Homework Assignment #1
Prob-Stats, Fall 2018
Due date: Monday, September 24

A. C&B §1.7, exercise 1.9.
B. C&B §1.7, exercise 1.12.
C. C&B §1.7, exercise 1.24.
D. C&B §1.7, exercise 1.35.
E. HB §1.11, exercise 7 and CSZ Exercise 7.3.5.
F. CSZ Exercise 7.3.10.
G. HB §1.11, exercise 12.
H. HB §1.11, exercise 13.
I. Suppose that $x, y \in \Omega$ and a class of subsets of Ω, $\mathcal{E}$, has the property that for all $E \in \mathcal{E}$, $1_E(x) = 1_E(y)$. Show that for every $A \in \sigma(\mathcal{E})$, $1_A(x) = 1_A(y)$.
J. Borel-Cantelli says that if $\sum_n P(A_n) < \infty$, then $P([A_n \text{ i.o.}]) = 0$. This problems gives conditions (stronger than absolutely necessary) to show that if $\sum_n P(A_n) = \infty$, then $P([A_n \text{ i.o.}]) = 1$.

   Dfn: the events $A_1, A_2, \ldots$ are **independent** if for any finite set of indexes, $n_1, \ldots, n_k$, $k \in \mathbb{N}$, $P(\cap_{j=1}^k A_{n_k}) = P(A_{n_1}) \cdot P(A_{n_2}) \cdots P(A_{n_k})$.

   Suppose that the $(A_n)_{n=1}^\infty$ are independent and that $\sum_n P(A_n) = \infty$. Complete the following steps to show that $P([A_n \text{ i.o.}]) = 1$.
   1. Show that $([A_n \text{ i.o.}])^c = [A_n^c \text{ a.a.}]$, that is, show that $(\cap_n \cup_{i\geq n} A_i)^c = (\cup_n \cap_{i\geq n} A_i^c)$.
   2. Show that if for all n, $P(\cap_{i\geq n} A_i^c) = 0$, then $P(A_n^c \text{ a.a.}) = 0$.
   3. Show that $1 - x \leq e^{-x}$. [Ask the math-for-econ professor about convex functions and tangents.]
   4. Show that the events $(A_n^c)_{n=1}^\infty$ are independent.
   5. Show that for $n \leq m$, $P(\cap_{i=n}^m A_i^c) = \Pi_{i=n}^m (1 - P(A_i))$.
   6. Show that for $n \leq m$, $\Pi_{i=n}^m (1 - P(A_i)) \leq e^{-\sum_{i=n}^m P(A_i)}$.
   7. Show that for each n, $P(\cap_{i=n}^\infty A_i^c) = 0$.
K. Let X_n be a sequence of Borel measurable functions from $(\Omega, \mathcal{F})$ to $[0, 1]$.
   1. Show that $\{\limsup_n X_n > r\} = \cup_{q\in\mathbb{Q}, q>r} \cap_{m\in\mathbb{N}} \cup_{i\geq m} \{X_i > q\}$.
   2. Show that the function $X(\omega) := \limsup_n X_n(\omega)$ is measurable.
   3. Let $\Phi : [0, 1] \leftrightarrow [-\infty, +\infty]$ be a strictly increasing continuous function. Show that if Y_n is a sequence of Borel measurable, $\mathbb{R}$-valued functions, then $Y(\omega) := \limsup_n Y_n(\omega)$ is a measurable function from Ω to $[-\infty, +\infty]$.
   4. In the previous part of this problem, what do we know about the behavior of $Y_n(\omega)$ if $Y(\omega) = -\infty$? And what do we know about the behavior of $Y_n(\omega)$ if $Y(\omega) = +\infty$?
L. A $S \subset \mathbb{R}$ is a **Lebesgue null set** if for all $\epsilon > 0$, there exists a countable collection of sets (a_n, b_n), $n \in \mathbb{N}$, such that $S \subset \cup_n (a_n, b_n)$ and $\sum_n (b_n - a_n) < \epsilon$. If S is a Lebesgue null set, then it seems that S has to be "small." This problem ask you to show that there is a Lebesgue null set that is uncountable, that is, that has exactly the same cardinality as $\mathbb{R}$ itself.
   1. Any countable S is a Lebesgue null set.
   2. Any countable union of Lebesgue null sets is a Lebesgue null set.
   3. For each sequence $\vec{a} = (a_n)_{n\in\mathbb{N}} \in \{0, 2\}^{\mathbb{N}}$, define $r(\vec{a}) = \sum_n \frac{a_n}{3^n}$.
      a. Show that all of the sequences $n \mapsto \frac{a_n}{3^n}$ are summable.
      b. Show that $\{0, 2\}^{\mathbb{N}}$ is uncountable.
      c. Show that $\vec{a} \mapsto r(\vec{a})$ is one-to-one.

d. Show that $S := r(\{0,2\}^{\mathbb{N}})$ is an uncountable Lebesgue null set.

