NOTES FOR MICRO I: SINGLE PERSON AND MULTIPERSON DECISION THEORY

MAXWELL B. STINCHCOMBE

Contents

Date: Fall, 2000. TuTh 8-9:30 and 9:30-11, BRB 2.136. Unique #'s 30153 and 30155.

0. Overview

Micro I is the first part of your two semester sequence on microeconomics. The text for this class is MWG, *Microeconomic Theory* by Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green.[1]

Micro I begins with single person decision theory. Single person decision theory has three parts, (1) consumer demand theory, material from Ch. 1-3 in MWG and §1-3 in these notes, (2) producer supply and demand theory, Ch. 5 in MWG and §5 in these notes, and (3) choice under uncertainty, Ch. 6 in MWG and §6 in these notes. We'll also take a detour through aggregation theory, Ch. 4 in MWG and §4 in these notes. The last third of Micro I will be multi-person decision theory, also known as game theory, Ch. 7-9 in MWG and §7 in these notes.

0.1. **Single person decision theory.**

As noted, this has three parts, consumer demand theory, producer theory, and choice under uncertainty.

0.1.1. *Consumer demand theory.*

This first part of single person decision theory is consumer demand theory. We will both cover and expand on the neoclassical consumer demand theory you should already have seen — the utility maximization derivation of demand curves, income expansion paths, and Engel curves. What may be new to you are the arguments that utility maximization is (mostly) equivalent to preference based approaches, and you may not have seen the derivation of demand functions from choice rules.

The next section, §1, covers the relations between preference maximization, utility maximization and internally consistent choice rules, all of this quite abstractly and operating in the mathematical context of finite sets of options. The following two sections specialize the general abstract treatment to more familiar Walrasian budget sets.

§2 covers the choice-based approach to demand theory, also known as revealed preference theory. §3 begins with a (hopefully) self-contained study guide to help you acquire a (hopefully) good working knowledge of constrained optimization techniques. It then covers the neoclassical preference-based approach to demand theory.

Our coverage of consumer theory will end in after a detour. §4 is a discussion of the (mostly negative) results concerning the aggregation of many individual demand functions.

0.1.2. *Producer theory.*

Producer theory is the topic of §5. Many of the tools and concepts from consumer theory carry over. As a result, we can spend much less time here while covering essentially the same amount of material.

0.1.3. *Choice under uncertainty.*

The last part of single person decision theory is choice under uncertainty. This is a fascinating topic by itself, it is also the foundations for the economics of information, and game theory, two of the central parts of modern microeconomics. The crucial background is a firm understanding of probability spaces, random variables, and conditional expectations.

[1]Other books to consult include Hal Varian's *Microeconomic Analysis*, 2'nd or 3'rd ed., and David Kreps' *A Course in Microeconomic Theory.* In previous years, these two books have been used as the primary textbooks for the class.

0.2. **Game theory, aka multi-person decision theory.** It is usually the case that one's actions affect others. Game theory is a systematic attempt to elucidate the interaction between individuals optimizing when their actions affect others and vice versa. §7 of these notes covers Part Two of MWG, and (hopefully) a great number of examples of game theoretic analysis of economic phenomena.

0.3. **Organization.** The notes contains the homework assignments and due dates. Most, but not all sections contain only one homework assignment. It is my intention that homeworks on a topic should be due at the beginning of lecture that follows the last class spent on that topic. The due dates may be pushed back if needed.

As noted above, §3 begins with a (hopefully) self-contained development of constrained optimization. It is set up as homework problems for you to work through. These problems can and should be well started on it before we start in on the economics contained in §3.

The homework assignments are crucial to the class. There are a lot of them. Do not fall behind.

Each of your 8 homeworks will count for 9% of the grade, the final exam will count for the remaining 28%.

1. Abstract Preferences and Choices

Dates: Thurday August 31, Sept. 5, 7.
Text Material: MWG, Introduction and Ch. 1, pp. 3-16.

There are two basic approaches to consumer demand theory. The first assumes that consumers have preferences over their possible choices and make their most preferred choice. In this approach, we make (hopefully) reasonable assumptions about the preferences and work out the implications for observed choices. The second approach makes assumptions about the internal consistency of observed choices. Here we get an introduction to the abstract versions of both approaches and to the basic relations between them.

Crudely speaking, that is, without the qualifications that make the following statements actually true, the essential results are:

1. preference maximization is equivalent to utility maximization, and
2. revealed preference is equivalent to preference maximization.

1.1. Basic Framework.

The starting point is a set X of options. In the section, X **is assumed to be finite**, and the case of infinite X's will be used to get counter-examples. 2^X denotes the set of all subsets of X.[2]

The second part of the basic set-up is a collection of possible budgets $\mathcal{B}$, $\mathcal{B} \subset 2^X$. Finally, we are interested in the choices people make when faced with a $B \in \mathcal{B}$. This is captured in a function $C : \mathcal{B} \to 2^X$. Putting it together, the intended interpretation is that when faced with the set $B \subset X$, a person chooses $C(B)$. The function $C(\cdot)$ can be generated by a preference maximization story, or by utility maximization, or it can satisfy certain internal consistency requirements. The "essential results" above can be rephrased as a statement of the form:

> $C(\cdot)$ is generated by preference maximization if and only if it is generated by utility maximization if and only if it satisfies the weak axiom of revealed preference.

At this level of abstraction, X can be anything that people need to choose from. For example, X could be the set of all possible meals, $B \subset X$ might be the set of choices available in a particular restaurant. For another example, X could be the set of possible jobs, $B \subset X$ the ones that are presently available. If we want to get grandiose, X can be the of possible legal systems, $B \subset X$ can be the set of small changes we're presently considering. In the 3 sections following this one, X will be recognizable from intermediate microeconomics — it will be the set of bundles of commodities, and we will be interested in choices when the consumer faces a subset $B \in \mathcal{B}$ of X, the Walrasian budget set.

1.2. Preference Based Approaches.

The basic idea here is that consumers have preferences over their possible choices, given by a set $B \subset X$, and $C(B)$ is their most preferred choice(s). Mapping out how $C(B)$ changes as B changes is mapping out 'demand' functions.

You should note that much of the essential intellectual work is now complete, we just gave the bones of the theory. All that's left is putting some flesh on the bones.

[2]Detours: Note that if X has (say) two elements and we write X as 2, then we have 2^2 which is four, exactly the number of subsets of a set of size 2. If X has N elements and we write X as N, then $2^X = 2^N$, and the number 2^N is exactly the number of subsets of a set having N elements.

Notation: $X \times X = \{(x, y) : x \in X, y \in X\}$. For example, $\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ is the set of vectors with both components being real numbers. This is often written as $\mathbb{R}^2$ for obvious notational reasons.

The next object is a preference relation $\succeq$. This is a **binary relation**, that is, a subset of $X \times X$, written $\succeq \subset X \times X$. For $x, y \in X$, $(x, y) \in \succeq \subset X \times X$ is written $x \succeq y$, and we say "x is at least as preferred as y". If $x \succeq y$ and $\neg[y \succeq x]$, then we write $x \succ y$ and we say "x is strictly preferred to y".

Example 1.2.1. *$X = \{0, 1, 2, 3, 4\}$, and I prefer smaller numbers to larger numbers. We can represent $\succeq$ by*

$$X = \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4), \\ (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

This is also written, using the notation "$\succeq$" in two different ways (a practice frowned upon by real mathematicians), as $\succeq = \{(x, y) \in X \times X : x \succeq y\}$. With the usual convention that x is on the horizontal axis and y on the vertical, $\succeq$ and $\succ$ can be graphically represented by

4	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$	$\otimes$	
2	$\otimes$	$\otimes$	$\otimes$		
1	$\otimes$	$\otimes$			
0	$\otimes$				
$y \uparrow / x \rightarrow$	0	1	2	3	4

and

4	$\otimes$	$\otimes$	$\otimes$	$\otimes$	
3	$\otimes$	$\otimes$	$\otimes$		
2	$\otimes$	$\otimes$			
1	$\otimes$				
0					
$y \uparrow / x \rightarrow$	0	1	2	3	4

We're not willing to allow all binary relations to be preference relations, only the ones we call **rational**. We're going to go through the two conditions that make up this kind of rationality by looking at what happens to the simple-minded choice theory outlined above when the conditions are violated.

1.2.1. *Completeness.* Suppose you're at a restaurant and you have the choice between four meals, Pork, Beef, Chicken, or Fish, each of which costs the same. Suppose that your preferences $\succeq$ are given by

pork				$\otimes$
beef			$\otimes$	
fish		$\otimes$		
chic	$\otimes$			
	chic	fish	beef	pork

Remember the basic story, you choose that option that you like best. It is difficult to say what choice you're going to make here, do you like fish better than chicken? chicken better than fish? neither? both? If you never make up your mind, you starve to death, and we like to think that starving to death is not rational behavior.[3]

This binary relation violates **completeness**.

[3]This does not say that hunger strikes are irrational.

Definition 1.2.1. *A binary relation* $\succeq$ *on* X *is* **complete** *if*

$$(\forall x, y \in X)[[x \succeq y] \text{ or } [y \succeq x]].$$

One thing to be clear about, the "or" is not exclusive, a statement "A or B" means "either A is true, or B is true, or both are true."

Graphically, a relation $\succeq$ is complete if, when you take the union of the graph of $\succeq$ and its rotation around the 45° line you get all of $X \times X$.

1.2.2. *Transitivity.* Suppose you're at a restaurant and you have the choice between four meals, Pork, Beef, Chicken, or Fish, each of which costs the same. Suppose that your preferences $\succeq$ are given by

pork	$\otimes$			$\otimes$
beef			$\otimes$	$\otimes$
fish		$\otimes$	$\otimes$	$\otimes$
chic	$\otimes$	$\otimes$	$\otimes$	
	chic	fish	beef	pork

so that $\succeq$ is complete, and $\succ$ is given by

pork	$\otimes$			
beef				$\otimes$
fish			$\otimes$	$\otimes$
chic		$\otimes$	$\otimes$	
	chic	fish	beef	pork

Remember the basic story, you choose that option that you like best. Here $p \succ b \succ f \succ c \succ p$. Look at what happens — you start by thinking about c, discover you like f better so you switch your decision to f, but you like b better, so you switch again, but you like p better so you switch again, but you like c better so you switch again, coming back to where you started. You get dizzy and then starve to death before you make up your mind.

Definition 1.2.2. *A binary relation* $\succeq$ *on* X *is* **transitive** *if*

$$(\forall x, y, z \in X)[\{[x \succeq y] \ \& \ [y \succeq z]\} \Rightarrow [x \succeq z]].$$

Note that $[p \succeq f]$, and $[f \succeq c]$ but $\neg[p \succeq c]$ in the previous example.

1.2.3. *Rational Preferences.* So, a minimal pair of assumptions needed to get the basic preference-based demand story to work is that preferences be complete and transitive. We give this pair of properties a name.

Definition 1.2.3. *A binary relation* $\succeq$ *on* X *is* **rational** *if it is both complete and transitive.*

Let us be clear, preferences that are not rational arise in many different contexts, we just can't use them for a preference-based choice theory. Here are two of my favorite examples of non-rational preference relations, the second one will reappear when you look at social choice theory next semester.

