensive form games with infinite choice sets.

1.2. Extensive form games

Suppose that Γ is the normal form representation of an extensive form game in which player 1 makes a pick s_1 in an infinite set S_1, s_1 is subsequently observed by player 2, who then picks an action a in a set $A = \{a, b\}$, and that player 2's choice of a or b always makes at least a utility difference of at least 1 to some player. Most extensive form games involve at least this much dynamic interaction between players. While it is not at all clear what set of strategies should be considered for player 2, a minimal requirement is that the class of functions, $S_2 \subset A^{S_1}$, constituting player 2's strategy set, must be dense in the product topology.[1]

The denseness implies that for all $s_1 \neq t_1 \in S_1$, there exists an $s_2 \in S_2$ such that $s_2(s_1) \neq s_2(t_1)$, implying that $d_{\mathcal{U}_1}(s_1, t_1) \geqslant 1$. Also, if $s_2 \neq t_2$ iff there exists an s_1 such that $s_2(s_1) \neq t_2(s_1)$ so that $d_{\mathcal{U}_2}(s_2, t_2) \geqslant 1$. The normal form of this game is therefore not ufa. By the cited equivalence result, Γ is neither integrable nor ncc.

[1] This is equivalent to 2's strategies allowing arbitrary patterns of response at all finite subsets of S_1. More explicitly, if F_1 is a finite subset of S_1, then for every vector $x_2 \in A^{F_1}$, there is a strategy in S_2 that agrees with x_2 at the points in F_1.

1.3. Finitistic equilibria

A generalized sequence (net), A_i^α, of finite subsets of S_i converges to S_i if for all finite F_i, $F_i \subset A_i^\alpha$ for all sufficiently large α. Finitistic equilibria are the limits of approximate equilibria taken along convergent generalized sequences of finite subsets of the strategy spaces. Points must be added to the strategy spaces in order to represent the limits of these sequences.

The exhaustive star-finite sets of nonstandard analysis, compactifications, and finitely additive strategies are three methods of adding these limit points. For ncc games, the three methods are equivalent (Harris et al., in press). They are not equivalent for the class of games considered here.

An exhaustive star-finite version of a set X contains every x in X but behaves logically as if it were finite. There is a strong similarity between such sets and the generalized sequences of finite sets, A_i^α, that eventually contain every point in S_i. Theorem 3.2 will show that the exhaustive star-finite sets of nonstandard analysis provide direct representations of finitistic equilibria.

Any compactification of a space can be represented as a collection of equivalence classes of any exhaustive star-finite set.[2] Detail is lost in the many-to-one surjection from exhaustive sets to compactifications. This loss of detail makes the representation issues considerably more complex, requiring selection.

1.4. Selection equilibria

Compactification delivers a game with compact strategy sets and utilities defined on dense subsets, the setting of selection equilibria (Simon and Zame, 1990). Selection equilibria are defined as a pair (v, μ) where v is a utility function and μ a strategy profile. The utilities, v, are equal to u at continuity points of u.[3]

If s is a discontinuity point of u, the utility $v(s)$ must belong to the convex hull of the set of possible limit utilities in the neighborhood of s. The choice of the value of $v(s)$ must contain the detail lost in moving from star-finite sets to compactifications.

Selection equilibria can play strictly dominated strategies, and may introduce spurious correlation between players' choices. Some correlation is needed to replace the lost detail. The spurious correlation may come from the use of the convex hull, and it may come from the process of selection itself. Theorem 3.3 and its corollaries show that finitistic equilibria are selection equilibria without these drawbacks. Drawbacks aside, selection equilibria contain a set-valued theory of integration for non-measurable functions.

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[2] A brief treatment of the compactifications of a space X using equivalence classes of ${}^*X$ can be found in Anderson (1982). Machover and Hirschfeld (1969) and Hurd and Loeb (1985) contain more detailed treatments. Replacing ${}^*X$ with an exhaustive star-finite set changes nothing in their constructions.

[3] For ncc games, every point is a continuity point.

1.5. Finitely additive strategies and integration

Finitely additive probabilities are equivalent to countably additive probabilities on compactifications. The games under study fail to be integrable. The expected utilities achieved by selection provide a set-valued definition of the integral of non-measurable functions. Since there are many selections, the integration theory is tightly linked to, and illuminated by, the integration of correspondences.

1.6. Roadmap

The next section provides an overview through a number of examples. The following gives and proves the results relating finitistic equilibria and selection equilibria. The two major interpretational results are Theorem 3.2, which shows that the star-finite sets of nonstandard analysis provide a direct interpretation of finitistic equilibria, and Theorem 3.3, which shows that finitistic equilibria can be understood as selection equilibria in compactifications. Subsidiary results in this section establish that finitistic equilibria do not have the drawbacks that selection equilibria usually have.

The set-valued theory of integration for non-measurable functions contained in the selection approach is covered in Section 4. This background is used in the study of finitely additive equilibria, the content of Section 5.

2. Overview of the major issues and results

Systematic study of the equilibrium existence question for infinite nfgs began with Fan (1952) and Glicksberg (1952), who proved that compactness and metrizability of the S_i and joint continuity of the u_i guarantee the existence of Nash (1950) equilibria. Continuity can be relaxed in a number of directions, assuming special "diagonal" discontinuities (e.g. Dasgupta and Maskin, 1986; Simon, 1987), or special monotonicities (e.g. Vives, 1990). Such approaches lead to deep insights into the structure of useful classes of games, but do not lead to a general theory of infinite games. By contrast, selection equilibria (Simon and Zame, 1990) exist for compact metric space games with arbitrary utility functions.

The first example of this section shows that compactification of individual strategy spaces is easy, but a jointly continuous extension of the utilities is generally impossible. This leads to the definition of selection equilibria. The second example demonstrates that selection equilibria may play a strictly dominated strategy. The definition of finitistic equilibria makes it clear that they do not have this drawback. The third example demonstrates how correlation arises when representing finitistic equilibria by selection. The fourth example demonstrates one of the two ways in which selection can introduce spurious correlation. The final example suggests the centrality of normal form analyses of infinite extensive form games.

2.1. Compactness of the strategy spaces

With the addition of some limit points, compactness of strategy spaces can be guaranteed, though joint continuity of the utilities may be impossible.

Example 2.1 (*An infinite coordination game*). Γ is specified by $I = \{1, 2\}$, $S_i = \mathbb{N}$, and the symmetric utility functions,

$$u_i(s_i, s_j) = \begin{cases} 1 & \text{if } s_i = s_j, \\ 0 & \text{if } s_i \neq s_j. \end{cases} \tag{2}$$

Player i's *utility sections* are the functions $\mathcal{U}_i = \{s_i \mapsto u_k(s\backslash s_i)\colon k \in I,\ s \in S\}$. Because $\lim_{s_i \uparrow} u_k(s\backslash s_i)$ exists (and is identically equal to 0 in this simple game) for all s, the sections continuously extended to the one-point compactification, $\widehat{S_i} = \mathbb{N} \cup \{\infty\}$ with e.g. the metric $d_i(s_i, s_j) = |\mathrm{e}^{-s_i} - \mathrm{e}^{-s_j}|$ with $\mathrm{e}^{-\infty} := 0$. The S_i are dense in the compact space $\widehat{S_i}$, which defines $\widehat{S_i}$ being a compactification of S_i.

It is *not* possible to extend u_i to a jointly continuous $\hat{u}_i$ on the joint compactification, $\widehat{S} = \times_{i \in I} \widehat{S_i}$. If it were, the continuous mapping $s_j \to \hat{u}_i(\cdot, s_j)$ from the compact $\widehat{S_j}$ to $C(\widehat{S_i})$ would have a compact range.[4] This contradicts the observation that $\|\hat{u}_i(\cdot, s_j) - \hat{u}_i(\cdot, s'_j)\| = 1$ for the infinitely many pairs of $s_j \neq s'_j$.

It is always possible to compactify an S_i so that any collection of bounded functions have unique continuous extensions to the compactification (for sketches of and references to the constructions, see Section 4.2.1). As seen in Harris et al. (in press) and below in Example 2.3, for the study of infinite games, the relevant collection of bounded functions is the class of utility sections. The points added by compactification guarantee that every utility section achieves its maximum, they represent the limits of approximate optima against pure strategies.

For ncc games, the compactification can be taken to be metrizable. In general this is not possible, and the compactifications are quite large. Despite the size of the compactifications, for non-ncc games, the addition of extra points is not sufficient to guarantee equilibrium existence because one must replace the detail lost in moving from finitistic sets down to compactifications.

2.2. Selection equilibria

For each $i \in I$, let T_i be a dense subset of the compact space S_i. Let $u : T \to \mathbb{R}^I$, $T = \times_{i \in I} T_i$, be a bounded function. Having $T_i = S_i$ is usual for the analysis of compact games, having T_i be a proper subset of S_i is crucial for the compact imbedding analysis of nfgs. The function u and the set $T \subset S$ define a *pre-game*, $\Gamma_T(S, u)$.

Let $\Phi = \Phi_u$ be the correspondence from S to $\mathbb{R}^I$ having as graph the closure (in the product topology) of the set $\{(t, u(t))\colon t \in T\}$. Because T is dense in S, Φ is non-empty valued. Since u is bounded, Φ is single-valued at s if and only if u has a unique continuous extension from T to S at s. (For ncc games, Φ is always single-valued.)

[4] For a compact X, $C(X)$ is the set of continuous functions on X with the sup-norm.

At points s where u can not be continuously extended from T to S, $\Phi(s)$ contains many points. Let Ψ be a closed graph correspondence satisfying $\Phi(s) \subset \Psi(s) \subset \mathrm{co}\,\Phi(s)$ for each $s \in S$.

Definition 2.1 (*Simon and Zame*). A *Ψ-selection equilibrium* is a pair (v, μ) where v is a measurable everywhere selection from Ψ, and $\mu = (\mu_i)_{i \in I}$ is an equilibrium profile of countably additive strategies for the game with compact strategy spaces S_i and measurable utility functions v.

When $\Psi(s) = \mathrm{co}\,\Phi(s)$ for each $s \in S$ and the S_i are metric spaces, Simon and Zame (1990) show that $\mathrm{co}\,\Phi$-selection equilibria exist. These are equilibria where the utilities at discontinuities are chosen as limits of utilities in the convex hulls of nearby utilities. Selection equilibria may involve play of strictly dominated strategies.

Example 2.2. Two players simultaneously pick in their action spaces, $S_i = T_i = [0, 1]$, and the utility functions are

$$u_1(s_1, s_2) = \begin{cases} 2 & \text{if } s_1 = 0, \\ s_1 & \text{if } s_1 > 0, \end{cases} \qquad u_2(s_1, s_2) = \begin{cases} 2 - s_2 & \text{if } s_1 = 0, \\ s_2 & \text{if } s_1 > 0. \end{cases}$$

Play of $(0, 0)$ is the unique equilibrium of this game, giving utilities $(2, 2)$. The unique continuous selection from Φ is $v(s_1, s_2) = (s_1, s_2)$. Play of $(1, 1)$ is the unique equilibrium of $(S_i, v_i)_{i \in I}$. The selection v fails to capture the crucial strategic aspect of $s_1 = 0$, player 1's ability to guarantee her/himself a payoff of $u_1 = 2$. Finitistic equilibria capture $s_1 = 0$ being available to player 1.

2.3. Finitistic equilibria

A generalized sequence (net) A_i^α converges to T_i if for all finite $F_i \subset T_i$, for all sufficiently large α, $F_i \subset A_i^\alpha$.

Definition 2.2. A mixed strategy $\mu = (\mu_i)_{i \in I}$ is a *finitistic equilibrium of the pre-game $\Gamma_T(S, u)$* if it is the limit of a generalized sequence μ^α where μ^α is an ϵ^α-equilibrium of the game $(A_i^\alpha, u_i)_{i \in I}$, $A_i^\alpha \to T_i$, $\epsilon^\alpha \to 0$.

The only finitistic equilibrium for Example 2.2 is $(0, 0)$—for all sufficiently large α, A_1^α will contain the point 0, and all ϵ^α equilibria put mass at least $1 - \epsilon^\alpha$ on $(0, 0)$. This is an instance of Corollary 3.3, which shows that finitistic equilibria do not ignore profitable deviations.

Theorem 3.2 shows that exhaustive star-finite versions of the T_i, that is, finitistic versions of the T_i, perfectly represent the generalized sequences of A_i^α's. There is a surjective map from any finitistic version of T_i to any compactification of T_i.[5] Some of the correlation in selection equilibria arises to represent the information/details lost in moving down from

[5] See fn. 2 for the details.

finitistic sets to compacts. The amount of correlation is limited, Theorem 3.3 below shows that every finitistic equilibrium is a selection equilibrium from a correspondence strictly smaller than $\mathrm{co}\,\Phi$.

2.4. Correlation, information, and the size of compactifications

The following modified matching pennies game is ncc. Therefore, if it is compactified so as to make all utility sections continuous, the resulting game is compact and continuous. If compactified in a smaller fashion, one that does not make the utility sections continuous, selection can replace the lost information. This is analogous to what happens in the surjection from finitistic sets to compactifications.

Example 2.3. Two players simultaneously pick in $T_i = \mathbb{N}$, with utility functions

$$u_i(t_i, t_j) = \begin{cases} (+1, -1) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i, t_j \text{ are both even or both odd,} \\ (-1, +1) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i, t_j \text{ are of different parity.} \end{cases} \tag{3}$$

The T_i are dense in the compact metric spaces (S_i, d_i), $S_i := \mathbb{N} \cup \{\infty\}$, $d_i(s_i, s_i') := |\mathrm{e}^{-s_i} - \mathrm{e}^{-s_i'}|$, $\mathrm{e}^{-\infty} := 0$. The (S_i, d_i) are the one-point compactifications of the T_i. Not even the utility sections extend continuously to $S_i \times S_j$, $\liminf_{s_i \uparrow} u_k(s \backslash s_i) < \limsup_{s_i \uparrow} u_k(s \backslash s_i)$. Further, $\Phi(t_i, \infty) = \{(+1 - 1/t_i, -1), (-1 - 1/t_i, +1)\}$, $\Phi(\infty, t_j) = \{(+1, -1 - 1/t_j), (-1, +1 - 1/t_i)\}$, and $\Phi(\infty, \infty) = \{(+1, -1), (-1, +1)\}$.

Finitistic equilibria involve both players picking infinitely large even and odd integers with probability infinitely close to $1/2$ each. This gives utilities of $(0, 0)$. This selection equilibrium corresponding to this finitistic equilibrium is (v, μ) where μ is point mass on (∞, ∞), $v(t_i, \infty) = (0 - 1/t_i, 0)$, $v(\infty, t_j) = (0, 0 - 1/t_j)$, and $v(\infty, \infty) = (0, 0)$. By putting mass $1/2$ each on the end points of the Φ, the selection utilities encode the lost information that the players are putting mass $1/2$ each on the evens and odds.

