Sketches of Solutions to Assignment #6
Managerial Economics, Fall 2015
Due: 2 p.m., Monday December 7'th, 2015

Readings. Baye textbook coverage of: mixed/randomized equilibria; repeated games; competition between firms with market power.

A. Practice with 2×2 games.

1. Find all of the equilibria and their associated equilibrium utility levels for the following game.

	Left	Right
Up	$(25, 45)$	$(38, 78)$
Down	$(3, 63)$	$(71, 55)$

Ans. Checking pure strategy best responses shows that the only equilibrium is mixed. To find it, we find $\alpha = Pr(1 \text{ play UP})$ so that 2 is indifferent and $\beta = Pr(2 \text{ play Left})$ so that 1 is indifferent.

$$45\alpha + 63(1-\alpha) = 38\alpha + 71(1-\alpha),\ 7\alpha = 8(1-\alpha),\ \alpha^* = \frac{8}{15},$$

$$25\beta + 38(1-\beta) = 3\beta + 71(1-\beta),\ 22\beta = 33(1-\beta),\ \beta^* = \frac{33}{55} = \frac{3}{5}.$$

2. Find all of the equilibria and their associated equilibrium utility levels for the following game.

	Left	Right
Up	$(52, 42)$	$(93, 16)$
Down	$(46, 18)$	$(89, 89)$

Ans. Checking pure strategy best responses shows that Up dominates Down for 1 (because $52 > 46$ and $93 > 89$), hence the only equilibrium is (Up, Left).

3. Find all of the equilibria and their associated equilibrium utility levels for the following game.

	Left	Right
Up	$(46, 4)$	$(94, 53)$
Down	$(91, 29)$	$(92, 22)$

Ans. Checking pure strategy best responses shows (Down, Left) and (Up, Right) are equilibria. From the Harsanyi theorem mentioned in lecture, you know that there is a third equilibrium, it is mixed. To find it, we find $\alpha = Pr(1 \text{ play UP})$ so that 2 is indifferent and $\beta = Pr(2 \text{ play Left})$ so that 1 is indifferent.

$$4\alpha + 29(1-\alpha) = 53\alpha + 22(1-\alpha),\ 49\alpha = 7(1-\alpha),\ \alpha^* = \frac{7}{56} = \frac{1}{8},$$

$$46\beta + 94(1-\beta) = 91\beta + 92(1-\beta),\ 45\beta = 2(1-\beta),\ \beta^* = \frac{2}{47}.$$

4. Find all of the equilibria and their associated equilibrium utility levels for the following game.

	Left	Right
Up	$(37, 86)$	$(58, 3)$
Down	$(16, 47)$	$(68, 99)$

Ans. Checking pure strategy best responses shows (Up, Left) and (Down, Right) are equilibria. To find the mixed equilibrium,

$$86\alpha + 47(1-\alpha) = 3\alpha + 99(1-\alpha),\ 83\alpha = 42(1-\alpha),\ \alpha^* = \frac{42}{125},$$

$$37\beta + 58(1-\beta) = 16\beta + 68(1-\beta),\ 21\beta = 10(1-\beta),\ \beta^* = \frac{10}{31}.$$

B. Monitoring games.

1. Consider the following game

	Audit	Other
Divert	$(\pi - f, B - C)$	$(\pi + b, 0)$
Clean	$(\pi, -C)$	$(\pi, 0)$

We assume that $B > C > 0$, $\pi, f, b > 0$. The interpretation is the one given in class: player 1 may divert corporate funds to their own uses or run a clean operation; the auditors can audit player 1's operation or do the other parts of their job. The opportunity cost of auditing player 1 is $C > 0$, the benefit to catching diversion is $B > C$, etc.

a. Give the unique equilibrium for this game.

Ans. Checking pure strategy best responses shows that there is no pure strategy equilibrium. To find the mixed equilibrium,

$$(B-C)\alpha + (-C)(1-\alpha) = 0\alpha + 0(1-\alpha),\ (B-C)\alpha = C(1-\alpha),\ \alpha^* = \frac{C}{B},$$

$$(\pi - f)\beta + (\pi + b)(1-\beta) = \pi\beta + \pi(1-\beta),\ f\beta = b(1-\beta),\ \beta^* = \frac{b}{b+f}.$$

b. For what values of f, b, B, and C is there little diversion despite infrequent auditing? Explain the economics of this.