Example 1.2.2. *I like basketball teams that win. In my league there are three teams. Team* A *is tall, slow, and clumsy, and they are beaten by team* B*, the graceful, medium height,* *medium speed team. Team B is in turn beaten by the short but incredibly fast team C. However, due to an extreme height advantage, team A beats team C.*

Example 1.2.3. *Imagine putting Trent Lott, Bill Clinton, and Jim Hightower*[4] *in a single room to vote over 3 budget options. Option A has the highest corporate subsidies and biggest tax breaks for the rich. Option B has fairly high corporate subsidies and the biggest tax breaks for the upper middle class. Option C has the low corporate subsidies and tax breaks for the working class. Clinton's preferences are*

$$B \succ_{Cl} A \succ_{Cl} C,$$

Lott's are

$$A \succ_{Lo} C \succ_{Lo} B,$$

($C \succ_{Lo} B$ because B is the middle road that Lott has sworn to avoid (all of those dead armadillos) and because C is Clinton's favorite), and Hightower's are

$$C \succ_{Hi} B \succ_{Hi} A.$$

If all three vote on pairs, pairwise majority rule, then B beats A by 2 to 1, C beats B by 2 to 1, and A beats C by 2 to 1. Majority rule voting gives a complete but not transitive set of preferences in this case. Since liberals no longer have any voting power in budget negotiations, take Hightower out. Majority rule again gives complete preferences (check), but $A \succ C$, the other choices are split votes, $A \sim B$ and $B \sim C$, so that if the majority rule preferences were transitive, then we would have both $C \sim A$ and $A \succ C$.

In summary, pairwise majority rule 'works' when there is one rational person, but not when there are 2 or more.

Given a rational preference relation $\succeq$, define $C^*_{\succeq}(B)$ as the set of prefered points in the set B,

$$C^*_{\succeq}(B) = \{x \in B : (\forall y \in B)[x \succeq y]\}.$$

The conditions that $C^*_{\succeq}(\cdot)$ ought to satisfy if we are to have a reasonable theory of preference based choice are:

1. For all $B \in 2^X$, $C^*_{\succeq}(B) \subset B$,
2. If $B \neq \emptyset$, then $C^*_{\succeq}(B) \neq \emptyset$, and
3. If for some $B \in 2^X$, $x, y \in B$ and $x \in C^*_{\succeq}(B)$, then for any $B' \in 2^X$, if $x, y \in B'$ and $y \in C^*_{\succeq}(B')$, then $x \in C^*_{\succeq}(B')$.

The first two requirments are that peoples choices belong to their set of options and that they make some choice. The third is the requirement that if x is ever chosen when y is available, then there can be no budget set containing both x and y where y is chosen but x is not chosen. This is a consistency idea: in the simplest case, if $C^*_{\succeq}(\{x, y\}) = \{x\}$, then we cannot have $C^*_{\succeq}(\{x, y, z\}) = \{y\}$.

Theorem 1.2.1. *If $\succeq$ is rational and X is finite, then $C^*_{\succeq}(\cdot)$ satisfies conditions 1 through 3 just given.*

[4] I couldn't think of an effective liberal in Congress so I used Hightower.

Proof: Condition 1 is satisfied by definition of $C^*_{\succeq}(B)$.

Proving Condition 2 takes a bit more work. Pick an arbitrary $B \subset 2^X$, $B \neq \emptyset$. We will show that $C(B) \neq \emptyset$. Since B is an arbitrary non-empty set, this will complete the proof.

For each $x \in B$, define $P_B(x) = \{y \in B : y \succeq x\}$. By completeness, $x \in P_B(x)$ so that for all $x \in B$, $P_B(x) \neq \emptyset$. We will prove two separate claims that will complete the proof for Condition 2.

Claim A: $C^*_{\succeq}(B) = \bigcap_{x \in B} P_B(x)$.

Claim B: $\bigcap_{x \in B} P_B(x) \neq \emptyset$.

Proving Claim A is a homework problem.

To see that Claim B is true, since B is a non-empty subset of a finite set, it can be enumerated as $B = \{x_1, \ldots, x_N\}$ for some integer N. We need to show that $\bigcap_{m=1}^{N} P_B(x_m) \neq \emptyset$. We will do this by induction:

We know that $\bigcap_{m=1}^{1} P_B(x_m) \neq \emptyset$ because completeness implies that $x_1 \in \bigcap_{m=1}^{1} P_B(x_m)$. Suppose now that $\bigcap_{m=1}^{n} P_B(x_m) \neq \emptyset$. The inductive step is to show that $\bigcap_{m=1}^{n+1} P_B(x_m) \neq \emptyset$ — if we show this for any integer $n \geq 1$, we can conclude that $\bigcap_{m=1}^{N} P_B(x_m) \neq \emptyset$.

There are two cases: either $x_{n+1} \in \bigcap_{m=1}^{n} P_B(x_m)$ or $x_{n+1} \notin \bigcap_{m=1}^{n} P_B(x_m)$. In the first case, we automatically know that $\bigcap_{m=1}^{n+1} P_B(x_m) \neq \emptyset$ since it contains x_{n+1}. The second case is equivalent to

(a) $(\exists m,\ 1 \leq m \leq n)[x_{n+1} \notin P_B(x_m)]$ – by the definition of intersection, which implies

(b) $\neg[x_{n+1} \succeq x_m]$ – by the definition of $P_B(\cdot)$, which in turn implies

(c) $x_m \succ x_{n+1}$ – by completeness and the definition of $\succ$, which in turn implies

(d) $P_B(x_m) \subset P_B(x_{n+1})$ – by transitivity and the definition of $P_B(\cdot)$, which in turn implies

(e) $\bigcap_{m=1}^{n+1} P_B(x_m) = \bigcap_{m=1}^{n} P_B(x_m)$ – by the definition of intersections, which in turn implies

(f) $\bigcap_{m=1}^{n+1} P_B(x_m) \neq \emptyset$ – by the inductive assumption.

Proving Condition 3 is a homework problem. ∎

1.2.4. *Utility Function Representations of $\succeq$.* Often, choosing the most preferred object in a set is that same as picking the one with the highest utility. Indeed, this is the modern definition of utility.

Definition 1.2.4. *We say that the* **utility function** $u : X \to \mathbb{R}$ **represents** $\succeq$ *if*

$$[x \succeq y] \Leftrightarrow [u(x) \geq u(y)].$$

In terms of choice, if u represents $\succeq$, then

$$C^*_{\succeq}(B) = \{y \in B : (\forall x \in B)[u(y) \geq u(x)]\},$$

i.e. $C^*_{\succeq}(B)$ is just the set of utility maximizers in B.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, that is, $x > y$ implies $f(x) > f(y)$. Then if $u(\cdot)$ represents $\succeq$, then so does $f(u(\cdot))$. In other words, utility cannot measure a quantity in any of our usual intuitive senses of the word quantity — twice as big no longer means anything, if $x \succ y$ and $r \succ s$, we cannot use $u(x) - u(y) > u(r) - u(s)$ to say that the person likes the change from y to x more than they like the change from s to r.

Such $f(u(\cdot))$'s are called **monotonic increasing transformations**, and if I forget to say "increasing", fill it in yourself.

Note that if $u : X \to \mathbb{R}$, we can define $\succeq_u$ by

$$[x \succeq_u y] \Leftrightarrow [u(x) \geq u(y)].$$

One implication of the following is that $\succeq_u$ must be rational.

Theorem 1.2.2. *If X is finite, then $\succeq$ can be represented by a utility function if and only if $\succeq$ is rational.*

Proof: Suppose that $\succeq$ is represented by u. We must show that $\succeq$ is complete and transitive.

Completeness Pick arbitrary $x, y \in X$, we must show that $x \succeq y$ or $y \succeq x$. However, we know that

$$u(x) \geq u(y) \text{ or } u(y) \geq u(x)$$

because $u(x)$ and $u(y)$ are numbers and $\geq$ is complete. Because u representes $\succeq$, this last is equivalent to

$$x \succeq y \text{ or } y \succeq x.$$

Transitivity Pick arbitrary $x, y, z \in X$ and suppose that $x \succeq y$ and $y \succeq z$. We must show that $x \succeq z$. Because u represents $\succeq$, this is equivalent to showing that $u(x) \geq u(z)$. Because u represents $\succeq$, we know that

$$u(x) \geq u(y) \text{ and } u(y) \geq u(z).$$

This implies that $u(x) \geq u(z)$.

Note that this first half of the proof did not use the finiteness of X.

Now suppose that $\succeq$ is rational. We must show that it can be represented by a utility function.

For each $x \in X$, define $L(x) = \{y \in X : x \succeq y\}$. Define $u(x) = \#L(x)$. Verbally, $u(x)$ is the number of elements of X that x beats or ties. We must show that $x \succeq y$ if and only if $u(x) \geq u(y)$.

Suppose that $x \succeq y$. It is sufficient to show that $L(y) \subset L(x)$ because this implies $u(x) \geq u(y)$. Let z be an arbitrary point in $L(y)$. This implies that $y \succeq z$. We know that $x \succeq y$. Therefore transitivity implies that $x \succeq z$ so that $z \in L(x)$.

Now suppose that $\neg[x \succeq y]$. We must show that $\neg[u(x) \geq u(y)]$. This last statement is equivalent to $u(y) > u(x)$. By completeness, $\neg[x \succeq y]$ means that $y \succ x$. We know (from the previous step) that this means that $L(x) \subset L(y)$. We are now going to show that $L(y)$ contains strictly more points than $L(x)$. In particular, $y \in L(y)$ but $y \notin L(x)$. Therefore $\#L(y) > \#L(x)$, i.e. $u(y) > u(x)$ as required. ∎

There are two additional points to be made here: 1) we'll never observe a preference relation, *a fortiori* never observe a utility function; 2) for our purposes utility functions and preferences are going to be interchangeable (with rare exceptions discussed for their mathematical rather than economic interest).

1.3. **Choice Based Approaches.** It is sometimes argued that a drawback to the preference based approach is that no-one has ever seen a preference relation.[5] However, we have seen a great deal of choice behavior. Mathematically, the basic story of the preference based approach is to look at a set of options $B \subset X$, and to define

$$C^*_{\succeq}(B) = \{x \in B : (\forall y \in B)[x \succeq y]\}.$$

[5]This is clearly a straw man kind of argument — I've never seen an electron either.

Notation alert: $C^*_{\succeq}(B)$ and $C^*(B, \succeq)$ will be used interchangeably. $C^*(B, \succeq)$ is the set of choices that a person with preferences $\succeq$ would make if they faced the set of choices B. The choice based approach is simply an abstract version of this structure.

Definition 1.3.1. *A* **choice structure** *is a collection* $\mathcal{B} \subset 2^X$ *and a function* $C : \mathcal{B} \to 2^X$ *such that* $C(B) \subset B$.

The interpretation is going to be that, faced with the choice set $B \in \mathcal{B}$, the person with choice structure $(\mathcal{B}, C(\cdot))$ is going to make a choice in the set $C(B)$. We are going to have to make assumptions that parallel the completeness and the transitivity assumptions above.

When X is finite, the substitute for completeness takes the form $C(B) \neq \emptyset$ if $B \neq \emptyset$. It is important to note that, unless the possibility is explicitly noted, we will assume that $C(B) \neq \emptyset$ if $B \neq \emptyset$.[6]

The substitute for transitivity is called the **weak axiom of revealed preference**, WARP for short.[7] It is easier to express WARP after we have the following definitions.

Definition 1.3.2. *Fix a choice structure* $(\mathcal{B}, C(\cdot))$.