M. The random variable X takes values in the two point space $\mathbb{X} = \{x_1, x_2\}$. Before learning the value of X, the decision maker must choose a point in $A = \{a_1, a_2, a_3\}$. Their utilities, $u : A \times \mathbb{X} \to \mathbb{R}$ are given by

	x_1	x_2
a_1	10	2
a_2	7	7
a_3	2	10

1. Give, as a function of $\beta \in \Delta(\mathbb{X})$, $a^*(\beta) := \arg\max_{a_i \in A} \sum_{x_n \in \mathbb{X}} u(a_i, x_n)\, \beta(x_n)$.
2. Give, as a function of $\beta \in \Delta(\{x_1, x_2\})$, $V(\beta) := \max_{a_i \in A} \sum_n u(a_i, x_n)\, \beta(x_n)$.
3. The epigraph of the function $V(\cdot)$ is the set $\{(\beta, r) \in \Delta(\mathbb{X}) \times \mathbb{R} : V(\beta) \geq r\}$. A function is convex if its epigraph is a convex set. Show that the epigraph of $V(\cdot)$ is a convex set.
4. The prior distribution of X is the point $\beta^\circ \in \Delta(\mathbb{X})$ defined by $\beta^\circ(X = x_1) = \beta^\circ(X = x_2) = \frac{1}{2}$. Before choosing $a \in A$, the decision maker observes the random signal $S \in \mathbb{S} := \{s_1, s_2, s_2\}$ where the conditional distribution of S are given by

$x \downarrow$	$P(S = s_1 \vert x)$	$P(S = s_2 \vert x)$	$P(S = s_3 \vert x)$
x_1	0.1	0.2	0.7
x_2	0.6	0.2	0.2

   a. Give the conditional beliefs, $\beta(\cdot|s_i)$, $i = 1, 2, 3$.
   b. Give the marginal distribution of S, $P(S = s_i)$, $i = 1, 2, 3$.
   c. Show that $\sum_{s \in \mathbb{S}} \beta(\cdot|s)\, P(S = s) = \beta^\circ(\cdot)$.
   d. Give $a^*(\beta(\cdot|s_i))$, $i = 1, 2, 3$.
   e. Give $V(\beta(\cdot|s_i))$, $i = 1, 2, 3$.
   f. Give the value of S, $\mathbb{V}(S) := \sum_{s \in \mathbb{S}} V(\beta(\cdot|s))\, P(S = s)$.
5. A randomized strategy for S is a function $\sigma : \mathbb{S} \to \Delta(A)$. A randomized strategy for S induces a joint distribution on $\mathbb{X} \times A$ defined by $\mu_\sigma(x, a) = \beta^\circ(x) \cdot \sum_{s \in \mathbb{S}} \sigma_s(a)\, P(S = s|x)$. The set of inducible joint distributions for S is

$$D(S) := \{\mu_\sigma : \sigma \text{ is a randomized strategy for } S\}.$$

   Give $D(S)$ and show that it is a convex set.
6. Now supose that T is the Markov scramble of S with $P(T = t_j | S = s_i)$ given in the matrix M

	t_1	t_2	t_3
s_1	1/3	0	2/3
s_2	1/2	1/2	0
s_3	0	1/4	3/4

   a. Give the conditional beliefs, $\beta(\cdot|t_i)$, $i = 1, 2, 3$
   b. Give the marginal distribution of T, $P(S = t_i)$, $i = 1, 2, 3$.
   c. Show that $\sum_{t \in \mathbb{S}} \beta(\cdot|t)\, P(T = t) = \beta^\circ(\cdot)$.
   d. Give $a^*(\beta(\cdot|t_i))$, $i = 1, 2, 3$.
   e. Give $V(\beta(\cdot|t_i))$, $i = 1, 2, 3$.
   f. Give the value of T, $\mathbb{V}(T) := \sum_{t \in \mathbb{S}} V(\beta(\cdot|t))\, P(T = t)$. Check your calculations by showing that $\mathbb{V}(T) \leq \mathbb{V}(S)$.