The T_i are dense in the compact metric spaces (R_i, ρ_i), $R_i := \mathbb{N} \cup \{\infty_{\mathrm{even}}, \infty_{\mathrm{odd}}\}$,

$$\rho_i(s_i, s_i') := \begin{cases} |\mathrm{e}^{-s_i} - \mathrm{e}^{-s_i'}| & \text{if } s_i, s_i' \text{ are both even or both odd,} \\ 1 + |\mathrm{e}^{-s_i} - \mathrm{e}^{-s_i'}| & \text{if } s_i, s_i' \text{ are of different parity.} \end{cases} \tag{4}$$

The utilities have unique continuous extensions from $T_i \times T_j$ to $R_i \times R_j$. The unique equilibrium in the resulting compact and continuous game has both players playing ∞_{even} and ∞_{odd} with probability $1/2$ each.

Selection can encode information lost in the passage to the limit, but they can also encode spurious information. Finitistic equilibria avoid the spurious correlation that can arise in selection equilibria.

2.5. Spurious correlation

Spurious correlation arises from two distinct aspects of the definition of selection equilibria, the use of the full convex hull, and the use of selection itself.

2.5.1. Correlation arising from the convex hull

The next game is a version of "pick the largest integer" in which one is rewarded more highly for beating the opponents pick by 2 or more, and punished for tying them. The T_i are dense subsets of compact S_i, the utility sections have unique continuous extensions, but the game is not ncc. For large t_i, t_j, the payoffs are essentially constant along lines parallel to the diagonal.

Example 2.4. Two players simultaneously pick in $T_i = \mathbb{N}$, and the utility functions are symmetric,

$$u_i(t_i, t_j) = \begin{cases} (10, -10) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i \geqslant t_j + 2, \\ (8, 4) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i = t_j + 1, \\ (-2, -2) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i = t_j, \\ (4, 8) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i = t_j - 1, \\ (-10, 10) - (\frac{1}{t_i}, \frac{1}{t_j}) & \text{if } t_i \leqslant t_j - 2. \end{cases} \tag{5}$$

The T_i are dense in the one-point compactifications used above. Continuity gives $\Phi(t_i, \infty) = \{(-10, 10)\}$ and $\Phi(\infty, t_j) = \{(10, -10)\}$. The only discontinuity point for u happens at (∞, ∞). The convex hull of the limits of the possible at payoffs at (∞, ∞) is

$$V = \operatorname{co} \Phi(\infty, \infty) = \operatorname{co}\bigl\{(10, -10), (8, 4), (-2, -2), (4, 8), (-10, 10)\bigr\}. \tag{6}$$

Any selection from V combined with play of (∞, ∞) is a selection equilibrium for $\Gamma_T(S, u)$. In particular, the utility levels $\alpha(8, 4) + (1 - \alpha)(4, 8)$, $\alpha \in (0, 1)$ belong to $\operatorname{co} \Phi(\infty, \infty)$ and can occur as a $\operatorname{co} \Phi$-selection equilibrium. However, these payoffs require that α of the time, i plays one higher than j plays and $1 - \alpha$ of the time s/he plays one lower. This requires perfect correlation and strictly positive randomization. This cannot arise from independent play.

Finitistic equilibria will have as utilities the limits of independent randomization by the players. The *Nash hull* correspondence studied below satisfies $\Phi(s) \subset \operatorname{Nh} \Phi(s) \subset \operatorname{co} \Phi(s)$ for all $s \in S$ and captures this independence. Theorem 3.3 shows that finitistic equilibria are always $\operatorname{Nh} \Phi$-selection equilibria.

2.5.2. Correlation arising from selection

In a two player game, $\operatorname{Nh} \Phi(s)$, the Nash hull at a point $s = (s_1, s_2)$ consists of the set of limits of payoffs to generalized sequences of independent randomizations that converge to the product of point masses on s_1 and s_2. At continuity points, $\operatorname{Nh} \Phi$ is a singleton set. When there are two discontinuities, a selection may pick different randomization at the different points, effectively correlating players' choices.

In more detail, the two $\operatorname{Nh} \Phi(s_1, s_2)$ and $\operatorname{Nh} \Phi(s_1, t_2)$, $s_2 \neq t_2$ are formed by independent randomization in the neighborhoods of (s_1, s_2) and (s_1, t_2). If independence is to be respected, then the play of 1 should be the same in both of these neighborhood (systems), 1's play should not depend on 2's choices. Selections have no such consistency restrictions.

It is not the Nash hull that is to blame. Example 5.1 gives a Φ with multiple discontinuities. Selections from Φ can introduce correlation between players' choices. Corollary 5.1.1 shows that it is multiple discontinuities that are to blame.

2.6. The need for normal form analyses

One approach to extensive form games with infinite choice sets at some node(s) is to specify a set of strategies and to use a finitistic analysis of the resulting infinite nfg. A second approach is to replace the infinite choice sets with finitistic versions and analyze the resulting game. This second approach may change essential informational structures in the game, suggesting the centrality of normal form analyses.

Example 2.5. At time $t = 0$, Nature picks $\omega = (\omega_1, \omega_2) \in \{-1, +1\} \times \{H, T\}$ according to a strictly positive distribution P. At $t = 1$, player 1 picks $a_1 \in A_1 = [0, 1]$ without observing any aspect of ω. At $t = 2$, player 2 observes a_1, but no aspect of ω, and picks $a_2 \in A_2 = [0, 1]$.

If $\omega_2 = H$, then at $t = 3$, player 1 observes a_2 and can change her mind, picking some other $a_1 \in [0, 1]$, but if $\omega_2 = T$, nothing happens at $t = 3$.

At $t = 4$, players 3, 4, and 5 observe the continuous signals $s_3 = \omega_1 \cdot |a_1 - a_2|$, $s_4 = \omega_1 \cdot |(a_1)^2 - a_2|$, and $s_5 = \omega_1 \cdot |a_1 - (a_2)^2|$.

After observing their respective signals, 3, 4, and 5 pick a_3, a_4, and a_5 in non-trivial sets. Payoffs are arranged so that players 1 and 2 have different interests in which of the later players are informed about ω_1.

The extensive form game just given has a clear strategic structure: depending on ω_2, if one of the first two players chooses either 0 or 1, then the other player can pick whether all later players or none of them know the value of ω_1; in a similar fashion, if one of the first two players chooses in the interval $(0, 1)$, then the other player can pick any one of the later players to be uninformed of the value of ω_1, or else can choose all of them to be informed.

(1) If the action sets A_1 and A_2 are replaced by finitistic sets, the game does not have this strategic structure.

Proof: Let F_1 and F_2 be finitistic versions of A_1 and A_2. If player 1 picks $t_1 \neq 0, 1$ and $\omega_2 = T$, then, in order for player 2 to have the choice of which of the three later players does not know the value of ω_1, F_2 must contain t_1, $(t_1)^2$, and $\sqrt{t_1}$. In exactly the same way, if $\omega_2 = H$, then for every $t_2 \in F_2 \setminus \{0, 1\}$, F_1 must contain t_2, $(t_2)^2$, and $\sqrt{t_2}$. Thus, three incompatible conditions must be simultaneously satisfied, $F_1 = F_2$, $F_1^2 = F_2$ and $F_1 = F_2^2$.

(2) Finitistic replacements of the normal form can replicate the strategic structure.

Proof: In the normal form, player 1's strategy set is the product of $[0, 1]$ and a large subset of $[0, 1]^{[0,1]}$ while 2's strategy set is a large subset of $[0, 1]^{[0,1]}$. Provided the large subsets contain the continuous functions, finitistic versions of the strategy sets contain the functions $f(x) = x$, $f(x) = x^2$, and $f(x) = \sqrt{x}$.

There are still difficult open questions for the development of a general theory of extensive form games. It is not clear that finitistic replacement of normal forms will always respect informational structures. If one takes a normal form approach, one must discover the extensive form implications of Nash hull selection equilibria.

2.7. Notation used throughout

In order to discuss finitely supported mixtures, for each $i \in I$, $\mathcal{S}_i$ is a field of subsets of S_i containing the singleton sets,[6] e.g. 2^{S_i}. The set of mixed strategies for $i \in I$ is Δ_i. Strategies are always assumed to be finitely additive on $\mathcal{S}_i$. There is no assumption that utilities are integrable. If $\mu = (\mu_i)_{i \in I} \in \Delta := \times_{i \in I} \Delta_i$ is a vector of mixed strategies, the product measure on $\times_{i \in I} \mathcal{S}_i$, the smallest field containing the measurable rectangles, is denoted $\text{prod}(\mu)$. If the μ_i are countably additive and the $\mathcal{S}_i$ are σ-fields, the unique extension of $\text{prod}(\mu)$ to the product σ-field, $\bigotimes_{i \in I} \mathcal{S}_i := \sigma(\times_{i \in I} \mathcal{S}_i)$, is again denoted $\text{prod}(\mu)$.

3. Finitistic and selection equilibria for compact games

For this section, each S_i is a non-empty compact Hausdorff space (cHs) with the Borel σ-field of subsets, $\mathcal{S}_i$. By assumption, the mixed strategies are the unique countably additive extensions of Baire measures to $\mathcal{S}_i$.[7]

Finitistic equilibria are the standard parts of the equilibria on games in which the S_i have been replaced by exhaustive, *-finite sets A_i. By transfer, such equilibria exist. Theorem 3.3 shows that finitistic equilibria are a subset of the Nash hull selection equilibria for games with arbitrary cHs strategy spaces. This generality enables the interpretation of the finitistic equilibria of nfgs through compact imbedding and selection. However, selection equilibria should be regarded as a useful interpretation of finitistic equilibria rather than as an independent solution concept.

Recall that Φ-selection equilibria constitute the smallest possible set of selection equilibria that use limit values for utilities. Example 2.2 shows that Φ-selection equilibria may ignore profitable deviations, Corollary 3.3 shows that finitistic equilibria do not. Example 5.1 shows that Φ-selection equilibria add spurious correlation to the finitistic equilibria. Endogenous sharing rule equilibria are $\text{co}\,\Phi$-selection equilibria. Example 2.4 shows that endogenous sharing rule equilibria add spurious correlation to the Nash selection equilibria. In sum, selection equilibria may ignore profitable deviations, any kind of selection can add spurious correlation, and $\text{co}\,\Phi$-selection adds the most.

[6] Without the singleton sets, the mixed strategies interpreted as being pure are the ones satisfying $\mu_i(E_i) \in \{0, 1\}$ for all $E_i \in \mathcal{S}_i$. Purely finitely additive $\{0, 1\}$-valued measures can be difficult to integrate against general u.

[7] This clears up a potential ambiguity in Definition 2.1. The Baire σ-field is the smallest making the continuous functions measurable. In metric spaces, the Baire σ-field is the Borel σ-field. Urysohn's lemma on the approximation of indicators of closed sets by continuous functions implies that Baire measures have unique countably additive extensions to the Borel σ-field for a cHs. There exist cHs with Borel measures that are not the extensions of Baire measures.

3.1. *Definition of selection equilibria*

For each $i \in I$, let T_i be a dense subset of S_i. Let $u : T \to \mathbb{R}^I$, $T = \times_{i \in I} T_i$, be a bounded function. Having $T_i = S_i$ is usual for the analysis of compact games, having T_i be a proper subset of S_i is crucial for the compact imbedding analysis of nfgs. The function u and the set $T \subset S$ define a *pre-game*, $\Gamma_T(S, u)$.

Let Φ be the correspondence from S to $\mathbb{R}^I$ having as graph the closure (in the product topology) of the set $\{(t, u(t))\colon\ t \in T\}$. Because T is dense in S, Φ is non-empty valued. Since u is bounded, Φ is single-valued if and only if u has a unique continuous extension from T to S. At points s where u does not extend continuously from T to S, $\Phi(s)$ contains many points.

3.1.1. *Correspondences derived from Φ*

Let $E = \times_{i \in I} E_i$ be a non-empty, measurable subset of S. The leading class of E's will have each E_i open. With $\operatorname{cl} A$ denoting the closure of the set A, the *point mass values of u on E* are

$$P_E = \operatorname{cl}\left\{\int u(s)\,\mathrm{d}\delta_t(s)\colon\ \delta_t \text{ point mass on some } t \in T \cap E\right\}, \tag{7}$$

the *Nash hull of the values of u on E* are

$$N_E = \operatorname{cl}\left\{\int u(s)\,\mathrm{d}\mu(s)\colon\ \mu \text{ a finitely supported, product measure on } T \cap E\right\}, \tag{8}$$

and the *correlated hull of the values of u on E* are

$$C_E = \operatorname{cl}\left\{\int u(s)\,\mathrm{d}\eta(s)\colon\ \eta \text{ a finitely supported measure on } T \cap E\right\}. \tag{9}$$

Point masses are product measures, and product measures are measures, so that $P_E \subset N_E \subset C_E$.

Let $\mathcal{O}(s)$ be a neighborhood basis for s consisting of sets of the form $G = \times_{i \in I} G_i$, G_i open in S_i. With $\operatorname{co} A$ denoting the convex hull of the set A, for all s, the compactness of the graph of Φ guarantees

$$\Phi(s) = \bigcap\big\{P_G\colon\ G \in \mathcal{O}(s)\big\}, \quad \text{and} \quad \operatorname{co}\Phi(s) = \bigcap\big\{C_G\colon\ G \in \mathcal{O}(s)\big\}. \tag{10}$$

Definition 3.1. The *Nash hull* of Φ is the correspondence $\operatorname{Nh}\Phi$ defined by

$$\operatorname{Nh}\Phi(s) = \bigcap\big\{N_G\colon\ G \in \mathcal{O}(s)\big\}. \tag{11}$$

If s is an isolated point or if u extends continuously to s from T, then $\Phi(s) = \operatorname{Nh}\Phi(s) = \operatorname{co}\Phi(s)$ is a singleton set.

3.1.2. *Representations of the correspondences derived from Φ*

The Nash hull of Φ has two useful representations. For the first one, let $\mathcal{C}$ be the class of finite open covers of S by sets of the form $\times_{i \in I} G_i$, G_i open in S_i. For each finite open cover $C \in \mathcal{C}$ and each $s \in S$, $C(s)$ denotes the open set $\bigcap\{G \in C\colon\ s \in G\}$. Let $\operatorname{Nh}\Phi_C$ be the correspondence having as graph the closure of the set $\{(s, v)\colon\ v \in N_{C(s)}\}$.

Lemma 3.1. *The graph of* $\mathrm{Nh}\,\Phi$ *is equal to the intersection over* $C \in \mathcal{C}$ *of the graphs of* $\mathrm{Nh}\,\Phi_C$.

Proof. If (s,u) is in the graph of $\mathrm{Nh}\,\Phi$, then for all G containing s, $u \in N_G$. For all $C \in \mathcal{C}$, $C(s)$ is an open neighborhood of s which implies that (s,u) belongs to the graph of $\mathrm{Nh}\,\Phi_C$. Since this is true for each C, (s,u) belongs to the intersection of the graphs of $\mathrm{Nh}\,\Phi_C$.