1. *A budget* $B \in \mathcal{B}$ *reveals that* x *is at least as good as* y, *written* $x \succeq^*_B y$, *if* $x, y \in B$, *and* $x \in C(B)$.
2. *A budget* $B \in \mathcal{B}$ *reveals that* x *is strictly better than* y, *written* $x \succ^*_B y$, *if* $x \succeq^*_B y$ *and* $\neg(y \succeq^*_B x)$, *i.e. if and only if* $x, y \in B$, $x \in C(B)$ *and* $y \notin C(B)$.
3. *Define the binary relation* $\succeq^*$, *"revealed at least as good as," by* $\succeq^* = \cup_{B \in \mathcal{B}} \succeq^*_B$. *Equivalently, by*
$$[x \succeq^* y] \Leftrightarrow (\exists B \in \mathcal{B})[x \succeq^*_B y.$$
4. *Define the binary relation* $\succeq^*$, *"revealed strictly better than" by* $x \succ^* y$ *if* $x \succeq^* y$ *and* $\neg(y \succeq^* x)$, *i.e. if* $(\exists B \in \mathcal{B})[x \succeq^*_B y]$, *and* $\neg(\exists B \in \mathcal{B})[y \succeq^*_B x]$. *This last line is equivalent to* $(\forall B \in \mathcal{B})[\neg(y \succeq^*_B x)]$.

We read $x \succeq^* y$ as "x is revealed at least as good as y." This is because there is a situation in which both x and y are available and x is one of the choices. In principle, $x \succeq^* y$ is observable, we just give a person a choice set $B \supset \{x, y\}$, especially $B = \{x, y\}$. Intuitively, we take $x \succeq^* y$ as evidence for the conclusion that $x \succeq y$ in the preference based approach.

Definition 1.3.3. *A choice structure satisfies* **WARP** *(the weak axiom of revealed preference) if*
$$[x \succeq^* y] \Rightarrow [\neg(\exists B \in \mathcal{B})[y \succ^*_B x]].$$

Homework 1.5.5 asks you to show that a choice structure satisfies WARP if and only if for some $B \in 2^X$, $x, y \in B$ and $x \in C^*_{\succeq}(B)$, then for any $B' \in 2^X$, if $x, y \in B'$ and $y \in C^*_{\succeq}(B')$, then $x \in C^*_{\succeq}(B')$. In other words, the internal consistency Condition 3 of Theorem 1.2.1 is just satisfaction of WARP.

[6]For infinite X some more mathematical complications need to be added, essentially restrictions on the set of B for which we require that $C(B) \neq \emptyset$.
[7]Originally from P. Samuelson's dissertation, published in 1947 as *Foundations of Economic Analysis*.

In words, if we see x chosen when y is also available in one situation, we should not see y and not x being chosen in a situation where both are available. This is a really minimal kind of rationality assumption. There is a homework problem below asking you to examine the choice structure induced by a non-transitive preference ordering. At this time, it's probably a good idea to take a look at Homework 1.5.6

If $(\mathcal{B}', C(\cdot))$ is a choice structure and $\mathcal{B} \subset \mathcal{B}'$, then $(\mathcal{B}, C_{|\mathcal{B}}(\cdot))$ is also a choice structure. It is possible that $(\mathcal{B}, C_{|\mathcal{B}}(\cdot))$ satisfies WARP even if $(\mathcal{B}', C(\cdot))$ does not — from the other side, the bigger is $\mathcal{B}'$, the more conditions that WARP imposes, that is, WARP becomes harder to satisfy. However, it is not impossible to satisfy it.

Theorem 1.3.1. *If $\succeq$ is a rational preference relation on X, then for any $\mathcal{B} \subset 2^X$, the choice structure $(\mathcal{B}, C^*(\cdot, \succeq))$ satisfies* WARP.

This is a consequence of Theorem 1.2.1.

Theorem 1.3.2. *If $(\mathcal{B}, C(\cdot))$ satisfies* WARP *and $\mathcal{B}$ contains all $1, 2,$ and 3 point subsets of X, then $\succeq^*$ is the unique rational preference relation satisfying $C(B) = C^*(B, \succeq^*)$ for all $B \in \mathcal{B}$.*

1.4. **Important Ideas and Points.** A list of the important ideas and points in this part of the course would have to include

1. binary relations,
2. rational preference relations,
3. representations of preferences by utility functions,
4. any preference represented by a utility function is rational,
5. for finite X, a preference relation is rational if and only if it can be represented by a utility function,
6. choice structures,
7. a choice structure gives rise to a binary relation called "revealed at least as good as", $\succeq^*$,
8. WARP,
9. preference based choice structures with a rational $\succeq$, i.e. $C^*(\cdot, \succeq)$, satisfy WARP,
10. if $\mathcal{B}$ is rich enough (e.g. contains all 3 point subsets of X), then a choice structure satisfying WARP is of the form $C(B) = C^*(B, \succeq^*)$.

1.5. **Homeworks.**
Due date: Tuesday September 12.
From MWG: Ch. 1, B.1-4, C.1, C.2, D.1, D.3, D.4.

Homework 1.5.1. *There are two goods, x_1 and x_2. There are only four possible combinations (bundles) are $X = \{(1,1),(1,2),(2,1),(2,2)\}$. Lexi's preferences, $\preceq$, over these bundles can be described by "More of good* 1 *is always better than any amount of good* 2*, but given two bundles with the same amount of good* 1*, more of good* 2 *is better than less." Fill in the set $\succeq = \{(x,y) \in X \times X : x \succeq y\}$.*

(2,2)				
(2,1)				
(1,2)				
(1,1)				
	(1,1)	(1,2)	(2,1)	(2,2)

Homework 1.5.2. *Graphically represent the non-transitive binary relations $\succeq$ and $\succ$ for Examples* 1.2.2 *and* 1.2.3.

Homework 1.5.3. *This question refers to Theorem* 1.2.1.

1. *Prove* Claim A*, specifically, suppose that $y \in C^*_{\succeq}(B)$ and show that $y \in \bigcap_{x\in B} P_B(x)$, then suppose that $y \in \bigcap_{x\in B} P_B(x)$ and show that $y \in C^*_{\succeq}(B)$.*
2. *Show by example that if X is infinite, then* Claim B *in this Theorem can fail.*
3. *Prove that Condition* 3 *is satisfied. You can use* Claim A *for this.*

Homework 1.5.4 (Optional)**.** *Show, by example, that there exist an infinite set X and a rational preference relation on X that cannot be represented by utility function. Thus, the finiteness in Theorem 1.2.2 is really needed.*

Homework 1.5.5. *Show that a choice structure satisfies WARP as defined in Definition* 1.3.3 *if and only if for some $B \in 2^X$, $x, y \in B$ and $x \in C^*_{\succeq}(B)$, then for any $B' \in 2^X$, if $x, y \in B'$ and $y \in C^*_{\succeq}(B')$, then $x \in C^*_{\succeq}(B')$. In other words, the internal consistency Condition* 3 *of Theorem* 1.2.1 *is just satisfaction of WARP.*

Homework 1.5.6. *This problem concerns WARP and* **Walrasian budget sets***. A Walrasian budget set is the set of goods affordable when prices are p and there is wealth w to spend. In the case of two goods, the typical Walrasian budget set is*

$$B_{p,w} = \{x \in \mathbb{R}^2_+ : p \cdot x \le w\}.$$

For this problem, assume that we have a single-valued choice structure,

$$C(B_{p,w}) = \{x(p,w)\}.$$

Throughout, assume that $p \cdot x(p,w) = w$.

1. *On the same graph, carefully draw the three budget sets $B_{p,w}$ when $(p^A, w^A) = ((4,2),72)$, $(p^B, w^B) = ((1,2),36)$, $(p^C, w^C) = ((1,1),20)$. (You may want this graph to fill as much as half a page.)*

2. *Pick three points* $x(p^A, w^A)$*,* $x(p^B, w^B)$*, and* $x(p^C, w^C)$ *that do* **not** *violate WARP. Explain your choices (recalling that* $p \cdot x(p, w) = w$*).*
3. *Pick three points* $x(p^A, w^A)$*,* $x(p^B, w^B)$*, and* $x(p^C, w^C)$ *with the property that any pair of them* **do** *violate WARP. Explain your choices (recalling that* $p \cdot x(p, w) = w$*).*
4. *If* (p^C, w^C) *is changed from* $((1, 1), 20)$ *to* $((1, 1), 30)$*, is it possible to find three points with the property that any pair of them violate WARP? Explain (recalling that* $p \cdot x(p, w) = w$*).*

Homework 1.5.7. *Suppose that* $X = \{x_1, \ldots, x_N\}$*. Single peaked preferences over* X *have a favorite option, say* x_n*, and for* $m \leq n$*,* $x_m \succ x_{m-1}$ *while for* $m > n$*,* $x_{m-1} \succ x_m$*. Preferences are strict if for all* $x \neq y$*,* $x \succ y$ *or* $y \succ x$ *(there is no indifference). There is an odd number,* M*, of people who vote on pairs of options in* X*, and each of the* M *people has strict, single peaked, rational preferences, though the peaks need not be the same. Show that the pairwise majority rule voting scheme given in Example 1.2.3 gives a rational preference ordering, and analyze the role of the median peak.*

Homework 1.5.8. *Find different choice structures for which* $\succeq^*$ *is*

1. *neither complete nor transitive,*
2. *complete but not transitive,*
3. *transitive but not complete.*

2. A Choice Structure Approach to Consumer Demand

Dates: Sept. 12, 14, & 19.
Material: MWG, Ch. 2, pp. 17-39, Appendices M.A, M.B, & M.D, pp. 926-929, 935-939.

This section specializes the abstract treatment of choice rules given above to cases of more economic interest. There are two results in this section.
(1) In the context of Walrasian budget sets, WARP implies the compensated law of demand. Since preference maximization implies WARP, this means that preference maximization implies the compensated law of demand.
(2) The compensated law of demand implies that the Slutsky matrix is negative semi-definite. From this we will derive the implication that Giffen goods must not only be inferior, they must be very inferior and staples.

Before we do this, some mathematical background is in order. With luck you will have seen most of this before. Beyond the notation for derivatives in many dimensions, the important definitions and results include homogenous functions, Euler's formula, and the negative and positive (semi)definiteness of matrices.

2.1. **Some Mathematics for Consumer Choice Structures.**

The assumption is that you have seen partial derivatives. (If you have not, then you should probably not be taking this course.) Remember that all vectors in $\mathbb{R}^N$ are column vectors (indistinguishable from an $N \times 1$ matrix), and that x^T is the $1 \times N$ transpose of the vector x. I'll try to stick to this convention, but after a while, will become somewhat slovenly.

Of particular use is the definition of the product (also known as the Cayley product), of two vectors $x = (x_1, \ldots, x_N)^T$ and y $y = (y_1, \ldots, y_N)^T$, $x \cdot y := \sum_{i=1}^N x_i y_i$. From the Pythagorean Theorem, the length of a vector x is $\|x\| = \sqrt{x \cdot x}$. From the definition of $\cos(\theta)$, $x \cdot y = \|x\| \, \|y\| \cos(\theta)$ where θ is the angle between x and y.

The notation 0 will mean either the number 0 in $\mathbb{R}$ or the vector $(0, \ldots, 0)^T$ in $\mathbb{R}^N$. You will be responsible for reasoning from context to figure out which is meant. For $x, y \in \mathbb{R}^N$, x_n is the n'th component of the vector x, we write $x \geq y$ if $x_n \geq y_n$, $n = 1, \ldots, N$, we write $x \gg y$ if $x_n > y_n$, $n = 1, \ldots, N$, The relation $\geq$ is **not** complete in $\mathbb{R}^N$ if $N \geq 2$.

2.1.1. *Matrix Notation for Derivatives and a Reminder.*

As a brief reminder, if $f : \mathbb{R} \to \mathbb{R}$ and

$$\lim_{|h| \downarrow 0} \frac{f(x^\circ + h) - f(x^\circ)}{h} = r,$$

then we say that r is the **derivative** of f at the point x°. This is written in a number of ways,

$$f'(x^\circ) = r, \ \frac{df(x^\circ)}{dx} = r, \text{ and } \frac{df(x)}{dx}_{|x=x^\circ} = r$$

being the most common. Geometrically, this means that the straight line with slope r through the point $(x^\circ, f(x^\circ))$ is a good linear approximation to the graph of the function f at the point x°. It is worth re-writing the definition with this in mind because it will be useful later. That's what the next paragraph does.