Ans. To have $\alpha = \frac{C}{B}$ low requires a high benefit to cost ratio for catching diversion. When this is true, it does not take much chance of catching a malfeasant to motivate the auditor to examine player 1, hence the equilibrium level of diversion will be low.

In a similar fashion, to have $\beta = \frac{b}{b+f} = \frac{1}{1+(f/b)}$ low requires that f/b be large, that is, to have the ratio of the benefit of diversion to cost of being caught be small. When this is true, the potential malfeasant will calculate that being caught is very dangerous, hence not be very motivated to divert, hence giving the auditor very little incentive to audit.

c. For what values of f, b, B, and C is there a great deal of diversion despite frequent auditing? Explain the economics of this.

Ans. To have $\alpha = \frac{C}{B}$ high requires a low benefit to cost ratio for catching diversion. When this is true, it takes a large chance of catching a malfeasant to motivate the auditor to examine player 1, hence the equilibrium level of diversion will be high.

In a similar fashion, to have $\beta = \frac{b}{b+f} = \frac{1}{1+(f/b)}$ high requires that f/b be small, that is, to have the ratio of the benefit of diversion to cost of being caught be large. When this is true, the potential malfeasant will calculate that being caught is not very dangerous compared to the benefits of diversion, hence be very motivated to divert, hence giving the auditor a large incentive to audit.

2. An office manager is concerned with declining productivity. Despite the fact that she regularly monitors her clerical staff four times each day — at 9:00 AM, 11:00 AM, 1:00 PM, and again at 3:00 PM, office productivity has declined 30 percent since she assumed the helm one year ago. Would you recommend that the office manager invest more time monitoring the productivity of her clerical staff? Explain.
   Ans. Four times a day, one hears "Everyone look busy! Here she comes for her regular inspection!!" At other times . . . . Or perhaps, "While the cat is away, the mice will play."

C. Repeated games

1. At a time when demand for ready-to-eat cereal was stagnant, a spokesperson for the cereal maker Kellogg's was quoted as saying, " . . . for the past several years, our individual company growth has come out of the other fellow's hide." Kellogg's has been producing cereal since 1906 and continues to implement strategies that make it a leader in the cereal industry. Suppose that when Kelloggs and its largest rival advertise, each company earns \$0 billion in profits. When neither company advertises, each company earns profits of \$8 billion. If one company advertises and the other does not, the company that advertises earns \$48 billion and the company that does not advertise loses \$1 billion. Under what conditions could these firms use Nash reversion trigger strategies to support the collusive level of advertising?
   Ans. The one-shot game is

	Advertise	Don't Advertise
Advertise	$(0,0)$	$(-1,48)$
Don't Advertise	$(-1,48)$	$(8,8)$

   The unique, dominant strategy Nash equilibrium is (Ad,Ad), with payoffs of $(0,0)$. Consider the trigger strategy "Don't advertise as long as no-one has every advertised in the past, otherwise advertise" and suppose that the firms are maximizing expected discounted payoffs with a discount factor β. We wish to examine when this might be a subgame perfect equilibrium. Suppose that each firm believes that the other firm is using this strategy.
   There are two kinds of subgames, those in which someone has advertised in the past, and those in which no-one has advertised. In the first case, both firms believes that the other firm's strategy calls for advertising forever into the future, and the unique best response is to also advertise. In the second case, suppose that we are at time t and one of the firms is considering deviating. Their payoff to deviating is

$$48\beta^t + 0\beta^{t+1} + 0\beta^{t+2} + \cdots = 48\beta^t,$$

   their payoff to following the trigger strategy is

$$8\beta^t + 8\beta^{t+1} + 8\beta^{t+2} + \cdots = 8\beta^t \frac{1}{1-\beta}.$$

   Thus, so long as $\beta > \frac{5}{6}$, following the trigger strategy is a subgame perfect equilibrium.