If (s,u) does not belong to the graph of $\mathrm{Nh}\,\Phi$, then there exists an open $G = \times_{i\in I} G_i \in \mathcal{O}(s)$ such that $u \notin N_G$. Since S is a compact Hausdorff space, for each $t \in S\backslash G$, there exists a pair of open sets G_t and H_t, $s \in H_t \subset G$, such that $G_t \cap H_t = \emptyset$. Because S is compact, there exists a finite subcover, C, of the open cover $\{G, \{G_t\colon t \in S\backslash G\}\}$. For this C, $C(s) \subset G$, so that (s,u) does not belong to $\mathrm{Nh}\,\Phi_C$. □

The second representation uses nonstandard analysis.[8] The Nash hull of Φ at s is the standard part of the integrals of *-finitely supported, product probabilities concentrated on the infinitesimal neighborhoods of s.

The finite subsets of T_i are denoted $\mathcal{P}_F(T_i)$. A star-finite (or *-finite) subset, A_i, of T_i is an element of $^*\mathcal{P}_F(T_i)$. When $s_i \in S_i$, the *monad of* s_i is the set $m_i(s_i) = \bigcap\{^*G_i\colon s_i \in G_i \in \mathcal{O}_i(s_i)\}$ where $\mathcal{O}_i(s_i)$ is the neighborhood basis for s_i in S_i. For any cHs, X, monads are Loeb measurable subsets of $^*X$ (Anderson and Rashid, 1978).

Lemma 3.2. *For all* $s \in S$, $\mathrm{Nh}\,\Phi(s) = \mathrm{st}\{\int {^*u}(a)\,\mathrm{d}\,\mathrm{prod}((\eta_i)_i \in I)(a)\}$ *where each* η_i *is supported by a* **-finite subset of* T_i*, and its associated Loeb measure satisfies* $L(\eta_i)(m_i(s_i)) = 1$.

Proof. For any subset E of a Hausdorff topological space, the standard part of $^*E$ is the closure of E. Since moving an infinitesimal amount of mass cannot affect the integral of a bounded function, this implies that each N_G is the standard part of the set $\{\int u(a)\,\mathrm{d}\,\mathrm{prod}((\eta)_i \in I)(a)\}$ where each η_i is a *-finitely supported measure satisfying $\eta_i(^*(T_i \cap G_i)) \simeq 1$. Definition 3.1 and the Loeb measurability of monads complete the proof. □

3.2. The existence of selection equilibria

Let Ψ be a non-empty valued, closed graph, bounded correspondence from S to $\mathbb{R}^I$. Each such Ψ defines a *multigame* $\Gamma_T(S,\Psi)$. A measurable function $\psi : S \to \mathbb{R}^I$ is an everywhere selection from Ψ if for all s, $\psi(s) \in \Psi(s)$.[9]

[8] We work in a κ-saturated, nonstandard enlargement of a superstructure $V(Z)$ where Z contains each S_i as well as $\mathbb{R}$, and κ is a cardinal greater than the cardinality of $V(Z)$. The most accessible introductions to nonstandard analysis that I have found are Lindstrøm (1988) and Anderson (1991).

[9] Everywhere selections exist because Ψ is a measurable closed-valued correspondence (Klein and Thompson, 1984, Definition 13.1.1, Theorem 14.2.1).

Definition 3.2. A strategy profile $\mu^* \in \Delta$, is a *selection equilibrium* for the multigame $\Gamma_T(S, \Psi)$ if there exists an everywhere selection, ψ, from Ψ such that μ^* is an equilibrium for the game $(S_i, \psi_i)_{i\in I}$.

A selection equilibrium for the multigame $\Gamma_T(S, \mathrm{co}\,\Phi)$ is called an *endogenous sharing rule equilibrium* in Simon and Zame (1990), which contains a proof of

Theorem 3.1. *If each S_i is a compact metric space, then a selection equilibrium exists for the multigame $\Gamma_T(S, \mathrm{co}\,\Phi)$.*

A *-finite $A_i \subset {}^*T_i$ is exhaustive if for all $t_i \in T_i$, $t_i \in A_i$. By κ-saturation, exhaustive star-finite sets exist.

Definition 3.3. A strategy profile μ^* is an *exhaustive, star-finite equilibrium for $\Gamma_T(S, u)$* if it is the weak* standard part of an ϵ-equilibrium, $\epsilon \simeq 0$, for the game $(A_i, {}^*u_i)_{i\in I}$ where each A_i is an exhaustive, *-finite subset of T_i.

The finite subsets of T_i, $\mathcal{P}_F(T_i)$, are partially ordered by $A_i \succeq_i A_i'$ if $A_i \supset A_i'$. Products $\times_{i\in I} A_i$ of finite subsets of $\times_{i\in I} S_i$ are partially ordered by $\times_{i\in I} A_i \succeq \times_{i\in I} A_i'$ if $A_i \succeq_i A_i'$ for each $i \in I$. From Definition 2.2, finitistic equilibria are the limits of approximate equilibria on large finite sets.

Theorem 3.2. *μ^* is a finitistic equilibrium for $\Gamma_T(S, u)$ if and only if it is an exhaustive star-finite equilibrium for $\Gamma_T(S, u)$.*

Proof. Let E' denote the set of finitistic equilibria, and E'' the let of limits described in the Lemma. Both E' and E'' are easily seen to be equivalent to the set E described below. For each $\times_{i\in I} A_i \in \times_{i\in I}\mathcal{P}_F(T_i)$ and $\epsilon > 0$, define $E(\times_{i\in I} A_i, \epsilon)$ as the weak* closure of the set $\{Eq^\epsilon((B_i, u_i)_{i\in I})\colon\ \times_{i\in I} B_i \succeq \times_{i\in I} A_i\}$. Define $E = \bigcap\{E(\times_{i\in I} A_i, \epsilon)\colon\ \times_{i\in I} A_i \in \times_{i\in I}\mathcal{P}_F(T_i),\ \epsilon > 0\}$. □

Corollary 3.2.1. *The set of finitistic equilibria is non-empty and compact.*

Proof. The class of compact sets $\{E(\times_{i\in I} A_i, \epsilon)\colon\ \times_{i\in I} A_i \in \times_{i\in I}\mathcal{P}_F(T_i),\ \epsilon > 0\}$, has the finite intersection property. Its non-empty, compact intersection is the set of finitistic equilibria. □

Theorem 3.3. *If each S_i is a compact Hausdorff space, then every finitistic equilibrium is a selection equilibrium for the multigame $\Gamma_T(S, \mathrm{Nh}\,\Phi)$.*

This generalizes Theorem 3.1 in four directions.

(1) It eliminates spurious correlation:
   (a) $\mathrm{co}\,\Phi$ is replaced with the smaller Nash hull, $\mathrm{Nh}\,\Phi$. This can lead to a much smaller set of equilibrium outcomes, Example 2.2.

(b) Selection itself, even from Φ, may introduce spurious correlation by violating the independence of players' choices, Example 5.1. Finitistic equilibria are interpretable as selection equilibria, but respect the independence of players' randomization, Theorems 5.1 and 5.2.

(2) The replacement of compact metric strategy spaces by general cHs allows treatment of finitely additive equilibria through compact imbedding.

(3) Robustness with respect to T_i deviations can be guaranteed by a limit interpretation of a *-finite construction in the proof—the strategies μ^* will be the limit of equilibria along a net of finite approximations to the game that includes each $t_i \in T_i$, $i \in I$. From Corollary 3.3, if $u_i(\cdot\backslash t_i)$ is $\mathrm{prod}((\mu_j^*)_{j\neq i})$-integrable and $T = S$, then

$$\int_S \psi_i(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*)(s) \geqslant \int_S u_i(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*\backslash\delta_{t_i})(s). \tag{12}$$

Example 2.2 gave a game $\Gamma_S(S,u)$ in which there is a unique continuous selection ψ from Φ, and the unique equilibrium for $\Gamma_S(S,\psi)$ fails Eq. (12).

(4) The set of finitistic equilibria depends upper hemicontinuously on the utility function, Corollary 3.3.2. Selection equilibria depend on measurable everywhere selections, and it is difficult to formulate hemicontinuity results for this class of selections.

Proof of Theorem 3.3. For each $i \in I$, let A_i be an exhaustive, star-finite subset of T_i. Since T_i is dense in S_i, the standard part mapping, $\mathrm{st}_i : A_i \to S_i$, is onto. By Anderson and Rashid (1978) and Loeb (1979), the weak*-standard part mapping, also denoted st_i, takes probabilities on A_i onto Δ_i.

Let $\eta = (\eta_i)_{i\in I}$ be an ϵ-equilibrium, $\epsilon \simeq 0$, for the internal game $(A_i, {}^*u_i)_{i\in I}$ played with the strategy sets $A_i \subset {}^*T_i$ and the utility function ${}^*u$. Let $\mu^* = (\mathrm{st}_i(\eta_i))_{i\in I}$. All that is left is to show that there is an everywhere selection, ψ, from $\mathrm{Nh}\,\Phi$ such that μ^* is an equilibrium for $(S_i, \psi_i)_{i\in I}$.

Let $L = L(\mathrm{prod}(\eta))$ denote the Loeb measure generated by the internal measure $\mathrm{prod}(\eta)$ on $A = \times_{i\in I} A_i$ with the Loeb σ-field $\mathcal{A}$ (the L-completion of the minimal σ-field containing the internal subsets of A). Let $\mathcal{F}$ denote the smallest sub-σ-field of $\mathcal{A}$ making the mapping $(a_i)_{i\in I} \mapsto (\mathrm{st}_i(a_i))_{i\in I}$ measurable (see Anderson and Rashid, 1978). Set $\psi(\cdot) = E({}^\circ{}^*u \mid \mathcal{F})(\cdot)$. Let $r = \int_A^\circ {}^*u(a)\,\mathrm{d}L(a) \in \mathbb{R}^I$. By iterated expectations and change of variable, $\int \psi(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*)(s) = r$.

Claim A: Measurably modifying ψ on a set of $\mathrm{prod}(\mu^*)$-measure 0 if necessary, for all $s \in S$, $\psi(s) \in \mathrm{Nh}\,\Phi(s) \cap \{v \in \mathbb{R}^I\colon v \leqslant r\}$.

The claim implies that μ^* is an equilibrium for $(S_i, \psi_i)_{i\in I}$—playing μ^* gives each i the expected payoff r_i, and $\psi(s) \leqslant r$ for all $s \in S$ implies that playing any s_i against μ^* must give i a payoff less than or equal to r_i.

Proof of Claim A:[10] Moving an infinitesimal amount of mass if necessary, we can guarantee that η is a $2\cdot\epsilon$-equilibrium such that for each $i \in I$ and each $a_i \in A_i$, $\eta_i(a_i) \in {}^*\mathbb{R}_{++}$, that is, each η_i is *-full support.

For a $P_i \in \Delta_i$, a P_i-continuity set is a set E_i with boundary ∂E_i satisfying $P_i(\partial E_i) = 0$. Because S_i is a cHs, for every open neighborhood G_i of s_i, there exists an open P_i-continuity set H_i such that $s_i \in H_i \subset G_i$.[11]

Let $\mathcal{C}_{\text{cont}}$ be the class of finite open covers of S by sets of the form $\times_{i\in I} G_i$, each G_i a μ_i^*-continuity set. Pick an arbitrary $C \in \mathcal{C}_{\text{cont}}$ and note that every $C_m \in C$ is a $\text{prod}(\mu^*)$ continuity set. Enumerate C as $C_1, \ldots, C_M$. Set $E_1 = \text{cl}\, C_1$. If E_n has been defined, set

$$E_{n+1} = \text{cl}\left(C_{n+1} \setminus \bigcup_{m=1}^{n} E_m\right) \Big\backslash \bigcup_{m=1}^{n} E_m. \tag{13}$$

Let $\mathcal{F}_C$ be the field generated by the E_ns.

Define an element $C \in {}^*\mathcal{C}_{\text{cont}}$ to have property (†) if for all non-empty $E \in \mathcal{F}_C$, ${}^*E \cap A \neq \emptyset$. Let ${}^*\mathcal{C}^{\dagger}_{\text{cont}}$ denote the internal subclass of ${}^*\mathcal{C}_{\text{cont}}$ with property (†). By construction, for a standard $C \in {}^*\mathcal{C}_{\text{cont}}$, each E_n is either empty or has non-empty interior. Therefore, because non-empty open sets meet the dense set T, and each $t \in T$ belongs to A, ${}^*\mathcal{C}^{\dagger}_{\text{cont}}$ contains all the standard elements of ${}^*\mathcal{C}_{\text{cont}}$.

For any $C \in {}^*\mathcal{C}^{\dagger}_{\text{cont}}$, and any $s \in {}^*S$, let $\mathcal{F}_C(s)$ be the smallest element of $\mathcal{F}_C$ that contains s. Since A meets each non-empty element of $\mathcal{F}_C$, the following internal function $\varphi_C : A \to {}^*\mathbb{R}^I$ is well defined:

$$\varphi_C(a) = \sum_{b\in\mathcal{F}_C(a)} {}^*u(b)\text{prod}(\eta)(b) \Bigg/ \sum_{b\in\mathcal{F}_C(a)} \text{prod}(\eta)(b). \tag{14}$$

The function $\psi_C = {}^\circ\varphi_C$ is a version of $E({}^{\circ *}u \mid \sigma(\mathcal{F}_C))$ (see Anderson, 1982). Because u is bounded, ψ_C is bounded.

Partially order the elements of ${}^*\mathcal{C}^{\dagger}_{\text{cont}}$ by $C' \succ C$ if for all $s \in {}^*S$, $\mathcal{F}_{C'}(s) \subset \mathcal{F}_C(s)$. The collection of ψ_C, C standard, is a uniformly bounded martingale closed by ψ, hence converges in L^1 norm to ψ. Further, the union of any finite collection of standard C in $\mathcal{C}^{\dagger}_{\text{cont}}$ is another standard element of $\mathcal{C}^{\dagger}_{\text{cont}}$. Therefore, by Overspill, there is a *-finite *-open cover $C' \in {}^*\mathcal{C}^{\dagger}_{\text{cont}}$ of ${}^*S$ with the properties that $C' \succ C$ for all standard C, that $\text{prod}(\eta)(C'(s)) > 0$ for each $s \in {}^*S$, and that ${}^*\|\varphi_{C'} - {}^*\psi\|_1 \simeq 0$. Therefore, the function ${}^\circ\varphi_{C'}$ is a version of $\psi(\cdot) = E({}^{\circ *}u \mid \mathcal{F})(\cdot)$. For all $s \in S$, $C'(s)$ is a subset of the monad of s because C' refines all finite open covers of S. Lemma 3.2 implies that for all $s \in S$, ${}^\circ\varphi_{C'}(s) \in \text{Nh}\,\Phi(s)$. Finally, because η is a $2\cdot\epsilon$-equilibrium, $\epsilon \simeq 0$, for all $a \in A$, ${}^{\circ *}u(a) \leqslant r$ in $\mathbb{R}^I$. By Eq. (14), this implies that for all $s \in S$, ${}^\circ\varphi_{C'}(s) \leqslant r$. □

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10 Two very different proofs are possible. The given proof uses Lemma 3.2 in forming a specific version of ψ from a nonstandard construction. This construction is crucial in the existence theorem of Section 5. There is a second proof that appeals to Lemma 3.1 in showing that $\mu^*(\psi \in \text{Nh}\,\Phi) = 1$, that $\mu^*(\psi \leqslant r) = 1$, and then uses an everywhere measurable selection theorem to modify ψ on the remaining set of measure 0.