A mapping $L : \mathbb{R} \to \mathbb{R}$ is linear if for all $x, y \in \mathbb{R}$ and for all $\alpha, \beta \in \mathbb{R}$,

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

This implies that $L(0) = 0$ — take $\alpha = \beta = 0$. Taking $\beta = 0$ and $x = 1$,

$$L(\alpha) = L(\alpha 1) = \alpha L(1),$$

so that a linear map from $\mathbb{R}$ to $\mathbb{R}$ is determined by its value at 1, which is its slope. We say that f has derivative r at the point x° if, for the linear function L with $L(1) = r$,

$$\lim_{|h|\downarrow 0} \frac{|[f(x^\circ + h) - f(x^\circ)] - L(h)|}{|h|} = 0.$$

If the given limit exists, we say that f is differentiable at x°, if it exists at all x°, we say that f is **differentiable**, and for a differentiable function f, if the mapping $x^\circ \mapsto f'(x^\circ)$ is continuous, we say that f is **continuously differentiable**. For our work, I am going to blur continuous differentiability and differentiability — if I make a statement of the form "$P(f)$ is true if f is differentiable", then if it requires continuous differentiability, that is must be what I meant, but if it requires only differentiability, that must be what I meant.

When $f : \mathbb{R}^N \to \mathbb{R}^M$, there are N possible directions to move in the domain, and any movement in the domain moves in M possible directions in the range. The convention is that all vectors are column vectors, so the function f takes N-dimensional column vectors and gives us M-dimensional column vectors. This is what matrix multiplication does too, if A is an $M \times N$ matrix and x is an $N \times 1$ vector, then Ax is and $M \times 1$ vector. Multiplication by an $M \times N$ matrix A is linear mapping from $\mathbb{R}^N$ to $\mathbb{R}^M$, i.e.

$$(\forall x, y \in \mathbb{R}^N)(\forall \alpha, \beta \in \mathbb{R})[A(\alpha x + \beta y) = \alpha Ax + \beta Ay].$$

It can be shown that all linear maps from $\mathbb{R}^N$ to $\mathbb{R}^M$ can be represented by $M \times N$ matrixes, and you should remember a number of lessons about choosing a basis.

The function f is **differentiable at** x° **with derivative** A if

$$\lim_{\|\eta\|\downarrow 0} \frac{\|[f(x^\circ + \eta) - f(x^\circ)] - A\eta\|}{\|\eta\|} = 0$$

where $\eta \in \mathbb{R}^N$.

Let e_n be the unit vector in the n'th direction in $\mathbb{R}^N$. A particularly interesting class of η's with $\|\eta\| \downarrow 0$ can be had by looking at $\eta = h \cdot e_n$ where $|h| \downarrow 0$. For $x \in \mathbb{R}^N$ and $m \in \{1, \ldots, M\}$, let $f_m(x)$ be the m'th component of the vector $f(x)$. If the limit

$$\lim_{|h|\downarrow 0} \frac{f_m(x^\circ + he_n) - f_m(x^\circ)}{h} = r_{m,n}$$

exists, it is called the **partial derivative** of f_m at x° in the n direction. It is written

$$\frac{\partial f_m(x^\circ)}{\partial x_n},$$

and there are variant notations that you (probably) will not see from me. It is the (m, n)'th entry in the matrix representing the derivative of f at x°.

The function f is **differentiable at** x° if all partial derivatives exist at all x°. Note that differentiability at x° requires that f be defined for all y such that $\|y - x^\circ\| < \epsilon$ for some strictly positive ϵ. This means that if I assume something is differentiable at a point x°, then I must have assumed that either there exists an $\epsilon > 0$ such that $\|y - x^\circ\| < \epsilon$ implies that f is defined at y. If the mappings from x° to $\frac{\partial f_m(x^\circ)}{\partial x_n}$ are all continuous, then the function is **continuously differentiable**. This is what I mean when I just say "differentiable."

The notation for the $M \times N$ matrix of derivatives at a point x is $Df(x)$. That is, for every $x \in \mathbb{R}^N$ there is an $M \times N$ matrix whose (m,n)'th entry is $\partial f_m(x^\circ)/\partial x_n$. (Remember that an $M \times N$ matrix is a rectangular array of numbers having M rows and N columns. Below we will talk about the logic of arranging numbers into these kinds of blocks.)

An example may well help things go down more smoothly.[8] Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$,

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1 \cdot x_2 + x_3 \\ x_3^2(x_1 + 2x_2) \end{pmatrix}.$$

The derivative of f at x° is the linear function given by multiplication by the 2×3 matrix

$$Df(x^\circ) = \begin{pmatrix} \frac{\partial f_1(x^\circ)}{\partial x_1} & \frac{\partial f_1(x^\circ)}{\partial x_2} & \frac{\partial f_1(x^\circ)}{\partial x_3} \\ \frac{\partial f_2(x^\circ)}{\partial x_1} & \frac{\partial f_2(x^\circ)}{\partial x_2} & \frac{\partial f_2(x^\circ)}{\partial x_3} \end{pmatrix} = \begin{pmatrix} x_2^\circ & x_1^\circ & 1 \\ (x_3^\circ)^2 & 2(x_3^\circ)^2 & 2x_3(x_1^\circ + 2x_2^\circ) \end{pmatrix}.$$

The mapping from x° to $Df(x^\circ)$ is as continuous. If the point $x^\circ = (7,5,1)^T$, then

$$Df(x^\circ) = \begin{pmatrix} 5 & 7 & 1 \\ 1 & 2 & 34 \end{pmatrix}.$$

Thus, if we move from $x^\circ = (7,5,1)^T$ to $x^\circ + dx$, $dx = (\epsilon, \delta, \gamma)^T$, df, the change in f, moving along the tangent plane to f at x° is

$$Df(x^\circ)\begin{pmatrix} \epsilon \\ \delta \\ \gamma \end{pmatrix} = \begin{pmatrix} 5 & 7 & 1 \\ 1 & 2 & 34 \end{pmatrix}\begin{pmatrix} \epsilon \\ \delta \\ \gamma \end{pmatrix} = \begin{pmatrix} 5\epsilon + 7\delta + 1\gamma \\ 1\epsilon + 2\delta + 34\gamma \end{pmatrix}.$$

Switching notation so that $dx = (dx_1, dx_2, dx_3)^T$,

$$Df(x^\circ)\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 1 \\ 1 & 2 & 34 \end{pmatrix}\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} 5dx_1 + 7dx_2 + 1dx_3 \\ 1dx_1 + 2dx_2 + 34dx_3 \end{pmatrix}.$$

When the vector dx is small, we will write

$$f(x^\circ + dx) \simeq f(x^\circ) + Df(x^\circ)dx,$$

which, in more familiar formal notation, means that

$$\lim_{\|dx\| \downarrow 0} \frac{\|[f(x^\circ + dx) - f(x^\circ)] - Df(x^\circ)dx\|}{\|dx\|} = 0.$$

If $f : \mathbb{R}^{N+K} \to \mathbb{R}^M$ is of the form $(x,y) \mapsto f(x,y)$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}^K$, then $D_x f(x,y)$ is the $M \times N$ matrix whose (m,n)'th entry is $\partial f_m(x,y)/\partial x_n$. In the previous example, if

[8] Mary Poppins said the same thing, but more musically.

$x = (x_1, x_2)^T$, $y = x_3$, then

$$D_x f(x^\circ) = \begin{pmatrix} \frac{\partial f_1(x^\circ)}{\partial x_1} & \frac{\partial f_1(x^\circ)}{\partial x_2} \\ \frac{\partial f_2(x^\circ)}{\partial x_1} & \frac{\partial f_2(x^\circ)}{\partial x_2} \end{pmatrix}.$$

2.1.2. *Homogenous Functions and Euler's Formula.*

Notation: $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x \geq 0\}$, $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : x \gg 0\}$.

Homogenous functions are defined either on $\mathbb{R}^N_+$ or $\mathbb{R}^N_{++}$.

Definition 2.1.1. *A function is* **homogenous of degree** r **(hd**(r)**)** *if for all $t > 0$ and all $x \in \mathbb{R}^N_+$, (or $\mathbb{R}^N_{++}$), $f(tx) = t^r f(x)$.*

Example: $f(x_1, x_2) = x_1 \cdot x_2$ is hd(2). From intermediate microeconomics you should remember that if $x_n(p, w)$ is the demand for good n at prices p and income w, then $x_n(\cdot, \cdot)$ is hd(0). This means that hd(0) is going to be important, so let's look at it in a bit more detail.

Suppose that $f(x) = f(x_1, \ldots, x_N)$ is hd(0). Then

$$f(\frac{1}{x_1}x) = f(1, \frac{x_2}{x_1}, \ldots, \frac{x_N}{x_1}) = \left(\frac{1}{x_1}\right)^0 f(x) = f(x).$$

In other words, all that $f(x)$ depends on are the $N-1$ ratios $(\frac{x_2}{x_1}, \ldots, \frac{x_N}{x_1})$. In particular, for the demand functions $x_n(p_1, \ldots, p_N, w)$,

$$x(p, w) = x(\frac{p_1}{w}, \ldots, \frac{p_N}{w}, 1) = x(1, \frac{p_2}{p_1}, \ldots, \frac{p_N}{p_1}, \frac{w}{p_1}).$$

The second term sets $w = 1$ and measures all prices as a proportion of income, the third term sets the price of good 1 equal to 1 and measures all prices as prices relative to p_1. We call good 1 the **numeraire** in this case.

From intermediate microeconomics you should also remember that we often assume that production functions are hd(1). The following result implies that the marginal products are hd(0).

Theorem 2.1.1. *If f is hd(r), then $\partial f / \partial x_n$ is hd$(r-1)$.*

Proof: For any $t > 0$, $f(tx) - t^r f(x) \equiv 0$ is an equivalence in x, take derivatives on both sides w.r.t. x_n and rearrange. ∎

Note that if the price paid to a factor of production is its marginal product in the hd(1) production function example, then, because marginal products are hd(0), proportional increases in all inputs of production have no effects on the prices paid to factors of production.

The next result is **Euler's formula**.

Theorem 2.1.2. *If f is hd(r), then at any $\overline{x}$,*

$$\underbrace{D_x f(\overline{x})}_{1\times L} \underbrace{\overline{x}}_{L\times 1} = \underbrace{r f(\overline{x})}_{1\times 1}.$$

Proof: Take the derivative w.r.t. t and evaluate at $t = 1$ in the previous proof. ∎

Going back to production function examples from intermediate microeconomics, if $D_x f(x^\circ) = p$ (where p is the vector of prices of factors of production), then Euler's formula implies that

$p \cdot x^\circ = f(x^\circ)$. In other words, if the wage rates paid to the factors of production are equal to their marginal value products, then the wage bill exactly account for the output. The assumption about wage rates is a **very** bad assumption empirically, but it is so seductive theoretically that it has had all too permanent an effect on economists' analyses.

We'll see other uses of these two results in our analysis of demand behavior.

2.1.3. *Matrices, Definite and Other.* One of the major results that we are going to derive in this section is that the Slutsky substitution matrix is negative semi-definite for any reasonable (read differentiable) choice structure approach to demand behavior. We're going to take a longer-than-strictly-necessary detour through some matrix algebra.