2. A producer can produce a high quality good or they can cut costs and produce a low quality good. The potential buyer cannot ascertain the quality of the good before they buy it. The associated payoffs are

	Low quality	High quality
Don't Buy	$(0,0)$	$(0,-2)$
Buy	$(-2,2)$	$(1,1)$

a. Give the unique equilibrium for the one-shot version of this game.
Ans. The producer's dominant strategy is Low, given this, the buyer's unique best response is Don't Buy and the unique equilibrium is (Don't Buy, Low quality).

b. Under what conditions could the buyer and producer use Nash reversion trigger strategies to support purchase of high quality products?
Ans. For the producer, using the strategy from any t at which (Buy, High) has always been played yields

$$1\beta^t + 1\beta^{t+1} + 1\beta^{t+2} + \cdots = \beta^t \frac{1}{1-\beta},$$

while deviating yields

$$2\beta^t + 0\beta^{t+1} + 0\beta^{t+2} + \cdots = 2\beta^t,$$

hence the trigger strategy is subgame perfect for the producer so long as $\beta > \frac{1}{2}$.
For the buyer, the comparison is between

$$1\beta^t + 1\beta^{t+1} + 1\beta^{t+2} + \cdots = \beta^t \frac{1}{1-\beta}$$

and

$$0\beta^t + 0\beta^{t+1} + 0\beta^{t+2} + \cdots = 0,$$

so deviating is never a best response.

D. A product differentiation game. Two firms, unimaginatively i and j, produce partially substitutable goods. The inverse demand functions are

$$q_i(p_i, p_j) = \alpha_i - \beta_i p_i + \gamma_i p_j, \text{ and}$$
$$q_j(p_i, p_j) = \alpha_j - \beta_j p_j + \gamma_j p_i$$

where the parameters, α_i, β_i, γ_i, α_j, β_j and γ_j are all strictly positive. The firms' cost functions are $C_i(q_i) = c_j q_i$ and $C_j(q_j) = c_j q_j$.

a. Find the firms' best response functions.
Ans. Solve

$$\max_{p_i} \, (p_i - c_i)[\alpha_i - \beta_i p_i + \gamma_i p_j].$$

Note that as long as $\alpha_i > c_i$, the solution to this yields a positive profit, and we will assume this. This is a quadratic in p_i opening downward, hence set the derivative equal to 0 and solve for

$$p_i^*(p_j) = m_i + s_i p_j$$

where $m_i = \frac{1}{2}(\frac{\alpha_i}{\beta_i} + c_i)$ and $s_i = \frac{1}{2}\frac{\gamma_i}{\beta_i}$. Changing the roles of i and j yields

$$p_j^*(p_i) = m_j + s_j p_i$$

where $m_j = \frac{1}{2}(\frac{\alpha_j}{\beta_j} + c_j)$ and $s_j = \frac{1}{2}\frac{\gamma_j}{\beta_j}$.

b. Find the Bertrand equilibrium.
Ans. Solving the simultaneous equations $p_i^*(p_j) = m_i + s_i p_j$ and $p_j^*(p_i) = m_j + s_j p_i$ yields

$$p_i^* = \frac{m_i + s_i m_j}{1 + s_i s_j} \text{ and } p_j^* = \frac{m_j + s_j m_i}{1 + s_j s_j}.$$

c. Suppose that both firms are acquired by a third firm and that the third firm picks both prices. Give the third firm's profits and explain why they charge higher prices for the two goods.
Ans. Let $\pi_i(p_i, p_j) = h_i(p_i) + g_i(p_i, p_j) = (p_i - c_i)[\alpha_i - \beta_i p_i] + (p_i - c_i)\gamma_i p_j$ and do the same for $\pi_j(p_i, p_j)$. The third firm's profits are

$$\pi_3^* = \max_{p_i, p_j} \; h_i(p_i) + g_i(p_i, p_j) + h_j(p_j) + g_j(p_i, p_j).$$