11 To see why, let g be a continuous function from S_i to $[0, 1]$ such that $g(s_i) = 0$ and $g(S_i \backslash G_i) = 1$. Such a function exists by Urysohn's lemma. The sets $g^{-1}(r)$ are disjoint and measurable, $0 < r < 1$, so that at most countably many of them have positive P_i-mass. For any r satisfying $P_i(g^{-1}(r)) = 0$, $H_i = g^{-1}(-\infty, r)$ is an open P_i continuity set containing s_i.

3.3. Properties of finitistic equilibria

Let $E_T(u)$ denote the set of Nash selection equilibria that arise as the standard parts of ϵ-equilibria, $\epsilon \simeq 0$, of the internal game played with an exhaustive star-finite strategy set $A_i \subset {}^*T_i$ and the utility function ${}^*u$.

Corollary 3.3.1. *If $T = S$, each $u_i(\cdot\backslash t_i)$ is measurable, and $(\psi, \mu^*) \in E_T(u)$, then for all $i \in I$ and all $t_i \in S_i$, $\int_S \psi_i(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*)(s) \geqslant \int_S u_i(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*\backslash\delta_{t_i})(s)$.*

Proof. Pick arbitrary $(\psi, \mu^*) \in E_T(u)$ and $t_i \in S_i$. Since the A_i are exhaustive, $t_i \in A_i$. Let η be an ϵ-equilibrium, $\epsilon \simeq 0$, such that $\mu^* = {}^\circ(\eta)$. By the definition of ϵ-equilibria,

$$\int_A {}^*u_i(a)\,\mathrm{d}\,\mathrm{prod}(\eta)(a) \geqslant \int_A {}^*u_i(a)\,\mathrm{d}\,\mathrm{prod}(\eta\backslash\delta_{t_i})(a) - \epsilon. \tag{15}$$

With $L = L(\mathrm{prod}(\eta))$ denoting the Loeb measure on $A = \times_{i\in I} A_i$, the previous inequality and the measurability of $u_i(\cdot\backslash t_i)$ implies that

$$\int_A {}^{\circ*}u_i(a)\,\mathrm{d}L(a) \geqslant \int_S u_i(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*\backslash\delta_{t_i})(s). \tag{16}$$

By iterated expectations and change of variable, $\int_S \psi_i(s)\,\mathrm{d}\,\mathrm{prod}(\mu^*)(s)$ is equal to $\int_A {}^{\circ*}u_i(a)\,\mathrm{d}L(a)$. $\square$

Metrize the set of utilities on T, $\mathcal{U}_T$, with the sup-norm, ρ. Since Δ is compact, the following is an upper hemicontinuity result for the finitistic equilibrium correspondence (the simple proof is omitted).

Corollary 3.3.2. *For fixed T, $\{(u, \mathrm{proj}_\Delta E_T(u))\}$ is a closed subset of $\mathcal{U}_T \times \Delta$.*

The next section provides a thorough examination of the relations between compact imbedding and selection for general sets of discontinuities. Since finitistic sets are exhaustive, they avoid the selection problem. As will now be seen, they are also an indispensable tool for examining selections.

4. Set-valued integrals of non-measurable functions

Implicit in the above approach to games with utilities defined on a dense subsets is a theory of integration for non-measurable functions.

(1) Fix an arbitrary set, X, and field $\mathcal{X}$ of subsets. Imbed it as a dense of a compact space, $\widehat{X}$, choosing the space $\widehat{X}$ so that any finitely additive μ on $\mathcal{X}$ has a unique extension, $\hat{\mu}$, to the Borel σ-field on $\widehat{X}$.
(2) Any bounded $\mathbb{R}^k$-valued function f on X can be identified with its graph, $\{(x, f(x))\colon x \in X \subset \widehat{X}\}$, in $\widehat{X} \times \mathbb{R}^k$. Closing the graph gives a non-empty valued correspondence $\Phi = \Phi_f$ with compact graph in $\widehat{X} \times \mathbb{R}^k$.

(3) Integrating the correspondence Φ, or a correspondence derived from Φ against $\hat{\mu}$ gives a set-valued integral of f against μ.

The two basic integrals, $E_c^\mu f$ and $E_x^\mu f$, are called the convex and the extremal integral of (a function) f with respect to (a measure) μ. Though here defined more transparently through sets of extensions of measures, both have integral-of-correspondence representations, $E_c^\mu f = \int \mathrm{co}\, \Phi_f \, \mathrm{d}\hat{\mu}$ and $E_x^\mu f = \int \Phi_f \, \mathrm{d}\hat{\mu}$.

The next section discusses the Nash and the product integral in game theoretic contexts where the domain has a product structure. The Nash integral is defined as $\int \mathrm{Nh}\, \Phi_f \, \mathrm{d}\hat{\mu}$. Since $\Phi_f \subset \mathrm{Nh}\, \Phi_f \subset \mathrm{co}\, \Phi_f$, $E_x^\mu f \subset E_N^\mu f \subset E_c^\mu f$. As seen above, the Nash integral yields a non-empty set of selection equilibria, which may be a strict subset of the selection equilibria when $\mathrm{co}\, \Phi$ is used, i.e. of the endogenous sharing rule equilibria.

Example 5.1 will show that selections, even from Φ_f, can involve correlation of the players' choices. Because of this, the whole concept of a selection equilibrium is larger than need be, and the theory of integration of correspondences, being based on selections, is not the correct tool for infinite normal form games. The product integral, $E_p^\mu f$, satisfies $E_p^\mu f \subset E_N^\mu f$ and cannot generally be represented as the integral of a correspondence. However, it is the correct integral for the analysis of nfgs.

All four of the integrals, $E_c^\mu f$, $E_x^\mu f$, $E_N^\mu f$, and $E_p^\mu f$, have *-finite characterizations, though the characterization of $E_N^\mu f$ is a bit awkward. Table 1 organizes these observations and provides a partial map.

Sections 4.1–4.4 treat the two basic integrals, $E_x^\mu f$ and $E_c^\mu f$: Section 4.1 gives their measure extension definitions and shows that they exist; Section 4.2 contains the compactification/selection characterizations; Section 4.3 contains the *-finite characterizations; using these characterizations, Section 4.4 covers the essential properties of E_c^μ and E_x^μ.

Section 5.1–5.2 cover the two integrals that require product space domains for their definition, the Nash and the smaller product integral, $E_N^\mu f$ and $E_p^\mu f$: Section 5.1 defines the two integrals and gives their basic properties; Section 5.2 shows that all normal form games with bounded payoffs have equilibria in finitely additive strategies when expected payoffs are computed using the product integral.

Table 1

Integral (notation)	Domain restrictions	Correspondence, characterization	*-Finite characteriz.	Equilibrium existence
Convex ($E_c^\mu f$)	None	$\mathrm{co}\, \Phi_f$, Theorem 4.2	Theorem 4.4	For some compact games[a]
Extremal ($E_x^\mu f$)	None	Φ_f, Theorem 4.2	Theorem 4.4	Open
Nash ($E_N^\mu f$)	Product space	$\mathrm{Nh}\, \Phi_f$, Definition 5.1	Cor. 5.1.1 et seq.	Theorem 3.3, all compact games
Product ($E_p^\mu f$)	Product space	None, Example 5.1	Definition 5.2	Theorem 5.2, all compact games

[a] (Simon and Zame, 1990).

4.1. Notation, definitions, and existence

$B^k(\mathcal{X})$ denotes the set of uniform limits of simple $\mathcal{X}$-measurable, $\mathbb{R}^k$-valued functions on X, $\mathcal{X}$ a field of subsets of X. $\mathbb{P}(\mathcal{X})$ denotes the set of finitely additive probabilities on $\mathcal{X}$. Fix an $f \in B^k(2^X)$ and a $\mu \in \mathbb{P}(\mathcal{X})$.

$M(\mu)$ denotes the set of extensions of μ from $\mathcal{X}$ to 2^X. By the Hahn–Banach extension theorem, $M(\mu)$ is non-empty. The weak* topology on $\mathbb{P}(2^X)$ is the weakest topology making the mappings $\mu \mapsto \int g\,\mathrm{d}\mu$ continuous for all $g \in B^k(2^X)$. $M(\mu) = \{\nu \in \mathbb{P}(2^X)\colon (\forall g \in B^1(\mathcal{X}))[\int g\,\mathrm{d}\mu = \int g\,\mathrm{d}\nu]\}$ expresses $M(\mu)$ as the intersection of sets satisfying weak*-continuous, linear equalities. Since $\mathbb{P}(2^X)$ is weak* compact, $M(\mu)$ is compact and convex. By the Krein–Milman theorem (e.g. Dunford and Schwartz, 1957, Theorem V.8.4), $M(\mu)$ is the closed convex envelope of its necessarily non-empty set of extreme points.

Definition 4.1. The *convex integral of f with respect to μ* is

$$E_c^\mu f = \left\{ \int_X f(x)\,\mathrm{d}\nu(x)\colon\ \nu \in M(\mu) \right\}, \tag{17}$$

and the *extremal integral of f with respect to μ* is

$$E_x^\mu f = \left\{ \int_X f(x)\,\mathrm{d}\nu(x)\colon\ \nu \in \operatorname{extr} M(\mu) \right\} \tag{18}$$

where $\operatorname{extr} M(\mu)$ denotes the set of extreme points of $M(\mu)$.

If $f \in B^k(\mathcal{X})$, then $E_c^\mu f = E_x^\mu f = \{\int f\,\mathrm{d}\mu\}$. Theorem 4.5 (below) shows that $f \in B^k(\mathcal{X})$ iff $E_c^\mu f$ is a singleton set for all $\mu \in \mathbb{P}(\mathcal{X})$. In general, we have

Theorem 4.1. *$E_x^\mu f$ is a non-empty, compact subset of $\mathbb{R}^k$ and $E_c^\mu f$ is its convex hull.*

The proofs for this section are in Appendix A.

4.2. Compactification/selection characterizations

L is a sup-norm continuous $\mathbb{R}^k$-valued function on $B^k(\mathcal{X})$ if and only if for all $g \in B^k(\mathcal{X})$, $L(g) = \int g\,\mathrm{d}\mu$ for some finitely additive measure, μ, on $\mathcal{X}$ (e.g. Dunford and Schwartz, 1957, Theorem IV.4.1). This parallel with integration of continuous functions against countably additive Borel measures on compact spaces suggests that finitely additive μs can be understood as the trace of a countably additive measure on a larger, compact space.[12] In this way, the theory of integration of bounded measurable functions is subsumed by the theory of integration of continuous functions.

[12] This identification of integration of bounded measurable functions with integration of continuous functions on compact spaces can be used to extend star-finite representation theorems for Radon measure spaces to general measure spaces (Anderson, 1982). The trace interpretation resolves the paradoxes that arise from the use of finitely additive probabilities in stochastic process theory (Kingman, 1967), in the theory of choice under uncertainty

The larger space, denoted $\widehat{X} \mid B^k(\mathcal{X})$, is called the Stone space for $(X, \mathcal{X})$. X is imbedded in $\widehat{X} \mid B^k(\mathcal{X})$ in such a fashion that every $g \in B^k(\mathcal{X})$ has a unique continuous extension to $\widehat{X} \mid B^k(\mathcal{X})$. This imbedding carries $\mu \in \mathbb{P}(\mathcal{X})$ to a countably additive $\hat{\mu}$ on $\widehat{X} \mid B^k(\mathcal{X})$. The graph of a non-measurable $f \in B^k$ is a subset of $X \times \mathbb{R}^k$. Imbedding of X in $\widehat{X} \mid B^k(\mathcal{X})$ carries each point $(x, f(x))$ to the corresponding point in $\widehat{X} \mid B^k(\mathcal{X}) \times \mathbb{R}^k$. Define the correspondence Φ to have as graph the closure of the graph of f in $\widehat{X} \mid B^k(\mathcal{X}) \times \mathbb{R}^k$. Theorem 4.2 shows that integral of the correspondence Φ against $\hat{\mu}$ is exactly $E_x^{\mu} f$, and the integral of $\mathrm{co}\, \Phi$, the pointwise convex hull of Φ, is exactly $E_c^{\mu} f$.

4.2.1. Background

The following constructions and results can be found in many standard sources, e.g. Ash (1972), Dudley (1989), or Dunford and Schwartz (1957).

$B^k = B^k(2^X)$ denotes the set of bounded, $\mathbb{R}^k$-valued functions on X. Aside from some duplication of coordinates, $B^k = (B^1)^k$, that is, any $g \in B^k$ can equally be regarded as a k-length vector of points in B^1. For any function $g \in B^1$, let K_g denote a compact set in $\mathbb{R}^1$ containing $g(X)$. For any $\mathbb{G} \subset B^1$, each $x \in X$ can be imbedded as the vector $\tilde{x} \in \times_{g \in \mathbb{G}} K_g$ satisfying $\mathrm{proj}_g(\tilde{x}) = g(x)$ for all $g \in \mathbb{G}$. Let $\widehat{X} \mid \mathbb{G}$ denote the closure of $\widetilde{X}_{\mathbb{G}} = \{\tilde{x}\colon x \in X\}$ in the product topology. Let $\widehat{\mathcal{X}} \mid \mathbb{G}$ denote the trace of the Baire (i.e. the product) σ-field on $\widehat{X} \mid \mathbb{G}$. Taking $\mathbb{G} = B^1(\mathcal{X})$ gives the space $(\widehat{X} \mid B^1(\mathcal{X}), \widehat{\mathcal{X}} \mid B^1(\mathcal{X}))$, known as the Stone space for $(X, \mathcal{X})$. Other than some duplication of coordinates, there is no difference between the spaces $\widehat{X} \mid B^1(\mathcal{X})$ and $\widehat{X} \mid B^k(\mathcal{X})$.

For $\mathbb{G} \subset B^1$, let $\mathrm{proj}_{\mathbb{G}}$ denote the canonical projection of $\times_{g \in B^1} K_g$ onto $\times_{g \in \mathbb{G}} K_g$ so that $\widehat{X} \mid \mathbb{G}$ is the image of $\widehat{X} \mid B^1$ under $\mathrm{proj}_{\mathbb{G}}$. For $\mathbb{G} \subset B^1$, let $\mathbf{alg}(\mathbf{1}, \mathbb{G})$ denote the smallest, sup-norm closed algebra containing the constants and $\mathbb{G}$. Each $h \in \mathbf{alg}(\mathbf{1}, \mathbb{G})$ has a unique, continuous extension $\hat{h}_{\mathbb{G}}$ to $\widehat{X} \mid \mathbb{G}$. Further, $\sup_{x \in X} h(x) = \max_{\hat{x} \in \widehat{X} \mid \mathbb{G}} \hat{h}_{\mathbb{G}}(\hat{x})$, and any continuous function on $\widehat{X} \mid \mathbb{G}$ is the extension of some $h \in \mathbf{alg}(\mathbf{1}, \mathbb{G})$. Any $\hat{h}_{\mathbb{G}}$ has a canonical, continuous extension to $\widehat{X} \mid B^1$ defined by $\hat{h}(x) = \hat{h}_{\mathbb{G}}(\mathrm{proj}_{\mathbb{G}}(x))$.