An $M \times N$ matrix A is a collection of $M \cdot N$ numbers arranged in a rectangular box having M rows and N columns,

$$\underbrace{A}_{M \times N} = \begin{bmatrix} a_{1,1} & \dots & a_{1,N} \\ \vdots & \dots & \vdots \\ a_{M,1} & \dots & a_{M,N} \end{bmatrix}.$$

The i, j'th entry in the matrix A is $a_{i,j}$. This is the number in the i'th row and the j'th collum. The matrix can also be seen as a collection of M horizontal $1 \times N$ matrices $A_{i,\cdot}$, $i = 1, \dots, M$, or as a collection of N vertical $M \times 1$ matrices $A_{\cdot,j}$, $j = 1, \dots, N$.

We can add $M \times N$ matrices using the definition $C = A + B$ where $c_{i,j} = a_{i,j} + b_{i,j}$. Multiplication is more interesting, and is the major reason that matrices are so useful.

If A is an $M \times N$ matrix and x is a vector in $\mathbb{R}^N$, that is, x is an $N \times 1$ matrix, then Ax is the $M \times 1$ matrix (vector in this case) whose $i, 1$'th entry is $A_{i,\cdot} \cdot x$.

If A is an $M \times N$ matrix and B is an $N \times P$ matrix, then the (Cayley) product of the matrices A and B is denoted by AB is the $M \times P$ matrix whose i, j'th entry is $A_{i,\cdot} \cdot B_{\cdot,j}$.

It is important to remember that we write matrices and matrix products as if they were numbers, but they are not! Two examples:

1. If A is 2×3 and B is 3×2, then AB is 2×2 while BA is 3×3. Thus, $AB \neq BA$. You can (and should) find $N \times N$ matrices A and B for which $AB \neq BA$.
2. Let $0_{(2)}$ be the 2×2 matrix with 0's in each place. It is easy to find non-zero, 2×2 matrices A and B such that $AB = BA = 0_{(2)}$.

2.1.4. *A Preview of Linear Regression.* Suppose that we believe that the N values of a variable y that we observe depend on M other variables, $x_{i,m}$, $m = 1, \dots, M$, and the dependence is random and of the form

$$y_i = \beta_0^\circ + \beta_1^\circ x_{i,1} + \dots + \beta_M^\circ x_{i,M} + \epsilon_i$$

where the ϵ_i are independent, unobserved random variables having mean 0 and equal (finite) variance. Suppose also that we do not know what the β_i°'s are but are interested in finding a good guess as to their values. Stacking the y_i into an $N \times 1$ vector Y, stacking the ϵ_i into an $N \times 1$ vector ϵ, letting X denote the $N \times (M + 1)$ matrix

$$X = \begin{bmatrix} 1 & x_{1,1} & \dots & x_{1,M} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N,1} & \dots & x_{M,N} \end{bmatrix},$$

and letting β° be the $(M+1) \times 1$ matrix $(\beta_0^\circ, \beta_1^\circ, \ldots, \beta_M^\circ)^T$, we can write all of the dependences as

$$\underbrace{Y}_{N\times 1} = \underbrace{\underbrace{X}_{N\times(M+1)} \underbrace{\beta^\circ}_{(M+1)\times 1}}_{N\times 1} + \underbrace{\epsilon}_{N\times 1}.$$

Now, the problem is to find the vector β° when all that we observe are is Y and X. This is impossible, after all the ϵ_i are never observed.[9] We can, however, think about making some good guesses.

Note that $\epsilon = Y - X\beta^\circ$, so that $\epsilon^T\epsilon = (Y - X\beta^\circ)^T(Y - X\beta^\circ)$. Each guess for β gives a guess, $e(\beta)$, for ϵ, $e(\beta) = Y - X\beta$. One way to go about guessing a value for β° is to pick the β that minimizes

$$f(\beta) = \|e(\beta)\|^2 = e(\beta)\cdot e(\beta) = (Y - X\beta)^T(Y - X\beta).$$

There is some clear geometry here — find the β that makes the length of the error vector as small as possible. This corresponds to one interpretation of making the $x_{i,m}$'s do as much explaining of the y_i's as is possible. There are other reasonable interpretations of "making the $x_{i,m}$'s do as much explaining of the y_i's as is possible", e.g. picking β to minimize other measures of the length of the vector $e(\beta)$.

Note that $f : \mathbb{R}^{M+1} \to \mathbb{R}^1$ so that the derivative of f with respect to β is a $1 \times (M+1)$ matrix. Minimizing this involves setting the derivative equal to 0. To solve this problem, we need matrix inverses and the appropriate generalization of the second derivative test from elementary calculus.

If I_N is the $N \times N$ matrix with 1's down the diagonal and 0's off the diagonal, than $AI = IA$ for any $N \times N$ matrix A. If A and B are $N \times N$ matrices, and if $BA = AB = I$, then we write $B = A^{-1}$. Notice that if we know A, then there are N^2 equations in N^2 unknowns in the equation $AB = I$. Further, the equations are linear in the unknowns so we can solve this unless there is some degeneracy in the system.

Suppose that we know that x satisfies $Ax = b$. Suppose we then go out and find A^{-1}. Then we know that

$$A^{-1}(Ax) = A^{-1}b.$$

But this is the same as

$$Ix = A^{-1}b,$$

and $Ix = x$, so that we have expressed the solution as a matrix multiple of b. Now go and do Homeworks 2.7.12, 2.7.13, and 2.7.14.

Another aspect of $N \times N$ matrices is their definiteness. This is sort of like being a negative or a positive number.

Definition 2.1.2. *An $N \times N$ matrix A is* **negative semi-definite** *if $x^TAx \le 0$ for all $x \in \mathbb{R}^N$. It is* **negative definite** *if $x^TAx < 0$ for all $x \ne 0$.*

[9]Well, we might do it if the ϵ_i are degenerate random variables, that is, if there is absolutely no error in any of our observations. This means that we'll never do it even approximately in economics.

This is going to be important for a number of reasons. One of them is Taylor's theorem, if $f : \mathbb{R}^N \to \mathbb{R}$ is twice continuously differentiable, then for any x° and $x \simeq x^\circ$,

$$f(x) \simeq f(x^\circ) + Df(x^\circ)(x - x^\circ) + \tfrac{1}{2}(x - x^\circ)^T D^2 f(x^\circ)(x - x^\circ).$$

This can be re-written, and it may be that you have seen this formula with $dx = x - x^\circ$ so that

$$f(x^\circ + dx) - f(x^\circ) \simeq Df(x^\circ)dx + \tfrac{1}{2}dx^T D^2 f(x^\circ)dx.$$

If x° is a point where $Df(x^\circ) = 0$, then $D^2 f(x^\circ)$ being negative definite implies that $f(x) < f(x^\circ)$ for x close to x°. This is the multivariate version of the second derivative test for a local maximum.

Suppose that A is negative definite. Then for all x, $x^T A x = x^T(\frac{1}{2}A^T + \frac{1}{2}A)x$, and the matrix $(\frac{1}{2}A^T + \frac{1}{2}A)$ is symmetric.

Theorem 2.1.3 (Diagonalization)**.** *An $N \times N$ matrix A is a symmetric and negative definite if and only if there exists a matrix B such that $B^T B = I$, and a diagonal matrix Λ,*

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots\dots\dots & & & 0 \\ 0 & \lambda_2 & 0 & \dots\dots & & 0 \\ 0 & 0 & \lambda_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots\dots & & 0 & \lambda_{N-1} & 0 \\ 0 & \dots\dots & & & 0 & \lambda_N \end{bmatrix}$$

with each $\lambda_n < 0$ and $A = B^T \Lambda B$.

In particular, being negative definite implies the existence of an inverse, $A^{-1} = B^T \Lambda^{-1} B$. An observation that will matter to you in your coursework on linear regression is that if A is symmetric and positive definite, then you can define its square root as $B^T \sqrt{\Lambda} B$. If you allow imaginary numbers as entries in your matrices, then you can define the square root of any symmetric matrix.

2.2. **Commodities and Budget Sets.** There are L commodities. Okay, so this is a simplification, but along which dimensions? Let us think about commercial radio or television stations, what are they selling/producing? Insurance companies? The company that makes Rolex watches? Designer jeans?

Given that we have now assumed that there are a finite number, L, of commodities, the possible levels of consumption of each commodity are (to be catholic) $\mathbb{R}$, so that the possible commodity bundles are $\mathbb{R}^L$. Sometimes it is sensible to restrict the set of feasible bundles to some $X \subset \mathbb{R}^L$. Examples include (at least) integer constraints and capacity constraints.

The classical X is $\mathbb{R}^L_+$, the Walrasian budget set is $B_{p,w} = \{x \in X : p \cdot x \leq w\}$. Usually we restrict ourselves to $p \gg 0$ and $w > 0$. Draw various changes in $B_{p,w}$ for the case $L = 2$.

Many interesting empirical questions can not be answered using simple Walrasian budget sets:

1. Think about overtime and increasing tax rates.
2. Think about the decision to take a low-paying job w/o benefits as opposed to being covered by Medicaid.
3. Think about getting price breaks as you consume more.
4. Think about access to prescription pain-killers (the price goes up a great deal once you get beyond what the doctor is willing to prescribe for you).

2.3. **Demand Functions as Choice Functions.** The set $\mathcal{B}^W = \{B_{p,w} : p \gg 0,\ w > 0\} \subset 2^X$ so the Walrasian budget sets form the first part of a choice structure. We are going to assume (for simplicity) that the choice rule $C : \mathcal{B}^W \to 2^X$ always gives a singleton set. This means that there is a function $x(p,w)$ such that $C(B_{p,w}) = \{x(p,w)\}$. The function $x(p,w)$ is the demand function.

Theorem 2.3.1. *Any Walrasian choice structure $(\mathcal{B}^W, C(\cdot))$ is hd(0).*

Proof: For any $\alpha > 0$, $B_{\alpha p, \alpha w} = B_{p,w}$. ∎

In particular, by having the choice function $C(\cdot)$ depend only on the budget sets, we are assuming that our consumers do not suffer from money illusion.

Since $0 \in B_{p,w}$ (assuming that $X = \mathbb{R}^L_+$) for any p and w, $x(p,w) \equiv 0$ is (theoretically) possible. This is an extreme version of asceticism. We are going to rule it out because we are interested in the behavior of people who like something, anything.

Definition 2.3.1. *A Walrasian choice structure $(\mathcal{B}^W, C(\cdot))$ satisfies* **Walras' law** *if for all $p \gg 0$, $w > 0$, $p \cdot x(p,w) = w$.*

This should be thought of as a law that holds over a lifetime rather than period by period, sometimes we borrow and consume more than present income, sometimes we save and consume less than present income, but eventually we spend it all, either on our own consumption or the consumption of our heirs. With this interpretation, we have even included the extreme ascetics.

2.4. **Comparative Statics.** We are going to vary $B_{p,w}$ around and see how $C(B_{p,w})$ varies under the maintained assumption that the choice structure satisfies Walras' Law. To do this, we are going to assume (for convenience mostly) that $x(p,w)$ is differentiable. Being that we've already assumed single-valued, continuity and differentiability are only small steps.

Definition 2.4.1. *The* **Engel function at** $\overline{p}$ *is the mapping $w \mapsto x(\overline{p}, w)$. The set $E_{\overline{p}} = \{x(\overline{p}, w) : w > 0\}$ is the* **wealth expansion path**.

Definition 2.4.2. *Commodity ℓ is* **normal at** (p,w) *if*

$$\frac{\partial x_\ell(p,w)}{\partial w} \geq 0.$$

Commodity ℓ is **inferior at** (p,w) *if*

$$\frac{\partial x_\ell(p,w)}{\partial w} < 0.$$

Demand is **normal at** (p,w) *if*

$$D_w x(p,w) \geq 0.$$

Draw the various cases.

