Assignment #1 for **Mathematics for Economists II**
Spring 2010

Due date: Tue. Feb. 9.
Readings: CSZ (Corbae, Stinchcombe, Zeeman), Ch. 1, Ch. 2.10, 2.13, and Ch. 3.
Topics: statements as sets and vice versa; Russell's paradox and superstructures; infinite sets of different sizes; the real line, $\mathbb{R}$, as the completion of the rationals, $\mathbb{Q}$.

Problems

A. [With apologies to Lewis Carroll] We let X, the "universe of discourse," be the set of people. We define $B \subset X$ as the set of babies, $I \subset X$ as the set of illogical people, D the set of despised people, and M the set of people who can manage a crocodile.
   1. Express the following three statements as subset relations.
      (a) All babies are illogical.
      (b) Nobody is despised who can manage a crocodile.
      (c) Illogical persons are despised.
   2. Express the same three statements as implications.
   3. Prove that if the three statements are true, then no baby can manage a crocodile.

B. This question gives you three statements. Your job is to identify the appropriate setting, turn the statements into their subset relation form and into their logical "if-then" form with statements. The following is an example of what I am after.
   Statement: For complete transitive preferences, a larger sets of options is at least as good as a smaller set.
   The setting must include a set of options, call it X, the set of "budgets," i.e. $\mathcal{P}(X)$, and a complete transitive preference ordering, $\preceq$, on X Let $G = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : (\forall x \in A)(\exists y \in B)[x \preceq y]\}$. Equivalently, let $\mathbb{G}(A, B)$ be the statement "$(\forall x \in A)(\exists y \in B)[x \preceq y]$, so that $G = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathbb{G}(A, B)\}$. Let $L = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subset B\}$, and $\mathbb{L}(A, B)$ be the statement "$A \subset B$." The subset and the implication forms of the above statement are
$$L \subset G \;\text{ and }\; \mathbb{L} \Rightarrow \mathbb{G}.$$
   1. **Statement**: For any finite set X, a relation on X is complete and transitive iff there exists a utility function that represents it.
   2. **Statement**: Pareto optimality implies weak Pareto optimality.
   3. Let $X \subset \mathbb{R}$ and $T \subset \mathbb{R}$, $f : X \times T \to \mathbb{R}$ be a (utility) function, and $x^*(t) := \text{argmax}_{x \in X} f(x, t)$. **Statement**: Supermodularity of f implies a non-decreasing argmax.

C. **From Chapter 2.10**: 2.10.2 and 3 (p. 56), 2.10.11 and 13 (p. 58), and 2.10.17 (p. 60).
D. **From Chapter 3.3**: 3.3.2 (p. 75), 3.3.10 (p. 77), 3.3.14 and 17 (p. 78), and 3.3.22 (p. 80).
E. **From Chapter 3.4**: 3.4.2 (p. 83), and 3.4.12 (p. 84).
F. **From Chapter 3.7**: 3.7.12 (p. 96), 3.7.16 (p. 98), and 3.7.17 (p. 98).
G. **From Chapter 3.9**: 3.9.3 (p. 101) and 3.9.5 (p. 102).

Assignment #2 for **Mathematics for Economists II**
Spring 2010

Due date: Tue. Feb. 23.

Readings: CSZ, Ch. 4.1-8.

The basics of metric spaces are: convergence, completeness, closure, compactness, and continuity. This part of the course covers these topics for $\mathbb{R}^\ell$, but often in a fashion that generalizes directly to metric spaces. Specifically, most of the proofs can be understood using pictures from $\mathbb{R}^\ell$, but they are independent of the pictures, as they must (almost) always be, and independent of the special structures available to us in $\mathbb{R}^\ell$. Here are some of the metric spaces that we are interested in that do not appear in Chapter 4.

(1) $\mathcal{F}^\circ = \{\cup_{i\leq I}(a_i, b_i] : I \in \mathbb{N},\ a_i, b_i \in (0, 1]\}$ is the field of finite unions of half closed interval subsets of $(0, 1]$. For a fixed cdf, F, on $(0, 1]$, define $P_F((a, b]) = F(b) - F(a)$, and for $A = \cup_{i\leq I}(a_i, b_i]$ a disjoint union, define $P_F(A) = \sum_{i\leq I} P_F((a_i, b_i])$. The metric on $\mathcal{F}^\circ$ is $d_F(A, B) = P_F(A\Delta B)$ where $A\Delta B := (A \setminus B) \cup (B \setminus A)$. This is a metric except that $A \neq B$ and $d_F(A, B) = 0$ can both happen. We will fix this later. Also, this metric space is not complete, and completing it gives us the basic probability spaces that we use.
(2) For $A \subset \mathbb{R}^\ell$ and $\mathbf{x} \in \mathbb{R}^\ell$, define $d(\mathbf{x}, A) = \inf\{d(x, a) : a \in A\}$ and for $A, B \subset \mathbb{R}^\ell$, $d_H(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$. For A, B compact sets, this gives a metric, but it is not generally a metric for non-compact sets.
(3) $C_b(\mathbb{R}^\ell)$ is the set of continuous functions on $\mathbb{R}^\ell$ with the metric $d_\infty(f, g) := \sup\{|f(\mathbf{x}) - g(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^\ell\}$.
(4) For $p \in [1, \infty)$, $\ell^p := \{x \in \mathbb{R}^\mathbb{N} : \sum_{i=1}^\infty |x_i|^p < \infty\}$ with the distance $d_p(x, y) = \left(\sum_{i=1}^\infty |x_i - y_i|^p\right)^{1/p}$.
(5) For $p \in [1, \infty)$, $L^p([0, 1])$ is the set of functions on $(0, 1]$ with $\int_{[0,1]} |f(t)|^p\,dt < \infty$ with the distance $d_p(f, g) = \left(\int_{[0,1]} |f(t) - g(t)|^p\right)^{1/p}$.