7. The set of inducible joint distributions for T is $D(T)$ defined in a fashion directly paralleling the definition of $D(S)$ above. Give $D(T)$, show that it is convex, and show that $D(T) \subset D(S)$. Use this to give an alternative argument that $\mathbb{V}(T) \leq \mathbb{V}(S)$.
8. Refer to Theorem 4.11.9 (and the surrounding material) in Corbae et al. for the following. Suppose that the Markov scramble given by the matrix M above is applied repeatedly. Show that is the number of applied scrambles becomes large, the information structure becomes worthless, that is, it converges to $V(\beta^\circ)$.

N. Let A, $\mathbb{X}$, and $\mathbb{S}$ be finite sets of actions, possible realizations of X, and possible realizations of signals, let $\beta^\circ \in \Delta(\mathbb{X})$ be a prior distribution for X, and let $u : A \times \mathbb{X} \to \mathbb{R}$ be a utility function. For each $x \in \mathbb{X}$, suppose that the signal S has the conditional distribution $P(\cdot|X = x) \in \Delta(\mathbb{S})$, let $P(s) = \sum_x P(s|X = x)\beta^\circ(x)$ be the marginal distribution of S, and for each s with $P(s) > 0$, let $\beta(\cdot|S = s) \in \Delta(\mathbb{X})$ be the conditional distribution of X given that $S = s$. Let $a^*(\beta) = \arg\max_{a \in A} \sum_x u(a, x)\, \beta(x)$ and let $V(\beta) = \sum_x u(a^*(\beta), x)\, \beta(x)$. Define $\mathbb{V}(S) = \sum_s V(\beta(\cdot|s))\, P(s)$ and let $D(S) \in \Delta(\mathbb{X} \times A)$ be the set of inducible joint distributions.

Now suppose that the observations S are independently repeated, that is, suppose that (S, S') is observed and $P(S = s, S' = s'|X = x) = P(S = s|X = x) \cdot P(S' = s'|X = x)$. Show that $D(S) \subset D(S, S')$ and that for every utility function u, the decision maker with that utility function at least weakly prefers (S, S') to S.

O. [Doob's Theorem] Let $(\Omega, \mathcal{F})$ and let $(Y, \mathcal{Y})$ be non-empty sets and σ-fields of subsets. The most frequent class of sub-σ-fields, $\mathcal{G} \subset \mathcal{F}$, that we will encounter arise from a measurable $g : \Omega \to Y$, they are of the form $\mathcal{G} = g^{-1}(\mathcal{Y})$. Let $\mathcal{B}$ denote the Borel σ-field on $\mathbb{R}$, that is, the smallest σ-field of subsets of $\mathbb{R}$ containing the open subsets of $\mathbb{R}$. This problem shows that if $f : \Omega \to [0, 1]$ has the property that $f^{-1}(\mathcal{B}) \subset \mathcal{G}$, then there exists a measurable $h : Y \to \mathbb{R}$ such that $f(\omega) = h(g(\omega))$. Thus, the only $\mathcal{G}$-measurable functions are in fact functions of g, the function g contains everything (measurable) that one could ever get from $\mathcal{G}$.
1. Give an elementary proof of the assertion if $g(\Omega)$ is a finite set $G = \{y_1, \ldots, y_N\}$ and $\{y_n\} \in \mathcal{Y}$ for each n.
2. Show that $\mathcal{G}$ is a σ-field.
3. Show that $A_{i,n} := \{\omega : f(\omega) \in [i/2^n, (i + 1)/2^n)\}$ is of the form $g^{-1}(B_{i,n})$ for some $B_{i,n} \in \mathcal{Y}$.
4. Define the functions $f_n : \Omega \to [0, 1]$ and $h_n : Y \to [0, 1]$ by
$$f_n = \textstyle\sum_{i=1}^{2^n} \frac{i}{2^n} 1_{A_{i,n}} \text{ and } h_n = \sum_{i=1}^{2^n} \frac{i}{2^n} 1_{B_{i,n}}.$$
Show that for all ω, $f_n(\omega) = h_n(g(\omega))$.
5. Define $h(y) = \limsup_n h_n(y)$. Show that for all ω, $f(\omega) = h(g(\omega))$.
6. In the previous step, why couldn't we define $h(y) = \lim_n h_n(y)$?