Explicitly solving these is slightly messy, but we can get to the answer without doing the algebra. The intuition is as follows: before the firms were acquired, if one increased the profits, this increased the profits of the other firm, but neither cared about this effect; however, the monopolist internalizes this external effect. Given that $\partial g_i / \partial p_j > 0$ and $\partial g_j / \partial p_i > 0$, it seems clear that the monopolist will charge higher prices. Here is the complete argument.
The monopolist's FOCs are

$$\frac{\partial h_i}{\partial p_i} + \frac{\partial g_i}{\partial p_i} + \frac{\partial g_j}{\partial p_i} = 0$$

and

$$\frac{\partial h_j}{\partial p_j} + \frac{\partial g_j}{\partial p_j} + \frac{\partial g_i}{\partial p_j} = 0.$$

By contrast, the equilibrium FOCs are

$$\frac{\partial h_i}{\partial p_i} + \frac{\partial g_i}{\partial p_i} = 0$$

and

$$\frac{\partial h_j}{\partial p_j} + \frac{\partial g_j}{\partial p_j} = 0.$$

The first two terms are the same in both places, the monopolist's FOCs contains an extra, strictly positive terms, T_i and T_j respectively. Any solution to

$$\frac{\partial h_i}{\partial p_i} + \frac{\partial g_i}{\partial p_i} + T_i = 0$$
$$\frac{\partial h_j}{\partial p_j} + \frac{\partial g_j}{\partial p_j} + T_j = 0$$

must be strictly increasing in positive T_i and T_j.

d. Under what conditions could the two firms use Nash reversion trigger strategies to support the same higher prices and profits?
Ans. Let π_i^M and π_j^M be the profits of the two firms if they charge the third firms optimal prices, let π_i^B and π_j^B be their Bertrand profits, and let

π_i^d and π_j^d be their optimal profits if they optimally cut prices in a one-shot deviation from (π_i^M, π_j^M). The conditions for i are

$$\pi_i^d \beta^t + \pi_i^B \beta^{t+1} + \pi_i^B \beta^{t+2} + \cdots < \pi_i^M \beta^t \frac{1}{1-\beta}.$$

In slightly more detail, $\pi_i^d + \pi_i^B \frac{\beta}{1-\beta} < \pi_j^M \frac{1}{1-\beta}$, equivalently,

$$(1-\beta)\pi_i^d + \beta\pi_i^B < \pi_i^M, \text{ or } \beta > \frac{\pi_i^d - \pi_i^M}{\pi_i^d - \pi_i^B}.$$

This is possible because $\pi_i^d > \pi_i^M > \pi_i^B$. The parallel equation holds for firm j.

E. Bonuses and incentives. The next several problems refer to the following situation. Profits for the next fiscal year are a random variable Π. The distribution of Π depends on employee efforts, e. In particular, for any level of profits x and any $e' > e$, $Prob(\Pi > x|e') \geq Prob(\Pi > x|e)$. This means that higher efforts increase the probability that profits are higher than x for any and all values of x.

Employees are paid a salary S and a bonus B if profits reach or exceed a target T. Employee utility if they receive y dollars and put in effort level e is $u(y, e) = v(y) - c(e)$ where $v(\cdot)$ is increasing in y and $c(\cdot)$ is increasing in e. (You could think of e as hours spent on the job, but this is a very crude measure of effort, thought and imagination matter). Employee expected utility is

$$\max_{e \in [0, \overline{e}]} [v(S)(1 - P(\Pi > T|e)) + v(S+B)P(\Pi > T|e)] - c(e).$$

There are several reasons to make the bonus depend on Π rather than on e: the employees may know better than the managers what specific forms the effort should take; the managers may not be able to observe e; and we care about outputs, not inputs, a really talented employee may be able to achieve high levels of Π with minimal effort, or may make hard work look effortless.