The matrix $D_p x(p,w)$ contains the own-price effects and the cross-price effects.

Definition 2.4.3. *Commodity* ℓ *is a* **Giffen good** *if*

$$\frac{\partial x_\ell(p,w)}{\partial p_\ell} > 0.$$

Note that a good can be Giffen for some levels of w and not be Giffen at others. We think Giffen goods are atypical. We are going to see that for a good to be a Giffen good it must be inferior, indeed, it must be very inferior. The typical story of a Giffen good is . . . . This involves a conflation of price and income effects.

Theorem 2.4.1. $\underbrace{D_p x(p,w)}_{L\times L}\underbrace{p}_{L\times 1} + \underbrace{D_w x(p,w)}_{L\times 1}\underbrace{w}_{1\times 1} = \underbrace{0}_{L\times 1}.$

Proof: $x(p,w)$ is hd(0), apply Euler's rule. ∎

Detour: A 2 minute coverage of elasticities as unit-free measures of responsiveness. If $y = f(x)$ and both y and x measure something where units matter, then $dy/dx = f'(x)$ has units top and bottom. On the other hand, the **elasticity of** y **with respect to** x

$$\epsilon_{y,x} = \frac{dy/y}{dx/x} = \frac{dy}{dx}\frac{x}{y}$$

has no units. It is an exercise from elementary calculus to show that

$$\epsilon_{y,x} = \frac{d\ln f}{d\ln x},$$

which explains why you will see so many regressions of the form

$$\ln Y = \beta \ln X + \text{noise},$$

the estimated β's from this regression give the elasticity of Y with respect to X. You will calculate a number of elasticities later on.

In particular,

$$\epsilon_{\ell,k} := \frac{\partial x_\ell(p,w)}{\partial p_k}\frac{p_k}{x_\ell(p,w)}$$

and

$$\epsilon_{\ell,w} := \frac{\partial x_\ell(p,w)}{\partial w}\frac{w}{x_\ell(p,w)}$$

leads to

$$\sum_{k=1}^{L} \epsilon_{\ell,k} + \epsilon_{\ell,w} = 0.$$

Interpret.

Noting that $p \cdot x(p,w) \equiv w$ (by assumption), and taking derivatives on both sides with respect to p and then w, we can conclude that

$$pD_p x(p,w) + x(p,w)^T = 0^T,$$

and

$$pD_w x(p,w) = 1.$$

Letting $b_\ell(p,w) = p_\ell x_\ell(p,w)/w$, there are two famous elasticity formulas for the above equations. These are important because economists are forever going out and estimating elasticities, and these restrictions had better hold if the estimations are going to make any sense.

2.5. **WARP.** Let us translate WARP to the Walrasian context (remember that WARP is so general an idea that it can be used in any of the non-Walrasian budget sets discussed above). Recall, a choice structure satisfies WARP if

$$[x \succeq^* y] \Rightarrow \neg(\exists B \in \mathcal{B})[y \succ_B^* x].$$

For Walrasian structures, this translates to

If $p \cdot x(p',w') \leq w$ and $x(p',w') \neq x(p,w)$, then $p' \cdot x(p,w) > w'$.

Draw the pictures and talk through the cases to convince yourself that this is in fact the same old WARP. Remember, what we showed in the previous section was that $C^*(\cdot,\succeq)$ satisfies WARP (and that part of the proof did not rely on the finiteness of X), so that anything we manage to show as an implication of WARP ends up also being an implication of preference maximization.

Theorem 2.5.1 (The Compensated Law of Demand)**.** *Suppose that $x(p,w)$ is single valued, $hd(0)$ and satisfies Walras' law. Then $x(p,w)$ satisfies WARP if and only if for all (p,w), (p',w') such that $w' = p' \cdot x(p,w)$,*

$$(p'-p) \cdot [x(p',w') - x(p,w)] \leq 0,$$

with strict inequality when $x(p',w') \neq x(p,w)$.

Before we give a proof of the WARP $\Rightarrow$ Compensated Law of Demand part of this, let us examine the case where only one price changes in moving from p to p', say the price of good ℓ . . . . The picture for this is very informative.

Proof: The proof in that WARP $\Rightarrow$ Compensated Law of Demand is quite easy. If $x = x'$, then $(p'-p)(x'-x) = 0$. If $x \neq x'$, then we must show that $(p'-p)(x'-x) < 0$. Rewriting,

$$(p'-p)(x'-x) = p'(x'-x) - p(x'-x) = (w-w) - (px'-w) = w - px'.$$

But if $px' \leq w$ then WARP implies that x' rather than x should have been chosen at (p,w). ∎

This result is known as the **compensated law of demand**, roughly it says that prices and quantities move in opposite directions. Letting $\Delta p = (p'-p)$ and $\Delta x = (x'-x)$, the inequality can be rephrased as $\Delta p \cdot \Delta x \leq 0$. When we send the Δp and Δx to 0, we get the differential formulation that $dp \cdot dx \leq 0$.

We are going to back into the next result. If you had a good intermediate micro-economics class you might remember that the Slutsky substitution matrix is symmetric and negative semi-definite no matter what the utility function. We haven't introduced preferences yet, and all that we can get is that the Slutsky matrix is negative semi-definite.

Start at prices and income (p, w). Change prices to $p + dp$, dp a tiny vector. Change the wealth to $w + dw$, where dw is the tiny (1-dimensional) vector $dw = x(p, w) \cdot dp$, so that the old bundle is still affordable. For any infinitesimal (read tiny) dp and dw, the change in x is

$$\underbrace{dx}_{L\times 1} = \underbrace{D_p x(p,w)}_{L\times L} \underbrace{dp}_{L\times 1} + \underbrace{D_w x(p,w)}_{L\times 1} \underbrace{dw}_{1\times 1}.$$

For our particular case where $dw = x(p, w) \cdot dp$, we have

$$\underbrace{dx}_{L\times 1} = [\underbrace{D_p x(p,w)}_{L\times L} + \underbrace{D_w x(p,w)}_{L\times 1} \underbrace{x(p,w)^T}_{1\times L}] \underbrace{dp}_{L\times 1}.$$

But we know that $dp^T dx \leq 0$, so that

$$\underbrace{dp^T}_{1\times L} [\underbrace{D_p x(p,w)}_{L\times L} + \underbrace{D_w x(p,w)}_{L\times 1} \underbrace{x(p,w)^T}_{1\times L}] \underbrace{dp}_{L\times 1} \leq \underbrace{0}_{1\times 1}.$$

Let us name the matrix is the square brackets, calling it S for the Slutsky substitution matrix,

$$\underbrace{S}_{L\times L} = [\underbrace{D_p x(p,w)}_{L\times L} + \underbrace{D_w x(p,w)}_{L\times 1} \underbrace{x(p,w)^T}_{1\times L}].$$

The vector dp was arbitrary, meaning that we have established

Theorem 2.5.2. *Slutsky matrix S is negative semi-definite.*

We have not established that S is negative definite, indeed, we cannot do this. Suppose that dp is a small multiple of p so that moving to $p + dp$ corresponds to multiplying all prices by some constant. When we move w so that the old bundle is still affordable, we have not changed the budget set. This implies that $dx = 0$ in this case so that $dp^T S dp = dp^T dx = dp^T 0 = 0$. This comes directly from the fact that $x(p, w)$ is hd(0).

The ℓ, k element of S is

$$s_{\ell,k}(p,w) = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w).$$

The thing to understand is that $s_{\ell,k}(p, w)$ measures the differential change in the demand for commodity ℓ for a differential change in the price of commodity k that preserves the ability to consume the old bundle. The answer is **not** $\frac{\partial x_\ell(p,w)}{\partial p_k}$ because, algebraically, we are assuming that $p \cdot x(p, w) \equiv w$. This means that a change dp_k in the price of commodity k cannot be isolated the way that the partial derivative $\frac{\partial x_\ell(p,w)}{\partial p_k}$ is defined.

The classical infinitesimal analysis of a change dp_k in p_k is two part: 1) it leads to a change $(\frac{\partial x_\ell(p,w)}{\partial p_k}) dp_k$ if w is unchanged, and 2) it leads to a change $x_k(p, w) dp_k$ in w if the consumer is to be able to just afford the old bundle (the $p \cdot x(p, w) \equiv w$ part), meaning $(\frac{\partial x_\ell(p,w)}{\partial w}) x_k(p, w) dp_k$.

2.5.1. *Giffen Goods.* It is time to return to Giffen goods, remember, these are goods that satisfy

$$\frac{\partial x_\ell(p,w)}{\partial p_\ell} > 0.$$

Let us look at $s_{\ell,\ell}(p,w)$. Because S is negative semi-definite, we know that $s_{\ell,\ell} \leq 0$. Writing this out,

$$\frac{\partial x_\ell(p,w)}{\partial p_\ell} + \frac{\partial x_\ell(p,w)}{\partial w} x_\ell(p,w) \leq 0.$$

If good ℓ is a Giffen good, then it must be the case that

$$\frac{\partial x_\ell(p,w)}{\partial w} x_\ell(p,w) < 0.$$

In other words, Giffen goods must be not only inferior, but inferior enough to outweigh $\frac{\partial x_\ell(p,w)}{\partial p_\ell}$.

2.5.2. *An Example Due to Hicks.* Finally, there is Hicks' example that suggests that this choice based theory is different than a preference based theory. In Hicks' example, there are three bundles and prices arranged so that

$$x_1 \succ^* x_3 \succ^* x_2 \succ^* x_1.$$

If there were preferences $\succeq$ such that $C(\cdot) = C^*(\cdot, \succeq)$, then it seems that they should satisfy

$$x_1 \succ x_3 \succ x_2 \succ x_1.$$

However, this means that $\succeq$ violates transitivity. More specifically, the price wealth pairs and $(p^1, w) = ((2,1,2),8)$, $(p^2, w) = ((2,2,1),8)$, and $(p^3, w) = ((1,2,2),8)$ and the respective unique choices are $x^1 = (1,2,2)$, $x^2 = (2,1,2)$, $x^3 = (2,2,1)$. Draw the pictures. This is suggestive only. *Ad astra.*

2.6. **Important Ideas and Points.** A list of the important ideas and points in this part of the course would have to include, as a minimum,

1. budget sets, Walrasian and non-Walrasian,
2. homogeneity and Euler's Law,
3. comparative statics,
4. Engel functions,
5. wealth expansion paths,
6. normal and inferior goods,
7. WARP for Walrasian demands,
8. compensated law of demand,
9. the Slutsky substitution matrix is negative semi-definite

2.7. **Homeworks.**
Due date: Tuesday Sept. 26.
From MWG: Ch.2: D.2, 4; E.1-4, 8; F.2, 4, 5, 8, 10, 17.

Homework 2.7.1. *Show that when N is greater than or equal to 2, $\geq$ is not a complete relation on $\mathbb{R}^N$.*

Homework 2.7.2. *Show that $x \cdot y = \|x\| \, \|y\| \cos(\theta)$ where θ is the angle between x and y.*

Homework 2.7.3. *Find Ax when A is the 2×3 matrix*

$$\begin{bmatrix} 2 & 3 & 2 \\ 9 & 6 & 2 \end{bmatrix},$$

and

1. $x = (2, 1, 4)^T$,
2. $x = (1, -5, 0)^T$,
3. $x = (-3, 5, 8)^T$.

Homework 2.7.4. *Using the matrix A from the previous problem find*

1. $\{x : Ax = (2, 2)^T\}$,
2. $\{x : Ax \leq (3, 2)^T\}$,
3. $\{x : Ax \gg (0, 0)^T\}$.