The linear mapping $h \leftrightarrow \hat{h}_{\mathbb{G}}$ gives an isometric isomorphism between $\mathbf{alg}(\mathbf{1}, \mathbb{G})$ and $C(\widehat{X} \mid \mathbb{G})$ (both with the sup-norm topology). This means that continuous linear functions on $\mathbf{alg}(\mathbf{1}, \mathbb{G})$ and continuous linear functions on $C(\widehat{X} \mid \mathbb{G})$ can be identified. If $\mathbb{G} = B(\mathcal{X})$, the identification of continuous linear functions identifies the finitely additive μ in $\mathbb{P}(\mathcal{X})$ with the countably additive Baire probabilities $\hat{\mu}$ on $(\widehat{X} \mid \mathbb{G}, \widehat{\mathcal{X}} \mid \mathbb{G})$. Each such Baire probability has a unique extension to the Borel σ-field, and this extension is a Radon measure. No distinction will be made between Baire probabilities and their Borel extensions. Because Borel probabilities that are not Radon will not be considered, there is no reason to distinguish between Baire and Borel σ-fields. For $\mu \in \mathbb{P}(\mathcal{X})$, $\widehat{M}(\hat{\mu})$ denotes the set of countably additive extensions of $\hat{\mu}$ on $\widehat{X} \mid \mathbb{G}$ to countably additive probabilities on $\widehat{X} \mid B^1$. The identification delivers the following two equalities, valid for all $f \in B^k$ and all $\mu \in \mathbb{P}(\mathcal{X})$,

$$E_c^{\mu} f = \left\{ \int_{\widehat{X} \mid B^k} \hat{f}\, \mathrm{d}\nu\colon \nu \in \widehat{M}(\hat{\mu}) \right\}, \quad \text{and} \tag{19}$$

with finitely additive subjective probabilities (Stinchcombe, 1997), and it clarifies the structure of social choice possibility theorems in models infinitely many agents (Kirman and Sondermann, 1972; Armstrong, 1980, 1985).

$$E_x^\mu f = \left\{ \int_{\widehat{X}|B^k} \hat{f}\,\mathrm{d}\nu\colon\ \nu \in \operatorname{extr} \widehat{M}(\hat{\mu}) \right\}. \tag{20}$$

4.2.2. Compactification and selection is equivalent to extension and integration

For a given $\mu \in \mathbb{P}(\mathcal{X})$ there are many extensions to 2^X giving rise to many possible integrals of $f \in B^k$. For any given $f \notin B^k(\mathcal{X})$, there are points x in $\widehat{X} \mid B^k(\mathcal{X})$ where $\hat{f}$ is not continuous. Taking $\widehat{\mathcal{X}} \mid B^k(\mathcal{X})$-measurable selections from the possible limits at discontinuity points and then integrating the selection with respect to the unique extension $\hat{\mu}$ gives rise to many possible integrals of f. The content of Theorem 4.2 is that these two approaches are equivalent.

Pick arbitrary $f \in B^k$, and define the correspondence $\Phi = \Phi_f$ from $\widehat{X} \mid B^k(\mathcal{X})$ to $\mathbb{R}^k$ by its graph,

$$gr\,\Phi = \left\{ \left(\operatorname{proj}_{B^k(\mathcal{X})}(\hat{x}),\ \hat{f}(\hat{x})\right)\colon\ \hat{x} \in \widehat{X} \mid B^k \right\}, \tag{21}$$

and define the correspondence $\operatorname{co}\Phi$ by defining $\operatorname{co}\Phi(\hat{x})$ to be the convex hull of $\Phi(\hat{x})$.

There are three useful alternate representations of $gr\,\Phi$. The first is $gr\,\Phi = \widehat{X} \mid \mathbb{G}$ where $\mathbb{G} = \{B^k(\mathcal{X}), (f_i)_{i=1}^k\}$, implying that $gr\,\Phi$ and $gr \operatorname{co}\Phi$ are compact sets in the product topology. Second, for each $\hat{x} \in \widehat{X} \mid B^k(\mathcal{X})$, $\Phi(\hat{x})$ is the set of accumulation points of nets $\hat{f}(\hat{x}_\alpha)$ where $\operatorname{proj}_{B^k(\mathcal{X})}(\hat{x}_\alpha)$ converges to $\hat{x}$ in $\widehat{X} \mid B^k(\mathcal{X})$. Third, for each $\hat{x} \in \widehat{X} \mid B^k(\mathcal{X})$, $\Phi(\hat{x}) = \hat{f}(\operatorname{proj}^{-1}_{B^k(\mathcal{X})}(\hat{x}))$.

Because the correspondences Φ and $\operatorname{co}\Phi$ have closed graphs, they are measurable in the Borel σ-field, and have measurable everywhere (m.e.) selections (Klein and Thompson, 1984, Definition 13.1.1 and Theorem 14.4.1)). The integrals of Φ and $\operatorname{co}\Phi$ with respect to $\hat{\mu}$ are therefore the set of integrals of the m.e. selections,

$$\int \Phi\,\mathrm{d}\hat{\mu} = \left\{ \int \psi\,\mathrm{d}\hat{\mu}\colon\ \psi \text{ is an m.e. selection from } \Phi \right\}, \tag{22}$$

$$\int \operatorname{co}\Phi\,\mathrm{d}\hat{\mu} = \left\{ \int \psi\,\mathrm{d}\hat{\mu}\colon\ \psi \text{ is an m.e. selection from } \operatorname{co}\Phi \right\}. \tag{23}$$

Theorem 4.2. *$E_x^\mu f = \int \Phi\,\mathrm{d}\,\hat{\mu}$ and $E_c^\mu f = \int \operatorname{co}\Phi\,\mathrm{d}\hat{\mu}$ for all $f \in B^k$ and all $\mu \in \mathbb{P}(\mathcal{X})$.*

The value Φ at a point in $x \in \widehat{X} \backslash X$ is the set of accumulation points of $f(x^\alpha)$ as x^α converges to x in X. Examples 2.4 (above) and 4.2 (below) show that the set $E_x^\mu f$ can be quite large even when μ is $\{0,1\}$-valued and $\mathcal{X}$ separates points.

4.3. Star-finite characterizations

If $\mathcal{X} = \{\emptyset, X\}$ is the trivial field, then for the unique $\mu \in \mathbb{P}(\mathcal{X})$ and for any bounded f, $E_x^\mu f$ is $\operatorname{cl} f(X)$, the closure of the range of f. In particular, $E_x^\mu f$ can be any compact set so that $E_c^\mu f$ can be any compact convex set. It follows that if $\mathcal{X}$ is finite, then $E_x^\mu f = \sum_E \operatorname{cl} f(E)\mu(E)$ where the sum is over E in the partition of X generated by $\mathcal{X}$. For finite $\mathcal{X}$, this yields:

(1) $E_c^\mu f$ is a singleton set for all μ if and only if f is $\mathcal{X}$-measurable, and
(2) for $w \in \mathbb{R}^k$, $\sup w \cdot E_x^\mu f = \sup E_x^\mu (w \cdot f) = \sum_E \sup_{x \in E}(w \cdot f(x)) \cdot \mu(E)$ where the sum is over the E in the partition generated by $\mathcal{X}$,

The aim is to extend these and other comparably simple, finite analyses to the general case using *-finite representations of $\mathcal{X}$ and 2^X.

4.3.1. *The nonstandard setting*

As noted above, we work in a κ-saturated, nonstandard enlargement of a superstructure $V(Z)$ where Z contains X and $\mathbb{R}$, and κ is a cardinal greater than the cardinality of 2^Z. κ-saturation implies that any of the compactifications of X used above can be taken to be a collection of equivalence of $^*X$ (Hurd and Loeb, 1985, Section III.7, or Machover and Hirschfeld, 1969, Section 9.4). For nearstandard $r \in {}^*\mathbb{R}^k$, $^\circ r \in \mathbb{R}^k$ denotes the standard part of r.

For any $Y \in V(Z)$, $\mathcal{P}_Y$ denotes the finite subsets of Y, and $^*\mathcal{P}_Y$ is the collection of *-finite (read "star finite") subsets of $^*Y$. A *-finite Y_F is *exhaustive for Y* if, for all $y \in Y$, $^*y \in Y_F$ (identifying y and $^*y$). κ-saturation implies that there are exhaustive, *-finite versions of any $Y \in V(Z)$.

4.3.2. *The field-based nonstandard characterizations*

Let $\mathbb{F}(\mathcal{X})$ denote the finite sub-fields of $\mathcal{X}$, and pick an $\mathcal{X}' \in^* \mathcal{P}_{\mathbb{F}(\mathcal{X})}$ that is exhaustive for $\mathbb{F}(\mathcal{X})$, and let $\mathcal{X}_F$ be the *-field generated by $\mathcal{X}'$. This guarantees that if $E \in \mathcal{X}$, then $^*E \in \mathcal{X}_F$. In a similar fashion, pick a $\mathcal{Y}_F$ generated by a $\mathcal{Y}'$ that is exhaustive for $\mathbb{F}(2^X)$. Note that $\mathcal{X}' \cap \mathcal{Y}'$ belongs to $^*\mathcal{P}_{\mathbb{F}(\mathcal{X})}$ and is exhaustive for $\mathbb{F}(\mathcal{X})$.Therefore, there is no loss in assuming, as is done here, that $\mathcal{X}_F \subset \mathcal{Y}_F$.

The internal set of extensions of $^*\mu$ from $\mathcal{X}_F$ to $\mathcal{Y}_F$ is

$$H(^*\mu) = \{\nu \in {}^*\mathbb{P}(\mathcal{Y}_F)\colon (\forall E \in \mathcal{X}_F)[\nu(E) =^* \mu(E)]\}. \tag{24}$$

For $f \in B^k$, $^*f$ varies by at most an infinitesimal over any D in the $\mathcal{Y}_F$-partition. Therefore, for any $\nu \in {}^*\mathbb{P}(\mathcal{Y}_F)$, for any collections x_D, x'_D of points in D, D in the $\mathcal{Y}_F$-partition, $\sum_D {}^*f(x_D)\nu(D) \simeq \sum_D {}^*f(x'_D)\nu(D)$. This implies that the following integral is well defined.

Definition 4.2. The *M-integral* of $f \in B^k$ against $\nu \in \mathbb{P}(\mathcal{Y}_F)$ is defined by

$${}^M\!\!\int f \,\mathrm{d}\nu = {}^\circ\textstyle\sum_D {}^*f(x_D)\nu(D) \tag{25}$$

where the *-sum is taken over D in the $\mathcal{Y}_F$-partition, and for each D, $x_D \in D$.

The nonstandard definitions of the extremal and the convex integral are

$$D_c^\mu f = \left\{{}^M\!\!\int f \,\mathrm{d}\nu\colon \nu \in H(^*\mu)\right\}, \quad \text{and} \tag{26}$$

$$D_x^\mu f = \left\{{}^M\!\!\int f \,\mathrm{d}\nu\colon \nu \text{ is an } ^*\text{-extreme point of } H(^*\mu)\right\}. \tag{27}$$

For any D in the $\mathcal{Y}_F$-partition of ${}^*X$, let $\delta_D \in {}^*\mathbb{P}(\mathcal{Y}_F)$ be the probability assigning mass 1 to all $A \in \mathcal{Y}_F$ containing D and mass 0 to all other sets. Transfer of the corresponding statement for finite partitions proves the following.

Lemma 4.1. *ν is an *extreme point of $H({}^*\mu)$ if and only if it is of the form $\nu = \sum_E \delta_{D_E}{}^*\mu(E)$, where the sum is over E in the $\mathcal{X}_F$-partition and $D_E \subset E$ is an element of the $\mathcal{Y}_F$-partition.*

There are two convenient reformulations of Lemma 4.1:

(1) ν is an *-extreme point of $H({}^*\mu)$ if and only if for all E in the $\mathcal{X}_F$-partition and $A \in \mathcal{Y}_F$, $\nu(A|E)$ is equal either to 0 or to 1 (as $D_E \not\subset, \subset A$).
(2) ν is an *-extreme point of $H({}^*\mu)$ if and only if for all $A \in \mathcal{Y}_F$, ${}^*\min\{\nu(A\Delta E)\colon E \in \mathcal{X}_F\} = 0$. This is a finitistic version of (Lipecki et al., 1979, Theorem 3).

Integrating selections also works in the nonstandard context.

Theorem 4.3. *For $f \in B^k$ and $\mu \in \mathbb{P}(\mathcal{X})$, $r \in D_x^\mu f$ (respectively $r \in D_c^\mu f$) if and only if $r = {}^\circ\int g\,\mathrm{d}{}^*\mu$ for some $\mathcal{X}_F$-measurable g with the property that for all $x \in E$, E in the $\mathcal{X}_F$-partition of ${}^*X$, $g(x)$ belongs to the set ${}^*f(E)$ (respectively ${}^*\mathrm{co}\, f(E)$).*

The next result shows that standard and the nonstandard integrals are equivalent. This crucial result also shows that the nonstandard definitions are not dependent on the choice of $\mathcal{X}_F$ or $\mathcal{Y}_F$ because the definitions of $E_c^\mu f$ and $E_x^\mu f$ make no reference to nonstandard constructions.

Theorem 4.4. *For all $f \in B^k$ and all $\mu \in \mathbb{P}(\mathcal{X})$, $E_c^\mu f = D_c^\mu f$ and $E_x^\mu f = D_x^\mu f$.*

4.3.3. The finitistic nonstandard characterizations

The finitistic approach replaces standard infinite sets with exhaustive *-finite versions of the same set. The definition of the M-integral does the same.

Pick a point $x_D \in^* X \cap D$ for each D in the $\mathcal{Y}_F$-partition, and let X_F be the set of x_D. By replacing the δ_D with point masses on x_D, we can replace $({}^*X, \mathcal{Y}_F)$ with $(X_F, \mathcal{T})$ where $\mathcal{T}$ is the collection of internal subsets of X_F. Because $\mathcal{Y}_F$ is exhaustive for 2^X, it contains $\{x\}$ for every $x \in X$, implying that X_F is exhaustive for X. Therefore, replacing X by an exhaustive, *-finite version of itself and considering the *-finitely supported measures in $H({}^*\mu)$ leads directly to $E_c^\mu f$ and $E_x^\mu f$.

The next lemma shows that there may be many *-finitely supported ν in $H({}^*\mu)$, and that the exhaustiveness of X_F is only needed when μ has atoms. Recall that a finitely additive $\mu \in \mathbb{P}(\mathcal{X})$ is *non-atomic* if for all $E \in \mathcal{X}$ and all $a \in [0,1]$, there exists an $E_a \in \mathcal{X}$ such that $\mu(E_a) = a\mu(E)$.