1. Show that optimal effort, e^*, is increasing in the bonus level, B.
   Ans. The only term in $[v(S)(1 - P(\Pi > T|e)) + v(S+B)P(\Pi > T|e)] - c(e)$ that includes both B and e is $v(S+B)P(\Pi > T|e)$, and this is supermodular in B and e.
2. Suppose that $v(\cdot)$ demonstrates everywhere positive and everywhere decreasing marginal utility of income.
   a. Show that optimal effort decreases in S.
      Ans. It is sufficient to show that

      $$f(e, S) := [v(S)(1 - P(\Pi > T|e)) + v(S+B)P(\Pi > T|e)] - c(e)$$

      is submodular in e and S. For this, note that

      $$\partial f / \partial S = [v'(S+B) - v'(S)]P(\Pi > T|e)$$

      and

      $$\partial^2 f / \partial S \partial e = [v'(S+B) - v'(S)]h(e)$$

      where $h(e) = \partial P(\Pi > T|e)/\partial e > 0$. Because $v''(\cdot) < 0$, we know that $v'(S+B) < v'(S)$, so $\partial^2 f / \partial S \partial e < 0$, submodularity.

b. In professional baseball, we have the observation that, on average, in the year after signing a large new contract, MLB players tend to do worse than they had in the past. In what way is the previous analysis related?
Ans. If the size of the bonuses for performance do not change in the new contract, then the previous suggests that effort will decline.

c. [Harder] Suppose that bonuses, B, are kept constant as a proportion of salary, S, that is $B = \kappa S$ for a fixed number κ. Give conditions on $v(\cdot)$ that would make optimal effort <u>in</u>crease in S, and conditions that would make optimal effort decrease.
Ans. Set $f(e, S) = [v(S)(1 - P(\Pi > T|e)) + v((1+\kappa)S)P(\Pi > T|e)] - c(e)$ and check for the sign of $\partial^2 f/\partial S\partial e$. From the previous, this is the opposite of the sign of $[(1+\kappa)v'((1+\kappa)S) - v'(S)]$. Therefore a sufficient condition is that this expression is positive for all $\kappa > 0$. In terms of the derivatives of $v(\cdot)$, this is the requirement that $\frac{d(1+\kappa)v'((1+\kappa)S)}{d\kappa} > 0$, which is $v'(rS) + rv''(rS) > 0$ for all $r > 1$.

3. Setting the target too high or too low is counterproductive.

a. Explain why we would not expect optimal effort to be monotonic in T.
Ans. If T is unreasonably large, $\Pi > T$ will only happen by a miracle unrelated to effort, and effort is costly. As T increases, there must come a point where the marginal benefit is not worth the marginal cost.

b. Explain why the expected utility cannot be (strictly) supermodular in effort, e, and the target level, T.
Ans. Consider

$$h(e, T) := [v(S)(1 - P(\Pi > T|e)) + v(S+B)P(\Pi > T|e)] - c(e)$$

and ask about the supermodularity of $h(\cdot, \cdot)$ in e and T. Gathering the terms that contain both e and T yields $g(e, T) = (v(S+B) - v(S))P(\Pi > T|e)$. The term $(v(S+B) - v(S))$ is positive because $v(\cdot)$ is increasing. Now observe that $\partial g/\partial T = -\varphi(T|e)$ where $\varphi(\cdot|e)$ is the pdf of the random variable Π when effort is e. Since $\int \varphi(t|e)\, dt \equiv 1$, it is impossible for the sign of $\partial\varphi/\partial e$ to always have one sign.

4. The role of outside options. From the above, you might conclude that lowering S and compensating with a higher B is the optimal strategy for management. This may be short-sighted. Suppose that employees have an "outside option," that is, they can go work for someone else, or go start their own firm, or retire. Suppose that the outside option gives them expected utility of $\overline{v}$. This means that the employee's expected utility maximization problem is now

$$\max\{\underline{v}, \max_{e \in [0, \overline{e}]} [v(S)(1 - P(\Pi > T|e)) + v(S+B)P(\Pi > T|e)] - c(e)\}.$$

The interpretation is that if S and B and the optimal effort that they imply becomes too unrewarding, the employees will take their skills and abilities elsewhere.

a. Why might optimal effort increase in S?
Ans. If S is too low to keep the employee and $S' > S$ is large enough that they will stay, then effort is higher at S' than it is at S.

b. How does optimal effort depend on B?

Ans. The objective function is still supermodular in e and B, although the objective function may be flat when S and B lead the employee to take their outside option.