Chapter 4.1 contains all of the definitions, with very few examples. Chapter 4.2 applies all of the definitions to a rather trivial class of metric spaces, the so-called "discrete" ones. Being able to quickly apply all of the definitions is nice, but the real reason to study the discrete metric spaces is because they have really interesting classes of continuous functions. It is only when we are done with 4.3 through 4.8 that all of the terms in 4.1 should have become firmly established in your mind.

Problems

A. **From Chapter 4.3**: 4.3.6 (p. 116), 4.3.11 (p. 118), 4.3.15 (p. 120).
B. **From Chapter 4.4**: 4.4.3 (p. 120), 4.4.6 (p. 121), 4.4.12 and 13 (p. 123).
C. **From Chapter 4.5**: 4.5.2 (p. 124), 4.5.8, 9, 10, and 11 (p. 126).
D. **From Chapter 4.6**: 4.6.3 (p. 129).
E. **From Chapter 4.7**: 4.7.14 (p. 132), 4.7.3, 6 (p. 130).
F. **From Chapter 4.8**: 4.8.9 (p. 140), 4.8.10, changing "six" to "three" (p. 140), 4.8.13, pick two of the four (p. 141).

Assignment #3 for **Mathematics for Economists II**
Spring 2010

Due date: Wed. Mar. 10.

Readings: CSZ, Ch. 6.1 and 6.2.

Despite 'only' being two sections of Chapter 6, there is a great deal of material here. The three metric spaces we'll be looking at here are $(\mathcal{K}(Y), d_H)$, the class of non-empty compact subsets of the metric space (Y, d) with the Hausdorff metric d_H, the space $C_b(M)$ of continuous bounded functions on a metric space M, and $C(M)$, the space of continuous functions on M where M is σ-compact.

The metric space $(\mathcal{K}(Y), d_H)$ inherits its basic properties from (Y, d):

> $(\mathcal{K}(Y), d_H)$ is separable/complete/totally bounded/compact iff (Y, d) is separable/complete/totally bounded/compact.

There are two main reasons to study $(\mathcal{K}(Y), d_H)$. The first is the Theorem of the Maximum, which tells us that, provided there is enough continuity in the problem, the value function is continuous and the argmax correspondence is upper hemicontinuous. The second is that econometricians are beginning to estimate sets of parameters, and these are typically points in the space $(\mathcal{K}(Y), d_H)$.

There are many many reasons to study the spaces $C_b(M)$ and $C(M)$. We will see many of these as we go through the chapter.

A first observation is that the spaces $C_b(M)$ and $C(M)$ do not inherit their properties from the metric space M, because, for example, $C_b(M)$ is always a complete metric space, independent of the properties of M. In more advanced parts of mathematics, there are results that tell us that properties of compact M are reflected in $C(M)$ — that one can work from properties of $C(M)$ to find a homeomorphic copy of M.

Of particular importance for us will be characterizations of compact subsets of $C(M)$ when M is compact. These will extend to characterizations of compact subsets of $C_b(M)$.

Here, we will be interested in $C_b(M)$ as our first *Banach space*, that is, as our first complete normed vector space. If M is infinite, then this is an infinite dimensional vector space. Other Banach spaces, the ℓ^p spaces for $p \in [1, \infty)$, can be found as subspaces of $C_b(\mathbb{N})$. At the end of this chapter, we will turn to the other main class of Banach spaces of interest in economics, the $L^p(\Omega, \mathcal{F}, P)$ spaces, $p \in [1, \infty)$. These are spaces of random variables having finite moments of order p.

When M is not compact but is σ-compact, the spaces $C(M)$ have the defining properties of what are called *Frechet spaces*. These are metric vector spaces that are not normed. Approximation of a continuous function over all of $\mathbb{R}_+$ is important for thinking about continuous time stochastic process theory.

Problems

A. **From Chapter 6.1**: (problems on $(\mathcal{K}(Y), d_H)$) 6.1.6, 6.1.13, 6.1.15, 6.1.20, and 6.1.34.
B. **From Chapter 6.2**: (problems on separability, projection, and evaluation mappings) 6.2.6, 6.2.11, 6.2.12, and 6.2.14.
C. **From Chapter 6.2**: (a problem on the Levy distance) 6.2.22.
D. **From Chapter 6.2**: (a problem on Brownian motion) 6.2.24.
E. **From Chapter 6.2**: (problems on convergence in $C(M)$) 6.2.29 and 6.2.30.
F. **From Chapter 6.2**: (problems on dynamic programming) 6.2.34, 6.2.37 and 6.2.38.