Lemma 4.2. *If μ is non-atomic and X_F is a *-finite subset of X, then there exists a *-finitely supported $\nu \in H({}^*\mu)$ such that $\nu(X_F) = 0$.*

4.4. Basic properties of E_c^μ and E_x^μ

There are six basic properties of the convex and extremal integrals:

(1) They agree with usual integral for all μ if and only if $f \in B^k(\mathcal{X})$, Theorem 4.5.
(2) For non-measurable, $\mathbb{R}^1$-valued f, $E_c^\mu f$ is the convex interval between the upper and lower integrals, Theorem 4.6. (Recall that $E_x^\mu f$ can be an arbitrary compact set.)
(3) Neither $E_c^\mu f$ nor $E_x^\mu f$ satisfy any Riesz-like representation theorem, Example 4.1, though the integrals define continuous sub-linear mappings on $B^k(\mathcal{X})$, Theorem 4.7.
(4) It is sometimes possible to recover μ and $\mathcal{X}$ from the mapping $f \mapsto E_c^\mu f$. Theorem 4.8 shows that if μ is countably additive and $\mathcal{X}$ is a sigma-field, then μ and $\mathcal{X}$ are determined, up to null sets, by the values of $E_c^\mu f$ on the set $\mathcal{S}$ of those functions in $B^k(\mathcal{X})$ satisfying $\#E_c^\mu f = 1$.
(5) $E_x^\mu f$ is convex if μ is non-atomic, Theorem 4.9, and if $\mathcal{X} \subsetneq 2^X$, then there is a $\mu \in \mathbb{P}(\mathcal{X})$ and an $f \in B^1$ such that $E_x^\mu f$ contains exactly 2 points, Theorem 4.10. Finally,
(6) Fubini theorems fail rather completely, Example 4.2.

4.4.1. Comparison with the usual integral

The first result is

Theorem 4.5. *A function $f \in B^k$ belongs to $B^k(\mathcal{X})$ if and only if for all $\mu \in \mathbb{P}(\mathcal{X})$, $E_c^\mu f$ is a singleton set.*

4.4.2. Comparison with the upper and lower integrals

Non-measurable $\mathbb{R}^1$-valued functions are often bracketed above and below by an upper and a lower integral. Specifically, the *upper integral of $f \in B^1$ with respect to μ* is

$$I^+(f,\mu) = \inf\left\{\int h\,\mathrm{d}\mu\colon\ h \in B^1(\mathcal{X}),\ \text{and}\ (\forall x \in X)\big[f(x) \leqslant h(x)\big]\right\}, \tag{28}$$

while the *lower integral of $f \in B^1(2^X)$ with respect to μ* is

$$I_-(f,\mu) = \sup\left\{\int g\,\mathrm{d}\mu\colon\ g \in B^1(\mathcal{X}),\ \text{and}\ (\forall x \in X)\big[g(x) \leqslant f(x)\big]\right\}. \tag{29}$$

Theorem 4.6. *For $f \in B^1$, $E_c^\mu f$ is the interval $[I_-(f,\mu), I^+(f,\mu)]$.*

For many μ, there are functions $f \in B^1$ that are not in $B^1(\mathcal{X})$ and yet have $I_-(f,\mu) = I^+(f,\mu)$, e.g. when $\mathcal{X}$ is a σ-field, μ is countably additive, and f fails to be in $B^1(\mathcal{X})$ by a non-measurable μ-null set. This does not contradict Theorem 4.5 which concerns all possible μ.

The following is an immediate consequence of Theorem 4.6.

Corollary 4.6.1. *For $f \in B^k$,*

$$E_c^\mu f = \bigcap \big\{y \in \mathbb{R}^k\colon\ \big(\forall w \in \mathbb{R}^k\big)\big[I_-(w\cdot f,\mu) \leqslant w\cdot y \leqslant I^+(w\cdot f,\mu)\big]\big\}. \tag{30}$$

4.4.3. The failure of representation theorems

The most basic representation theorem for measures says that sup-norm continuous, linear functionals on $B^1(\mathcal{X})$ and integration against finitely additive measures on $\mathcal{X}$ can be identified. Compare

Example 4.1. Let $X = \{a, b\}$, $\mathcal{X} = \{\emptyset, X\}$ so that $\mathbb{P}(\mathcal{X})$ contains only one point, $f(x) = 1_{\{a\}}(x)$, $g(x) = 1 - f(x)$. Then $E_c^\mu f = E_c^\mu g = [0, 1]$ while $E_c^\mu(f + g) = \{1\}$.

The extremal (resp. convex) integral is sup-norm continuous, linear along rays from the origin, sublinear mapping to the class of compact (resp. compact convex) sets.

Theorem 4.7. *The mappings $f \mapsto E_x^\mu f$ and $f \mapsto E_c^\mu f$ from B^k to the class of compact subsets of $\mathbb{R}^k$ are continuous and satisfy $E_x^\mu \lambda f = \lambda E_x^\mu f$ and $E_c^\mu \lambda f = \lambda E_c^\mu f$ for any $\lambda \in \mathbb{R}$. The mappings are sublinear in the sense that for any $w \in \mathbb{R}^k$, the functional $L^w(f) := \max\{w \cdot y : y \in E_x^\mu f\}$ satisfies $L^w(f + g) \leqslant L^w(f) + L^w(g)$, and the functional $L_w(f) := \min\{w \cdot y : y \in E_x^\mu f\}$ satisfies $L_w(f + g) \geqslant L_w(f) + L_w(g)$.*

4.4.4. Recovering μ and $\mathcal{X}$

It is sometimes possible to recover μ and $\mathcal{X}$ from the mapping $f \mapsto E_c^\mu f$. Let $\mathcal{S}_\mu \subset B^1(2^X)$ denote the set of f such that $\#E_c^\mu f = 1$. For a σ-field $\mathcal{X}$ and a countably additive μ, let $\mathcal{X}^\mu$ denote the μ-completion of $\mathcal{X}$, here regarded as the domain of μ, and let $\mathcal{X}(\mathcal{S}_\mu)$ denote the minimal σ-field making all $f \in \mathcal{S}_\mu$ measurable. The omitted proof of the following is a routine application of the monotone class theorem.

Theorem 4.8. *If $\mathcal{X}$ is a σ-field and μ is countably additive, then $\mathcal{X}^\mu = \mathcal{X}(\mathcal{S}_\mu)$, and for all $A \in \mathcal{X}^\mu$, $E_c^\mu 1_A = E_x^\mu 1_A = \{\mu(A)\}$.*

In the presence of a countably additive μ on a σ-field, a.e. pointwise limits can be taken, extending the previous result to measurable functions.

4.4.5. When $E_x^\mu f$ is and is not convex

A $\mu \in \mathbb{P}(\mathcal{X})$ is *non-atomic on* $\mathcal{X}$ if for all $\epsilon > 0$, there is an $\mathcal{X}$-measurable partition $E_1, \ldots, E_n$ of X such that $\mu(E_i) < \epsilon$, $i = 1, \ldots, n$. The following result should be interpreted with two observations in mind:

(1) for some fields $\mathcal{X}$, e.g. the smallest field containing all finite subsets of an infinite set, there are no non-atomic probabilities in $\mathbb{P}(\mathcal{X})$; and
(2) in general, $E_x^\mu f$ may be an arbitrary compact set.

Theorem 4.9. *For non-atomic μ and all $f \in B^k(2^X)$, $E_x^\mu f = E_c^\mu f$.*

If $\mathcal{X} = 2^X$, then $E_x^\mu f$ is a (convex) singleton set for any μ.

Theorem 4.10. *If $\mathcal{X} \subsetneq 2^X$, then there is a $\mu \in \mathbb{P}(\mathcal{X})$ and an $f \in B^1(2^X)$ such that $E_x^\mu f$ contains exactly two points.*

4.4.6. The failure of Fubini theorems

When μ is a product measure, iterated integration of non-measurable functions does not lead to the integral, and performing iterated integration in different orders may lead to disjoint sets.

Example 4.2. First, the probability space: Let $X = S_1 \times S_2$, $S_1 = S_2 = \mathbb{N}$, $\mathcal{X}_1 = \mathcal{X}_2 = 2^{\mathbb{N}}$, $\mathcal{X}$ the smallest field containing sets of the form $E_1 \times E_2$, $E_1 \in \mathcal{X}_1$, $E_2 \in \mathcal{X}_2$, let μ_1 and μ_2 be purely finitely additive probabilities on $\mathcal{X}_1$ and $\mathcal{X}_2$ assigning mass 0 to all finite sets, finally, let $\mu = \mu_1 \times \mu_2$ be the product measure on $\mathcal{X}$. Second, the function: let $R = \{r_0\} \cup \{r_n\colon n \in \mathbb{N}\}$, $r_0 \neq 0$, be an otherwise arbitrary, bounded countable subset of $\mathbb{R}^k$, for each $n \in \mathbb{N}$, let T_n be the line of points starting at $(n, 2n)$ with slope 1, $T_n = \{(n+m, 2n+m)\colon m \in \mathbb{N}\}$, let T_0 be the part of the complement of $\bigcup_{n \geqslant 1} T_n$ that is above the diagonal, and set $f(x_1, x_2) = r_0 1_{T_0}(x_1, x_2) + \sum_{n\in\mathbb{N}} r_n 1_{T_n}(x_1, x_2)$. For each x_1, $f(x_1, \cdot)$ is measurable, for each x_2, $f(\cdot, x_2)$ is measurable,

$$(\forall x_1 \in S_1)\left[E_c^{\mu} f(x_1, \cdot) = \left\{\int f(x_1, s_2)\, \mathrm{d}\mu_2(s_2)\right\} = \{r_0\}\right], \quad \text{and} \tag{31}$$

$$(\forall x_2 \in S_2)\left[E_c^{\mu} f(\cdot, x_2) = \left\{\int f(s_1, x_2)\, \mathrm{d}\mu_1(s_1)\right\} = \{0\}\right]. \tag{32}$$

From (31) and (32), depending on the order of integration one gets the disjoint sets $\{r_0\}$ or $\{0\}$. By contrast, if μ_1 and μ_2 are both $\{0,1\}$-valued, then $E_x^{\mu} f$ is the closure of $R \cup \{0\}$.

Shrinking $\mathcal{X}_1$ and $\mathcal{X}_2$ in Example 4.2 gives more detail about iterated integration. Suppose that g is an arbitrary, bounded function on $X_1 \times X_2$ in Example 4.2, but that $\mathcal{X}_1$ and $\mathcal{X}_2$ are replaced by $\mathcal{F}_1$ and $\mathcal{F}_2$, the smallest fields containing the finite sets. Let $A^g_{x_1}$ denote the accumulation points of $g(x_1, \mathbb{N})$, A^g the accumulation points of $\{A^g_{x_1}\colon x_1 \in X_1\}$, $B^g_{x_2}$ the accumulation points of $g(\mathbb{N}, x_2)$, and B^g the accumulation points of $\{B^g_{x_2}\colon x_2 \in X_2\}$. Integrating selections from $x_1 \mapsto A^g_{x_1}$ against the unique $\{0,1\}$-valued purely finitely additive $\mu_1 \in \mathbb{P}(\mathcal{F}_1)$ gives A^g, integrating selections from $x_2 \mapsto B^g_{x_2}$ against the unique $\{0,1\}$-valued purely finitely additive $\mu_2 \in \mathbb{P}(\mathcal{F}_2)$ gives B^g. Iterated integration of g against $\mu_1 \times \mu_2$ gives A^g or B^g depending on the order of integration. By contrast, with $\mu = \mu_1 \times \mu_2$, $E_x^{\mu} g$ is the set of accumulation points of $\{g(x_1(m), x_2(m))\colon m \in \mathbb{N}\}$ as $x_1(m) \to \infty$ and $x_2(m) \to \infty$ so that $A^g, B^g \subset E_x^{\mu} g$. For the function f in Example 4.2, $A^f \neq B^f$, and both are one point subsets of $E_x^{\mu} f$.

5. Integrals on product spaces and finitely additive equilibria

For this section, the space X is replaced by $T := \times_{i\in I} T_i$ where I is the finite set (of players). The utility function to be integrated is bounded and takes values in $\mathbb{R}^k$, $k = I$. Each T_i comes with a field $\mathcal{T}_i$ of subsets. The field $\mathcal{X}$ is replaced by the field $\mathcal{T} = \times_{i\in I} \mathcal{T}_i$ generated by the measurable rectangles. Because Nash equilibria involve independent randomization, the relevant set of probabilities is $\mathbb{P}_{\text{prod}} \subset \mathbb{P}(\mathcal{T})$,

$$\mathbb{P}_{\text{prod}} = \left\{\mu \in \mathbb{P}(\mathcal{T})\colon \mu = \text{prod}\big((\mu_i)_{i\in I}\big),\ \mu_i \in \mathbb{P}(\mathcal{T}_i)\right\}. \tag{33}$$

The $\mathcal{T}_i$ are assumed to contain all singleton sets, and $B^1(\mathcal{T}_i)$ is assumed to contain the utility sections.[13]

S_i denotes $\widehat{T_i} \mid B^1(\mathcal{T}_i)$, and S denote $\times_{i\in I} S_i$. It can be shown that $S = \widehat{T} \mid B^1(\mathcal{T})$, and that the extension, $\hat{\mu}$, of any $\mu = \mathrm{prod}((\mu_i)_{i\in I})$ in $\mathbb{P}_{\mathrm{prod}}$ to S is of the form $\mathrm{prod}((\hat{\mu}_i)_{i\in I})$ where each $\hat{\mu}_i$ is the unique, countably additive, Borel extension of μ_i to S_i. As the T_i are dense in the S_i, this recovers the setting of selection equilibria for compact games of Section 3.

If the $\mathcal{T}_i$ are not part of the specification of the game, then $\mathcal{T}_i = 2^{T_i}$ can be used. If $\mathcal{T}_i = 2^{T_i}$, the spaces S_i are quite large and there is no measure extension of the individual μ_i being done to determine payoffs. Rather, it is only extension of $\mathrm{prod}(\mu)$ that is being done. If $\mathcal{T}_i \subsetneq 2^{T_i}$, then measure of extension of both the individual μ_i and $\mathrm{prod}(\mu)$ determines payoffs.

5.1. Integrals w.r.t. product measures

The two integrals considered here are the Nash and the product integral.

Definition 5.1. The *Nash integral of* $f \in B^k(2^T)$ *with respect to* $\mu = \mathrm{prod}((\mu_i)_{i\in I})$ is $E_N^{\mu} f = \int_S \mathrm{Nh}\,\Phi_f \,\mathrm{d}\hat{\mu}$.

For any $\mu_i \in \mathbb{P}(\mathcal{T}_i)$, $m_i(\mu_i)$ denotes the weak* monad of μ_i, that is, the set of probabilities $\nu_i \in {}^*\mathbb{P}(\mathcal{T}_i)$ such that for all $g \in B^k(\mathcal{T}_i)$, $|\int g\,\mathrm{d}\mu_i - {}^*\!\int {}^*g\,\mathrm{d}\nu_i| \simeq 0$. Because the finitely supported measures are weak*-dense in $\mathbb{P}(\mathcal{T}_i)$, $m_i(\mu_i)$ contains *-finitely supported measures.