Homework 2.7.5. *Use the matrix A from the previous problem.*

1. *Find the 2×2 matrix AA^T.*
2. *Find the 3×3 matrix $A^T A$.*
3. *Show that one of these matrices is positive definite while the other is only positive semi-definite.*
4. *Find a non-zero 2×3 matrix having both AA^T and $A^T A$ being positive semi-definite but not positive definite.*

Homework 2.7.6. *Find $N \times N$ matrices A and B for which $AB \neq BA$. Also, let $0_{(2)}$ be the 2×2 matrix with 0's in each place and find non-zero, 2×2 matrices A and B such that $AB = BA = 0_{(2)}$.*

Homework 2.7.7. *Show that if A is an $N \times N$, symmetric, negative definite matrix, then A^{-1} is an $N \times N$, symmetric, negative definite matrix. [Theorem 2.1.3 is helpful here.]*

Homework 2.7.8. *Suppose that $g(x_1, x_2) = x_1^2 + x_2$, $h(x_1, x_2) = x_1^2 \cdot e^{x_2}$, and $f(x_1, x_2) = g(x_1, x_2)h(x_1, x_2)$. Verify the product rule for the derivative of f using the matrix notation in eqn.* M.A.2.

Homework 2.7.9. *Suppose $g(x_1, x_2) = (x_1, x_2)^T$, $h(x_1, x_2) = (x_2, x_1)^T$, and $f(x_1, x_2) = g(x_1, x_2) \cdot h(x_1, x_2)$. Verify the product rule for the derivative of f using the matrix notation in eqn.* M.A.3.

Homework 2.7.10. *Suppose that $\alpha(x) = 9x^3$, $g(x) = (x, x^2)^T$ and $f(x) = \alpha(x)g(x)$. Verify the product rule for the derivative of f using the matrix notation in eqn.* M.A.4.

Homework 2.7.11. *Show that the matrix*

$$A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$$

does not have an inverse directly using the definition of an inverse. (For those of you with a background in linear algebra, showing that the determinant is 0 is not what I'm after here.)

Homework 2.7.12. *Suppose that* $Ax = b$ *where* A *is the* 2×2 *matrix*

$$\begin{bmatrix} 3 & 13 \\ 13 & 69 \end{bmatrix}.$$

Find x *in terms of* b *by explicitly finding* A^{-1}.

Homework 2.7.13. *Suppose that*

$$Y = \begin{bmatrix} 1.8 \\ 3.3 \\ 4.4 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{bmatrix},$$

and that $\beta = (\beta_0, \beta_1)^T$. *Solve the problem*

$$\min_{\beta} \underbrace{(Y - X\beta)^T}_{1\times 3} \underbrace{(Y - X\beta)}_{3\times 1},$$

and give the linear regression and the projection geometric interpretations of what you are doing.

Homework 2.7.14. *Given an* $(M + 1) \times N$ *matrix* X, *and an* $N \times 1$ *matrix* Y, *find the* $(M + 1) \times 1$ *matrix* β *that minimizes*

$$f(\beta) = (Y - X\beta)^T (Y - X\beta).$$

The steps include

1. *Find* $Df(\beta)$ *and give the equation* $Df(\beta) = 0$.
2. *Show that* f *is convex by showing that* $D^2 f$ *is positive semi-definite.*
3. *Show that if the matrix* X *is of full rank, then* f *is strictly convex by showing that* $D^2 f$ *is positive definite.*
4. *Solve the equation* $Df(\beta) = 0$ *for* β *as a function of the matrices* Y *and* X *assuming that* X *is of full rank.*

Homework 2.7.15. *Suppose that prices change from* p *to* p' *between period* 1 *and period* 2. *In period* 1, *the consumer has wealth* w *and chooses* $x(p, w)$ *satisfying Walras' Law and WARP. Suppose that in period* 2 *the consumer has wealth* $w' = p' \cdot x(p, w)$. *Show that either* $x(p', w') = x(p, w)$ *or else* $x(p', w')$ *is revealed strictly prefered to* $x(p, w,)$.

Dates: Sept. 21, 26, 28, Oct. 3, 5, & 10.
Material: MWG, Ch. 3 except sections F (to be covered later) and H (to be ignored), Appendices M.C, M.E-G, M.J-L.

There are two homework assignments for this Chapter/§. The first one, on the generalities of constrained optimization, is scattered throughout §3.1. It has about 50 questions. Aside from the ones marked as optional, the homework from Mathematics for Maximization is due Oct. 5. The homework on the specifics of utility maximization that economists pay attention to is due Tuesday Oct. 12.

This section specializes the abstract treatment of preference maximization to Walrasian budget sets. Before we do this, some (more) mathematical background is in order. With luck you will have seen some of the following list: convex sets, the separating and supporting hyperplane theorems, (quasi-)concave and (quasi-)convex functions, Lagrangeans, Kuhn-Tucker conditions. We will spend a week and a half of class time taking a helicopter tour of these topics.

3.1. **Mathematics for Maximization.** There are two aims to this homework/section: 1) for you to understand the Lagrangean technique of constrained optimization, 2) for you to understand the Kuhn-Tucker theorem (which explains why the Lagrangean technique works).

The start is some geometry that is crucial for everything that follow. Following the basic geometry are examples of how Lagrangean functions and multipliers help solve constrained maximization problems. These examples will have the same underlying geometry, and after understanding the examples, we (you) will turn to what are called Kuhn-Tucker conditions. These will first be presented as an extension of the reasoning used in Lagrangean multipliers. To really understand why, rather than how, the Kuhn-Tucker conditions work requires things called saddle points some more geometry. The basic geometrical tools are convex sets, hyperplanes, especially separating and supporting hyperplanes. After covering this material, we'll (you'll) turn back to constrained optimization, tying the geometry to Lagrangean functions and multipliers.

The homeworks are scattered throughout.

3.1.1. *Some Basic Geometry.* Recall that for $x, y \in \mathbb{R}^L$, $x \cdot y = x^T y := \sum_{k=1}^{L} x_k y_k$, that $x \cdot x$ is the square of the length of the vector $x \in \mathbb{R}^L$, that $\|x\| := \sqrt{x \cdot x}$ denotes the length of a vector x, and that $x \cdot y = \|x\| \, \|y\| \cos(\theta)$ where θ is the angle between x and y. From the previous, you can conclude that $x \cdot y = 0$ only when x and y are perpendicular. Notice that if $x \cdot v = 0$, then $x \cdot (y + v) = x \cdot y$. This means that the set of all vectors y such that $x \cdot y = r$ for some constant r is a line perpendicular to the line determined by the vector x.

Homework 3.1.1. *Time to draw and describe.*

1. *Draw the set of vectors* $y \in \mathbb{R}^2$ *that are perpendicular to* $x = (1, 2)^T$.
2. *Draw the set of vectors* $y \in \mathbb{R}^2$ *such that* $x \cdot y = 10$ *where* $x = (2, 3)^T$.
3. *Give a geometric description of the set of vectors* $y \in \mathbb{R}^2$ *such that* $y \geq (0, 0)^T$ *and* $x \cdot y \leq w$ *where* $x \gg (0, 0)^T$, $w > 0$.

4. *Give a geometric description of the set of vectors $y \in \mathbb{R}^3$ such that $y \geq (0,0,0)^T$ and $x \cdot y \leq 150$ where $x = (10, 15, 2)^T$.*

Homework 3.1.2. *Let $a = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and let $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find and geometrically describe the set*

$$C = \{c \in \mathbb{R}^2 : c = \lambda_1 a + \lambda_2 b,\ \lambda_1, \lambda_2 \geq 0\}.$$

A **cone** *is a subset of X of $\mathbb{R}^L$ such that $x \in X$ implies that $\lambda x \in X$ for all $\lambda > 0$. Verify that the C you found above is a cone. Also verify that a plane through the origin in $\mathbb{R}^3$ is a cone.*

3.1.2. *Lagrangeans and Constrained Optimization Problems.* Consider the following version of the (neo)classical consumer demand problem: A consumer has preferences over their non-negative levels of consumption of two goods. Consumption levels of the two goods is represented by $x^T = (x_1, x_2) \in \mathbb{R}^2_+$. We will assume that this consumer's preferences can be represented by the utility function

$$u(x_1, x_2) = x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}}.$$

The consumer has an income of 50 and faces prices $p^T = (p_1, p_2) = (5, 10)$. The standard behavioral assumption is that the consumer chooses among her affordable levels of consumption so as to make herself as happy as possible. This can be formalized as solving the constrained optimization problem:

$$\max_{(x_1,x_2)} \ x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \text{ subject to } 5x_1 + 10x_2 \leq 50,\ x_1, x_2 \geq 0,$$

which in turn can be re-written as

$$\max_x \ u(x) \text{ subject to } p \cdot x \leq w,\ x \geq 0.$$

The function being maximized is called the **objective function**.[10]

Homework 3.1.3. *The problem asks you to solve the previous maximization problem using a particular sequence of steps:*

1. *Draw the set of affordable points (i.e. the points in $\mathbb{R}^2_+$ that satisfy $p \cdot x \leq 50$).*
2. *Find the slope of the budget line (i.e. the slope of the line determined by the equation $p \cdot x = 50$).*
3. *Find the equations for the indifference curves (i.e. solve $x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} = c$ for $x_2(x_1, c)$).*
4. *Find the slope of the indifference curves.*
5. *Algebraically set the slope of the indifference curve equal to the slope of the budget line. This gives one equation in the two unknowns. Solve the equation for x_2 in terms of x_1.*
6. *Solve the two equation system that you get when you add the budget line to the previous equation.*
7. *Explain geometrically why the solution to the two equation system is in fact the solution to the constrained optimation problem.*

[10]From Webster, the second definition of the word "objective" is "2a: something toward which effort is directed :an aim, goal, or end of action."

8. *Explain economically why the solution you found is in fact the consumer's demand. Phrases that should come to your mind from intermediate microeconomics are "marginal rate of substitution," and "market rate of substitution."*

Homework 3.1.4. *Construct the* **Lagrangean function** *for the optimization problem given above,*

$$L(x_1, x_2, \lambda) = x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} + \lambda(50 - [5x_1 + 10x_2]) = u(x) + \lambda(w - p \cdot x),$$

and show that the solution to the three equation system

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0,$$

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0,$$

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0,$$

is the same as the solution you found in the previous problem. Be sure to solve for the extra variable, λ.

Homework 3.1.5. *Solve the previous problem for general* w, *i.e. find the demands* $x^*(w)$ *as a function of* w, *so that* $x^*(50)$ *gives you the previous answer. Define* $v(w) = u(x^*(w))$ *and find* $(\partial v/\partial w)_{|w=50}$. *Your answer should be the solution for* λ *that you found above. Interpret the derivative you just found economically. A phrase that should come to your mind from intermediate microeconomics is "marginal utility".*

Note that the gradient of the function defining the budget line is

$$D_x p \cdot x = p = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Homework 3.1.6. *Let* x^* *denote the solution in the previous two problems. Show that*

$$D_x u(x^*) = \lambda \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \lambda p$$

for the same $\lambda > 0$ *that you found in the previous two problems. Interpret this geometrically.*

Homework 3.1.7. *Draw the set of* x *such that* $u(x) > u(x^*)$ *in the previous problem. Show that it is disjoint from the set of affordable bundles. This is another way of saying that* x^* *does in fact solve the problem.*

Homework 3.1.8. *Suppose that* $u(x_1, x_2) = x_1^{0.5} \cdot x_2$. *Set up the Lagrangean function and use it to solve the problem*

$$\max_x \; u(x) \text{ s.t. } x \geq 0, \; p \cdot x \leq w$$

for $x^*(p, w)$ *and* $v(p, w) = u(x^*(p, w))$ *where* $p \gg 0$, $w > 0$. *Check (for yourself) that the geometry and algebra of the previous several problems holds, especially the separation of the "strictly better than" set and the feasible set, and* $\lambda = (\partial v/\partial w)$.