Definition 5.2. The *product integral of* $f \in B^k(2^T)$ *with respect to* $\mu = \mathrm{prod}((\mu_i)_{i\in I})$ is

$$E_p^{\mu} f = \mathrm{st}\left\{ \int\limits_{^*T} {}^*f(t)\,\mathrm{d}\,\mathrm{prod}((\eta_i)_i \in I)\colon\ \eta_i \in m_i(\mu_i),\ \eta_i\ ^*\text{-finitely supported} \right\}. \tag{34}$$

Theorem 5.1. *For all* $f \in B^k$ *and all* $\mu \in \mathbb{P}_{\mathrm{prod}}$, $E_p^{\mu} f \subset E_N^{\mu} f$, *and both integrals are non-empty and compact.*

Proof of Theorem 5.1. Fix arbitrary $f \in B^k$ and $\mu \in \mathbb{P}_{\mathrm{prod}}$. Because $E_N^{\mu} f$ is the integral of a non-empty valued correspondence with a compact graph, it is non-empty and compact.

By definition, all $r \in E_p^{\mu} f$ are the standard part of $\int_{^*T} {}^*f(t)\,\mathrm{d}\,\mathrm{prod}((\eta_i)_i \in I)$ for some vector of *-finitely supported $\eta_i \in m_i(\mu_i)$. For each $i \in I$, let A_i be a finitistic version of T_i supporting η_i. Pick $a_i^{\circ} \in A_i$ such that $\eta_i(a_i) \in {}^*\mathbb{R}_{++}$. Modify η_i by taking an infinitesimal proportion of $\eta_i(a_i^{\circ})$ and dividing it evenly over A_i so that for each $a_i \in A_i$, $\eta_i(a_i) \in {}^*\mathbb{R}_{++}$. After modification, $\int {}^*f\,\mathrm{d}\eta$ moves by at most an infinitesimal. Using such a vector of strictly positive $(\eta_i)_i \in I$, Claim **A** (in the middle of the proof of Theorem 3.3) shows that $r = \int \psi\,\mathrm{d}\hat{\mu}$ for some everywhere selection ψ from $\mathrm{Nh}\,\Phi$. This implies that $r \in \int \mathrm{Nh}\,\Phi\,\mathrm{d}\mu$.

[13] These assumptions guarantee that play of points $t_i \in T_i$ are integrable and that each $i \in I$ has a finitely additive best response to every pure strategy profile $t \in T$.

To show that $E_p^\mu f$ is non-empty and compact, for any function $g_i \in B^1(\mathcal{T}_i)$ and $\epsilon_i \in \mathbb{R}_{++}$, let $N_i(\mu_i\colon g_i, \epsilon_i)$ be the internal set of *-finitely supported η_i such that $|\int g_i \,\mathrm{d}\mu_i - \int {}^*g_i \,\mathrm{d}\eta_i| < \epsilon_i$. The standard part of the internal set $\{\int {}^* f \,\mathrm{d}\eta_i\colon \eta_i \in N_i(\mu_i\colon g_i, \epsilon_i)\}$ is closed, hence compact. The collection of such sets has the finite intersection property. $E_p^\mu f$ is the non-empty, compact intersection of this collection compact sets. □

The inclusion in Theorem 5.1 can be strict. In a two player game, consider the Nash hull at two points, $\mathrm{Nh}\,\Phi(s_1, s_2)$ and $\mathrm{Nh}\,\Phi(s_1, t_2)$, $s_2 \neq t_2$. By Lemma 3.2, these two sets are formed by having independent randomization on *-finite subsets of the product monads $m_1(s_1) \times m_2(s_2)$ and $m_1(s_1) \times m_2(t_2)$. If independence is to be respected, then the play of 1 should be the same in both of these product monads, it should not depend on 2's choices. Selections have no such restrictions.

Example 5.1. $I = \{1, 2\}$, $T_1 = T_2 = \mathbb{N}$, $\mathcal{T}_1$ is the smallest field containing the singleton sets, and $\mathcal{T}_2$ is the smallest field containing the singleton sets and the set of even numbers. μ_1 is the unique finitely additive probability assigning mass 0 to all finite sets, μ_2 is the unique finitely additive probability assigning mass $1/2$ to the evens and mass 0 to all finite sets. To be integrated against $\mu = \mathrm{prod}(\mu_1, \mu_2)$ is

$$f(t_1, t_2) = \begin{cases} (0, 0) & \text{if } t_1 \neq t_2, \\ (+1, (-1)^{t_1}) & \text{if } t_1 = t_2. \end{cases} \tag{35}$$

For all $t' \in T$, the mappings $t_i \mapsto f(t'\backslash t_i)$ are non-zero at only one point, hence belong to $B^1(\mathcal{T}_i)$.

Claim B: $(+1, 0) \in E_x^\mu f$, and

Claim C: $(+1, 0) \notin E_p^\mu f$.

Proof of Claim B: First, note that $S_1 = T_1 \cup \{\infty\}$ is the classical one-point compactification of T_1, while S_2 is $T_2 \cup \{\infty_{\mathrm{even}}, \infty_{\mathrm{odd}}\}$, with ∞_{even} being the limit of all even sequences going to infinity and ∞_{odd} is the limit of all odd sequences going to infinity. Second, $\mathrm{Nh}\,\Phi_f(\infty, \infty_{\mathrm{even}})$ is the line segment joining $(0, 0)$ to $(+1, +1)$ and $\Phi_f(\infty, \infty_{\mathrm{even}})$ contains the point $(+1, +1)$. In a similar fashion, $\mathrm{Nh}\,\Phi_f(\infty, \infty_{\mathrm{odd}})$ is the line segment joining $(0, 0)$ to $(+1, -1)$, and $\Phi_f(\infty, \infty_{\mathrm{odd}})$ contains the point $(+1, -1)$. Third, $\hat{\mu}_1$ is δ_∞, point mass on ∞, while $\hat{\mu}_2$ is $1/2\delta_{\infty_{\mathrm{even}}} + 1/2\delta_{\infty_{\mathrm{odd}}}$. Because it is equal to $1/2(+1, +1) + 1/2(+1, -1)$, the point $(+1, 0)$ belongs to $\int_S \Phi_f \,\mathrm{d}\,\mathrm{prod}(\hat{\mu}_1, \hat{\mu}_2)$.

Proof of Claim C: First, note that a *-finitely supported η_1 belongs to $m_1(\mu_1)$ if and only if $\eta_1(A) \simeq 0$ for all finite $A \subset \mathbb{N}$, while a *-finitely supported η_2 belongs to $m_2(\mu_2)$ if and only if $\eta_2(A) \simeq 0$ for all finite $A \subset \mathbb{N}$ and $\eta_2({}^*Evens) \simeq 1/2$ where $Evens$ is the set of even numbers. Second, for a point of the form $(+1, r)$ to belong to $E_p^\mu f$, it must be the case that $\mathrm{prod}(\eta_1, \eta_2)({}^*D) \simeq 1$ where D is the diagonal of $T_1 \times T_2$. Third, the only way that $\mathrm{prod}(\eta_1, \eta_2)({}^*D) \simeq 1$ can happen is if both η_1 and η_2 put mass infinitesimally close to 1 on the same point, call it t. If t belongs to ${}^*Evens$, then $\int {}^* f \,\mathrm{d}\,\mathrm{prod}(\eta_1, \eta_2) \simeq (+1, +1)$, otherwise it is equal to $(+1, -1)$.

Comment: Example 5.1 shows that even selections from Φ_f involve spurious correlation. To reach the utilities $(+1,+1)$, both players must be playing evens, to reach the utilities $(+1,-1)$, both players must be playing odds. When player 2 randomizes $(1/2; 1/2)$ over evens and odds, and player 1 is playing independently of player 2, then it is impossible to come anyplace close to the point $(+1,0) = 1/2(+1,+1) + 1/2(+1,-1)$. Integrating Φ_f allows choice of any selection. The selection that picks $(+1,+1)$ and $(+1,-1)$ to integrate against player of ∞ by 1 is allowing 1's choice of even or odd to depend on 2's choice.

It is easy to make a game out of Example 5.1 in which the multigame $\Gamma_T(S,\Phi)$ has equilibria involving too much correlation. Whether or not $\Gamma_T(S,\Phi)$ always has equilibria is an open question.[14]

Because $\Phi_f \subset \mathrm{Nh}\,\Phi_f \subset \mathrm{co}\,\Phi_f$, $E_x^\mu f \subset E_N^\mu f \subset E_c^\mu f$. In general, $E_p^\mu f$ does not fit into this scheme. When f has two (or more) discontinuity points on S, the logic of Example 5.1 shows that $E_x^\mu f$ need not be a subset of $E_p^\mu f$. By contrast,

Corollary 5.1.1. *If f has one discontinuity point in S, then for all $\mu \in \mathbb{P}_{\mathrm{prod}}$, $E_p^\mu f = E_N^\mu f$.*

Proof. Fix arbitrary $\mu \in \mathbb{P}_{\mathrm{prod}}$ and $r \in E_N^\mu f$. By Theorem 5.1, it is sufficient to show that $r \in E_p^\mu f$. Let s° denote the discontinuity point of f on S. If $\hat{\mu}(s^\circ) = 0$, then the a.e. continuity of f delivers equality of all of the four integrals. The remaining case has each $\delta_i := \hat{\mu}_i(s_i^\circ) > 0$.

Pick arbitrary star finitely supported $\eta_i' \in m_i(\mu_i)$ so that the Loeb measure $L(\eta_i')$ satisfies $L(\eta_i')(m_i(s_i^\circ)) = \delta_i$. The proof will be complete if there is a star finitely supported modification, η_i, of each η_i' such that $r \simeq \int {}^*f \,\mathrm{d}\,\mathrm{prod}((\eta_i)_{i\in I})$. Lemma 3.2 expresses $\Phi_f(s^\circ)$ as the standard part of the integral of ${}^*f$ against a product measure, $\mathrm{prod}((\nu_i)_{i\in I})$, concentrated on the monad of s°. Let E_i be an internal set such that $L(\eta_i')(E_i \Delta m_i(s_i^\circ)) = 0$. Defining $\eta_i(\cdot) = (1-\delta_i)\eta_i'(\cdot \mid E_i^c) + \delta_i \nu_i(\cdot)$ gives the requisite modification. □

Combined with Lemma 3.2, Corollary 5.1.1 shows the essential difference between the product integral and the Nash integral—the Nash integral fails to be a product integral because it allows the marginals of η_i to depend on $(s_j)_{j\neq i}$.

5.2. Finitely additive equilibria

Fix a game Γ with player set I, with each $i \in I$ picking an action in the set T_i, and with utility function $u : T \to \mathbb{R}^I$. Theorem 3.3 shows the existence of a selection equilibrium (ψ, μ^*) for the multigame $\Gamma_T(S, \mathrm{Nh}\,\Phi)$. The function ψ is a carefully chosen version of $E({}^{\circ*}u \mid \mathcal{F})$ where:

(1) $\mathcal{F}$ is the smallest σ-field of subsets of $A = \times_{i\in I} A_i$ making the product standard part mapping measurable, where each A_i is *-finite and exhaustive for T_i,

[14] My guess is no, this despite the observation that the existence of (so) many selections makes proving non-existence in any particular example quite difficult.

(2) expectation is taken with respect to the Loeb measure generated by $\mathrm{prod}(\eta)$,
(3) η is a *-full support ϵ-equilibrium, $\epsilon \simeq 0$.

By construction of the S_i, each μ_i^* can be identified with a unique finitely additive probability $\nu_i^* \in \mathbb{P}(\mathcal{T}_i)$. Hence, if the integral of u with respect to a vector $\nu \in \times_{i\in I}\Delta(\mathcal{T}_i)$ is understood as $\int_S \psi(s)\,\mathrm{d}\,\mathrm{prod}(\hat{\nu})(s)$, then Theorem 3.3 contains a general equilibrium existence result for arbitrary games with bounded payoffs and the Nash integral. The payoffs of the finitistic equilibria in Theorem 3.3 clearly belong to the product integral. Renormalizing the $^*\nu_{|\mathcal{F}_C}$ convex combinations of η delivers the next result, which shows that when players consider deviations, the payoffs to the deviation will also be product integrals.

Theorem 5.2. *For any $\nu \in \mathbb{P}_{\mathrm{prod}}$, $\int_S \psi(s)\,\mathrm{d}\,\mathrm{prod}(\hat{\nu})(s)$ belongs to $E_p^\nu u$.*

When $u \in B^I(\mathcal{T})$ so that $E_c^\mu u$ is a singleton set for all μ, the game is nearly compact and continuous. For such games, Harris et al. (in press) (the companion piece to this paper) and Marinacci (1997) contain a finitely additive equilibrium existence proof. Harris et al. (in press) shows that such games are equivalent to games with compact, metric space strategy sets and jointly continuous utility functions.

6. Summary

This paper develops a theory of equilibrium in normal form games with bounded utilities. The theory makes no topological or measure theoretic assumptions on the structure of the game. This is appropriate for a theory that is to be applied to the normal forms of infinite extensive form games.[15] Exhaustive star-finite sets provide a direct interpretation of the equilibrium strategies. Compactification and selection, or finitely additive probabilities and the theory of integration for non-measurable functions, provide indirect interpretations.

Exhaustive star-finite versions of a set X contain every x in X, but behave logically as if they were finite. There is a surjection from any exhaustive version of a set X to any compactification of X. For infinite normal form games, the surjection can lose payoff information. As often happen with lost information, it reappears as correlation between players' actions.

While selection of limit utilities can replace the information lost in the surjection, it has other, less desirable qualities. It can lose strategically important information, such as the existence of a dominant strategy, and it can encode spurious information. Selection equilibria provided a huge advance in the generality of the games covered by theory, and they provide interpretational tools for finitistic equilibria. However, their drawbacks mean that they should not, in general, be used as an independent solution concept.

Drawbacks aside, selection contains a set-valued theory of integration for non-measurable functions intimately tied to the integration of correspondences. Only a limited

[15] See Aumann (1964, 1961) for the measurability problems for infinite extensive form games.

form of correlation appears in finitistic equilibria. The limits on the forms of correlation reappear in the distinctions between the different integrals.

Finally, examples demonstrate that the study of infinite extensive form games will be a considerably more subtle undertaking than the present study of infinite normal form games.

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Appendix A. Proofs for Section 4

Proof of Theorem 4.1. The compactness and non-emptiness of $M(\mu)$, the Krein–Milman theorem and the continuity of integration in the weak* topology implies that $E_c^\mu f$ is non-empty and compact, and is equal to the convex hull of $E_x^\mu f$. The compactness of $E_x^\mu f$ will follow from Theorem 4.2. □

Proof of Theorem 4.2. Pick arbitrary $f = (f_i)_{i=1}^k \in B^k$ and $\mu \in \mathbb{P}(\mathcal{X})$. It is sufficient to prove that $E_x^\mu f = \int \Phi \, \mathrm{d}\hat{\mu}$ because the convex hull of $\int \Phi \, \mathrm{d}\hat{\mu}$ is equal to $\int \mathrm{co}\, \Phi \, \mathrm{d}\hat{\mu}$ and the convex hull of $E_x^\mu f$ is $E_c^\mu f$.

Set $\mathbb{G} = B^k(\mathcal{X})$, $\mathbb{G}' = \mathbb{G} \cup \{f_i\colon i = 1, \ldots, k\}$. The graph of Φ is $\widehat{X} \mid \mathbb{G}'$. $\widehat{M}(\hat{\mu})$, the set of (Radon) extensions of $\hat{\mu}$ from $\widehat{X} \mid \mathbb{G}$ to $\widehat{X} \mid \mathbb{G}'$, is the set of probabilities ν on $\widehat{X} \mid \mathbb{G}'$ having marginal equal to $\hat{\mu}$ on $\widehat{X} \mid \mathbb{G}$. Any measurable function φ from $\widehat{X} \mid \mathbb{G}$ to the Borel probabilities (with the Borel σ-field generated by the weak* topology) on $\times_{i=1}^k I_{f_i}$ with the property that $\varphi(x)(\Phi(x)) = 1$ for $\hat{\mu}$-almost all x gives rise to a probability $\nu \in \widehat{M}(\hat{\mu})$ defined by its values on measurable rectangles,

$$\nu_\varphi(A \times B) = \int_A \varphi(x)(B) \, \mathrm{d}\hat{\mu}(x). \tag{A.1}$$

Further, any $\nu \in \widehat{M}(\hat{\mu})$ gives rise to such a function φ_ν, e.g. (Dudley, 1989, Corollary 10.2.8). Identifying φs that vary only on sets of $\hat{\mu}$-measure 0, the mapping back and forth between νs and φs is linear, one-to-one, and onto. This means that the extreme points of $\widehat{M}(\hat{\mu})$ are the extreme points of the set of φs.

The extreme points of the set of probabilities on $\Phi(x)$ is the set of point masses on $\Phi(x)$. This implies that the extreme points of the set of φs are the functions that, on a set of x with $\hat{\mu}$-probability 1, have $\varphi(x)$ being point mass on a point $\psi(x) \in \Phi(x)$. For any extreme point ν of $\widehat{M}(\hat{\mu})$ and associated ψ such that $\varphi_\nu(x)$ is point mass on $\psi(x)$, $\int \hat{f} \, \mathrm{d}\nu = \int \psi(x) \, \mathrm{d}\hat{\mu}$. □

Comment: Because $\int \Phi \, \mathrm{d}\hat{\mu}$ is a closed set (Klein and Thompson, 1984, Theorem 18.3.2), $E_x^{\mu} f$ is compact, completing the proof of Theorem 4.1.

Comment: There is a more elementary (but longer) inductive construction of the function φ in the proof. It uses the characterization of extreme points of $\widehat{M}(\hat{\mu})$ given in (Lipecki et al., 1979, Theorem 3), which shows that ν is an extreme point of the extensions if and only if for all $A \in 2^X$, $\inf\{\nu(A \Delta E)\colon E \in \mathcal{X}\} = 0$. Taking f to be the indicator of a set A in the Theorem 4.2 and using properties of clopen sets in Stone spaces gives an alternate proof of this characterization of extreme points.

Proof of Theorem 4.3. The proof for $D_c^{\mu} f$ following directly from the proof for $D_x^{\mu} f$. Suppose that $r \in D_x^{\mathbb{F}} f$. By definition, there exists an *-extreme point ν of $H(^*\mu)$ and a collection of points $x_D \in D$, D the elements of the $\mathcal{Y}_F$-partition, such that $r =^{\circ} \sum_D^* f(x_D)\nu(D)$. By Lemma 4.1, for each E in the $\mathcal{X}_F$-partition, $\nu(D_E) > 0$ for at most one element $D_E \subset E$ of the $\mathcal{Y}_F$-partition. Therefore, changing the value of $^*f$ on other $D_E' \subset E$ does not change $\sum_D f(x_D)\nu(D)$. For $x \in E$, E in the $\mathcal{X}_F$-partition, define $g(x) =^* f(x_D)$ so that $g(x) \in^* f(E)$. Because $\nu \in H(^*\mu)$, $\int g \, \mathrm{d}^*\mu = \sum_D^* f(x_D)\nu(D)$.

Now pick an $\mathcal{X}_F$-measurable g such that for all $x \in E$, E in the $\mathcal{X}_F$-partition of $^*X$, $g(x) \in^* f(E)$. For each E, pick a $D_E \subset E$, D_E in the $\mathcal{Y}_F$-partition such that D_E contains a point x_E with $^*f(x_E) = g(x)$. This can be done because the $D_E \subset E$ partition E. By Lemma 4.1, the probability $\nu = \sum_E \delta_{D_E}{}^*\mu(E)$ is an extreme point of $H(^*\mu)$. By construction, $\int g \, \mathrm{d}^*\mu \simeq^M \int f \, \mathrm{d}\nu$. $\square$

Proof of Theorem 4.4. The equality of convex integrals follows directly from the equality of extremal integrals.

Pick an extreme point ν of $H(^*\mu)$, and define the probability $\gamma \in \mathbb{P}(2^X)$ as the weak* standard part of ν, that is, by $\gamma(A) =^{\circ} \nu(^*A)$ for $A \in 2^X$. Restricted to $\mathcal{X}$, $\gamma = \mu$ so that $\gamma \in M(\mu)$ (is an extension of μ). To show that $D_x^{\mu} f \subset E_x^{\mu} f$, it is sufficient to show that γ is an extreme point of $M(\mu)$.

Pick an arbitrary $A \in 2^X$. By (Lipecki et al., 1979, Theorem 3), it is sufficient to show that for all $\epsilon \in \mathbb{R}_{++}$, there exists an $E_\epsilon \in \mathcal{X}$ such that $\gamma(A \Delta E_\epsilon) < \epsilon$. Define the internal function $\psi : {}^*\mathbb{F}(\mathcal{X}) \to {}^*\mathbb{R}_+$ by $\psi(\mathcal{X}') = \min_{E \in \mathcal{X}'} \nu(^*A \Delta E)$. Since ψ achieves infinitesimal values on $^*\mathbb{F}$, Overspill implies that it achieves values less than any $\epsilon \in \mathbb{R}_{++}$ on the standard elements of $^*\mathbb{F}$. Since $A \in 2^X$, for any standard $\mathcal{X}'$ in $^*\mathbb{F}$, $\psi(\mathcal{X}') = \min_{E \in \mathcal{X}'} \gamma(A \Delta E)$, completing the first half of the proof.

From Theorem 4.2, $E_x^{\mu} f = \int \Phi \, \mathrm{d}\hat{\mu}$ where $\hat{\mu}$ is the countably additive extension of μ to $\widehat{X} \mid B^1(\mathcal{X})$ and the graph of Φ is the closure of the set $\{(x, f(x))\colon x \in X\}$ in $\widehat{X} \mid B^1(\mathcal{X}) \times \mathbb{R}^k$. Let $r = \int \psi \, \mathrm{d}\hat{\mu} \in \int \Phi \, \mathrm{d}\hat{\mu}$ where ψ is a $\widehat{\mathcal{X}}$-measurable, everywhere selection from Φ. By Lusin's theorem, for every $\epsilon \in \mathbb{R}_{++}$, there exists a continuous h_ϵ on the compact Hausdorff space $\widehat{X} \mid B^1(\mathcal{X})$ such that $\hat{\mu}(\{h_\epsilon = \psi\}) > 1 - \epsilon$. Since h_ϵ is continuous, it is of the form $\hat{h}$ for some $h \in B^k(\mathcal{X})$. Since $B^k(\mathcal{X})$ is the set of uniform limits of simple $\widehat{X}$-measurable functions, for every $\epsilon \in \mathbb{R}_{++}$, there is a simple, $\widehat{X}$-measurable g_ϵ such that $\hat{\mu}(|\hat{g}_\epsilon - \psi| > \epsilon) < \epsilon$. Because $\mathcal{X}_F$ is exhaustive, the Extension Principle implies that for some $\epsilon \simeq 0$, there is an $\mathcal{X}_F$-measurable, *-simple $g = g_\epsilon$ such that $^*\hat{\mu}(|^*\hat{g} -^* \psi| > \epsilon) < \epsilon$. Because ψ is an everywhere selection from Φ, for each x in each E in the $\mathcal{X}_F$-partition, ${}^*\psi(x) \in^* f(E)$. Therefore, except possibly for a set of E in the $\mathcal{X}_F$-partition having infinitesimal mass, each $g(E) \in {}^*\mathbb{R}^k$ is infinitesimally close to the set ${}^*f(E)$. By Theorem 4.3, ${}^\circ\int g\,\mathrm{d}^*\mu \in D_x^\mu f$. Since $\int g\,\mathrm{d}^*\mu \simeq \int^* \psi\,\mathrm{d}^*\hat{\mu} = \int \psi\,\mathrm{d}\hat{\mu} = r$, $r \in D_x^\mu f$, completing the proof. □

Proof of Lemma 4.2. For every standard finite X_F and every standard finite collection $E_1, \ldots, E_n$, of elements of $\mathcal{X}$, there is a standard ν that is finitely supported, satisfies $\nu(X_F) = 0$, and ν agrees with μ on $E_1, \ldots, E_n$. By saturation, this is sufficient.

Proof of Theorem 4.5. If $f \in B^k(\mathcal{X})$, then for all $\nu \in M(\mu)$, $\int f\,\mathrm{d}\nu = \int f\,\mathrm{d}\mu$, so that, by definition, $E_c^\mu f = E_x^\mu f = \{\int f\,\mathrm{d}\mu\}$.

Suppose now that $f \in B^k \backslash B^k(\mathcal{X})$. Pick $\mathcal{X}_F$ and $\mathcal{Y}_F$ as above. For any finite subfield $\mathcal{S} \in \mathbb{F}(\mathcal{X})$, define the oscillation of f over $\mathcal{S}$ by

$$osc(f, \mathcal{S}) = \sup\bigl\{\|f(s) - f(t)\|\colon\ s, t \in E,\ E \text{ in the } \mathcal{S}\text{-partition of } X\bigr\}. \tag{A.2}$$

Because $f \in B^k$, ${}^*osc({}^*f, \mathcal{Y}_F) \simeq 0$. Because $f \notin B^k(\mathcal{X})$, there exists an $\epsilon \in \mathbb{R}_{++}$ such that ${}^*osc({}^*f, \mathcal{X}_F) \geqslant 2\epsilon$. Pick an E in the $\mathcal{X}_F$-partition containing points x_1, x_2 such that ${}^\circ\|{}^*f(x_1) - {}^*f(x_2)\| > \epsilon$ and pick disjoint $D_1, D_2 \subset E$ in the $\mathcal{Y}_F$-partition containing x_1 and x_2. Define $\mu \in \mathbb{P}(\mathcal{X})$ by $\mu(A) = 0, 1$ as $E \not\subset, \subset {}^*A$. The two *-extreme points δ_{D_1} and δ_{D_2} of $H({}^*\mu)$ have integrals against ${}^*f$ that differ by more than ϵ, implying that $E_c^\mu f$ is not a singleton set. □

Proof of Theorem 4.6. Because $I_-(f, \mu) = -I^+(-f, \mu)$, it is enough to show that

$$I^+(f, \mu) = \sup\left\{{}^M\!\int f\,\mathrm{d}\nu\colon\ \nu \text{ is an }{}^*\text{extreme point of } H({}^*\mu)\right\}. \tag{A.3}$$

For each $x \in E$, E in the $\mathcal{X}_F$-partition, define $h(x) =^* \sup_{x \in E} f(x)$. Because $\mathcal{X}_F$ is exhaustive, $I^+(f, \mu) \simeq \int h\,\mathrm{d}^*\mu$. For each E, pick D_E in the $\mathcal{Y}_F$-partition such that $f(D_E) \simeq h(E)$. The probability $\nu = \sum_E \delta^*_{D_E}\mu(E)$ achieves the supremum on the right-hand side of (A.3), and ${}^M\int f\,\mathrm{d}\nu \simeq \int h\,\mathrm{d}^*\mu$.

Proof of Theorem 4.7. Suppose that $f^n \to f$ uniformly. For each $\nu \in \mathbb{P}(2^X)$, define the weak* continuous function $L^n(\nu)$ on by $\int f^n\,\mathrm{d}\nu$. The functions L^n converge pointwise to the continuous function $L(\nu) = \int f\,\mathrm{d}\,\nu$, so the convergence is uniform over the compact set $\mathbb{P}(2^X)$, proving the continuity statement. Directly from the definitions, $E_x^\mu \lambda f = \lambda E_x^\mu f$ and $E_c^\mu \lambda f = \lambda E_c^\mu f$.

Fix arbitrary $w \in \mathbb{R}^k$. Define $h_f \in {}^*B^k(\mathcal{X}_F)$ by $h_f(E) = {}^*\sup\{w \cdot {}^*f(x)\colon\ x \in E\}$ for E in the $\mathcal{X}_F$-partition, with parallel definitions for h_g and h_{f+g}. The standard part of ${}^*\int h_f\,\mathrm{d}^*\mu$ is equal to $L^w(f)$, and the same is true for g and $f+g$. For each E, $h_{f+g}(E) \leqslant h_f(E) + h_g(E)$ so that $L^w(f+g) \leqslant L^w(f) + L^w(g)$. Since $L^w(f) = -L_{-w}(f)$, this also proves the last statement of the theorem. □

Proof of Theorem 4.9. This follows from the *-finite Lyapunov theorem in Loeb (1973), or from the convexity of $\int F\,\mathrm{d}\hat{\mu}$ when $\hat{\mu}$ is non-atomic (Klein and Thompson, 1984, Proposition 18.1.1).

Proof of Theorem 4.10. Let A be a subset of X that is not in $\mathcal{X}$, and let $f(x) = 1_A(x)$. For any $\mathcal{S} \in \mathbb{F}$, the partition generated by $\mathcal{S}$ and A is strictly finer than the one generated by $\mathcal{S}$. By transfer, there exists an E in the $\mathcal{X}_F$-partition such that $E \cap^* A \neq \emptyset$ and $E \cap ({}^*X \backslash {}^*A) \neq \emptyset$. For $B \in \mathcal{X}$, define $\mu(B) = 0, 1$ as $E \not\subset, \subset {}^*B$. Because $\mathcal{Y}_F$ is exhaustive, it contains ${}^*A$, because it contains $\mathcal{X}_F$, it contains E. Therefore, the $\mathcal{Y}_F$-partition contains $D_E \subset E \cap^* A$ and $D'_E \subset E \cap ({}^*X \backslash {}^*A)$. Since ${}^*\mu(E) = 1$, the extreme points of $H({}^*\mu)$ are of the form δ_D for some $D \subset E$ in the $\mathcal{Y}_F$-partition. The integral of f against any such δ_D is either 0 or 1, the integral against δ_{D_E} is 1, the integral against $\delta_{D'_E}$ is 0.

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