Homework 3.1.9. *Suppose that* $u(x_1, x_2) = 7x_1 + 3x_2$. *Set up the Lagrangean function and use it to solve the problem*

$$\max_x \ u(x) \ s.t. \ x \cdot x \leq c$$

as a function of $c > 0$. *Check (for yourself) that the geometry and algebra of the previous several problems holds, especially the separation of the "strictly better than" set and the feasible set, and* $\lambda = (\partial v / \partial c)$.

In the next problem, the geometry is a bit trickier because the solution happens at a corner of the feasible set. You should solve this problem.

Consider the problem

$$\max_{(x_1,x_2)} \ x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \text{ subject to } 2x_1 + 1x_2 \leq 12, \ 1x_1 + 2x_2 \leq 12, \ x_1, x_2 \geq 0.$$

Find the optimum x^*, and note that

$$D_x u(x^*) = \lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where $\lambda_1, \lambda_2 > 0$. Interpret this geometrically using Homework 3.1.2. Construct the Lagrangean function

$$L(x_1, x_2, \lambda_1, \lambda_2) = x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} + \lambda_1(12 - [2x_1 + x_2]) + \lambda_2(12 - [x_1 + 2x_2]),$$

and look at the four equation system

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_1} = 0,$$

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_2} = 0,$$

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial \lambda_1} = 0,$$

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial \lambda_2} = 0.$$

Note that solving this set of equations gives x^*.

So far all of the questions have been consumer maximizations. There are also constrained optimization questions in producer theory.

Homework 3.1.10. *Here is an example of the simplest kind of production theory — one input, with level* $x \geq 0$, *and one output, with level,* $y \geq 0$. *One formalization of this runs as follows: an* (input,output) *vector* $(x, y) \in \mathbb{R}^2$ *is* **feasible** *if* $y \leq f(x)$, $x, y \geq 0$ *where* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *(this function is called the production function). The behavioral assumption is that the owner of the right to use the production technology represented by the feasible set takes the prices, p for output and w for input, as given, and chooses the profit maximizing feasible point. Letting* Π *denote profits, this gives the* Π*-maximization problem*

$$\max_{x,y} \ \Pi(x, y) = py - wx \text{ subject to } g(x, y) := y - f(x) \leq 0, \ x, y \geq 0.$$

For this problem, assume that $f(x) = \sqrt{x}$.

1. *The Lagrangean function for the optimization problem above is*
$$L(x, y, \lambda) = \Pi(x, y) + \lambda(0 - g(x, y)) = py - wx + \lambda(f(x) - y).$$
Solve the equations
$$\partial L/\partial x = 0,$$
$$\partial L/\partial y = 0, \text{ and}$$
$$\partial L/\partial \lambda = 0,$$
for $y^*(p, w)$ (the supply function), $x^*(p, w)$ (the input demand function), and for $\lambda^*(p, w)$.
2. *Find the gradient of the objective function, $\Pi(\cdot, \cdot)$, the gradient of the constraint function, $g(\cdot, \cdot)$, and draw the geometric relationship between the two gradients and λ^* at the solution. Also show the separation of the "strictly better than" set from the feasible set.*
3. *Define the profit function $\Pi(p, w) = py^*(p, w) - wx^*(p, w)$. Show that $\partial \Pi/\partial p = y^*(p, w)$ and $\partial \Pi/\partial w = x^*(p, w)$. [The cancellation of all of the extra terms is an implication of the envelope theorem.]*

Homework 3.1.11. *Another optimization problem with implications for producer theory involves producing some amount of output, y^0, using inputs, $x \in \mathbb{R}^L$. Suppose that $L = 2$, that the set of feasible (x, y) combinations satisfies $y \le f(x_1, x_2)$, $x, y \ge 0$, where (for this problem)*
$$f(x_1, x_2) = x_1^{0.2} x_2^{0.6}.$$
Assuming that the producer takes the prices $w^T = (w_1, w_2)$ of inputs as given, the cost minimization problem is
$$\min_{x_1, x_2} \; w_1 x_1 + w_2 x_2 \text{ subject to } f(x_1, x_2) \ge y^0, \; x_1, x_2 \ge 0.$$
To set this up as a standard maximization problem, multiply by -1:
$$\max_{x_1, x_2} \; -(w_1 x_1 + w_2 x_2) \text{ subject to } -f(x_1, x_2) \le -y^0, \; x_1, x_2 \ge 0.$$

1. *The Lagrangean function for the optimization problem above is*
$$L(x_1, x_2, \lambda) = -(w_1 x_1 + w_2 x_2) + \lambda(-y^0 - (-f(x_1, x_2)) = -w^T x + \lambda(f(x) - y^0).$$
Solve the equations
$$\partial L/\partial x = 0,$$
$$\partial L/\partial y = 0, \text{ and}$$
$$\partial L/\partial \lambda = 0,$$
for $x_1^*(w, y^0)$ and $x_2^*(w, y^0)$, the conditional factor demands, and for $\lambda^*(w, y^0)$.
2. *Find the gradient of the objective function, the gradient of the constraint function, and draw the geometric relationship between the two gradients and λ^* at the solution. Also show the separation of the "strictly better than" set from the feasible set.*
3. *Define the cost function $c(w, y^0) = w^T x^*(p, y^0)$. Show that $\lambda^* = \partial c/\partial y^0$.*

3.1.3. *Kuhn-Tucker Conditions and Geometry.* The time has come to talk of many things:[11] and to expand on them. The pattern has been that we have 1 non-zero multiplier for each constraint that is relevant to or **binding** on the problem. Further, the value of that multiplier tells us how the objective function changes for a small change in the constraint.

For example, our first problem,

$$\max_{(x_1,x_2)} x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \text{ subject to } 5x_1 + 10x_2 \leq 50,\ x_1, x_2 \geq 0,$$

has 3 constraints: first, $5x_1 + 10x_2 \leq 50$, second, $-x_1 \leq 0$, and third, $-x_2 \leq 0$. However, at any solution, the second and third constraints are not binding (that is, not relevant), and we used a Lagrangean with only one multiplier.

For another example, our last problem,

$$\max_{(x_1,x_2)} x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \text{ subject to } 2x_1 + 1x_2 \leq 12,\ 1x_1 + 2x_2 \leq 12,\ x_1, x_2 \geq 0,$$

has 4 constraints: first, $2x_1 + 1x_2 \leq 12$, second, $1x_1 + 2x_2 \leq 12$, third, $-x_1 \leq 0$, and fourth, $-x_2 \leq 0$. However, only the first two are binding at the solution, and we could solve the problem using only two multipliers.

The general pattern is that we can use a Lagrangean with multipliers for each of the binding constraints to solve our constrained optimization problems. We are going to include all of the multipliers, but set the irrelevant ones equal to 0.

The general form of the problem is

$$\max_{x \in X} f(x) \text{ subject to } g(x) \leq b$$

where $X \subset \mathbb{R}^L$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^M$.

For example, with $L = 2$ and $M = 4$, the previous 4 constraint problem had

$$g(x_1, x_2) = \begin{bmatrix} 2x_1 + 1x_2 \\ 1x_1 + 2x_2 \\ -x_1 \\ -x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 12 \\ 0 \\ 0 \end{bmatrix}.$$

The general form of the Lagrangean function is

$$L(x, \lambda) = f(x) + \lambda^T(b - g(x)).$$

With x_k's and g_m's this is

$$L(x_1, \ldots, x_L, \lambda_1, \ldots, \lambda_M) = f(x_1, \ldots, x_L) + \sum_{m=1}^{M} \lambda_m(b_m - g_m(x_1, \ldots, x_L)).$$

[11] of shoes—and ships—and sealing wax, of cabbages and Kings, and why the sea is boiling hot, and whether pigs have wings

The **Kuhn-Tucker** (K-T) conditions are

$$\frac{\partial L}{\partial x} = 0,$$

$$\frac{\partial L}{\partial \lambda} \geq 0, \text{ equivalently}$$

$$\begin{aligned} b - g(x) &\geq 0 \\ \lambda &\geq 0, \text{ and} \\ \lambda \cdot (b - g(x)) &= 0. \end{aligned}$$

Pay Attention to the Following

If the K-T conditions hold, then:
1) the only way that $\lambda_m > 0$ can happen is if $b_m - g_m(x) = 0$, and
2) if $b_m - g_m(x) > 0$, then $\lambda_m = 0$.

In words, only the binding constraints have positive multipliers, and non-binding constraints have 0 multipliers. This patterns is known as **complementary slackness**. When we think about the $\lambda = \partial v/\partial w$ results above, this is good.

Homework 3.1.12. *Write out the Lagrangean conditions with x_k's and g_m's.*

Because non-negativity constraints are so common, they often have their own separate notation for multipliers, μ. More explicitly, the four constraint problem

$$\max_{(x_1,x_2)} x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \text{ subject to } 2x_1 + 1x_2 \leq 12,\ 1x_1 + 2x_2 \leq 12,\ -x_1 \leq 0,\ -x_2 \leq 0$$

has the Lagrangean

$$\begin{aligned} L(x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2) = x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}} + \lambda_1(12 - [2x_1 + 1x_2]) + \lambda_2(12 - [1x_1 + 2x_2]) \\ + \mu_1(0 - (-x_1)) + \mu_2(0 - (-x_2)). \end{aligned}$$

This can be written as

$$L(x, \lambda, \mu) = f(x) + \lambda^T(b - g(x)) + \mu^T x$$

with the understanding that g does not contain the non-negativity constraints.

The **Kuhn-Tucker** (K-T) conditions with the μ's are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0, \\
\frac{\partial L}{\partial \lambda} &\geq 0, \text{ equivalently} \\
b - g(x) &\geq 0 \\
\lambda &\geq 0, \\
\lambda \cdot (b - g(x)) &= 0 \\
\frac{\partial L}{\partial \mu} &\geq 0, \text{ equivalently} \\
x &\geq 0 \\
\mu &\geq 0, \\
\mu \cdot x &= 0
\end{aligned}$$

Homework 3.1.13. *Write out this last version of the Lagrangean conditions with x_k's and g_m's.*

The next problem demonstrates complementary slackness in the K-T conditions.

Homework 3.1.14. *Let $c^T = (2, 12)$, $p^T = (1, 5)$, $w = 5$, $u(x) = -(x-c)^T(x-c)$ and consider the problem*

$$\max_x \; u(x) \text{ subject to } p^T x \leq w, \; -x \leq 0.$$

1. *Write out the Lagrangean both in vector and in x_k and g_m notation using $\mu^T = (\mu_1, \mu_2)$ for the multipliers of the non-negativity constraints.*
2. *Write out the K-T conditions.*
3. *Try to solve the K-T conditions on the assumption that only the first non-negativity constraint is binding, i.e. on the assumption that $\mu_1 > 0$ and $\lambda_1 = \mu_2 = 0$. Interpret.*
4. *Try to solve the K-T conditions on the assumption that only the second non-negativity constraint is binding, i.e. on the assumption that $\mu_2 > 0$ and $\lambda_1 = \mu_1 = 0$. Interpret.*
5. *Try to solve the K-T conditions on the assumption that only the budget constraint is binding, i.e. on the assumption that $\lambda_1 > 0$ and $\mu_1 = \mu_2 = 0$. Interpret.*
6. *Try to solve the K-T conditions on the assumption that the budget constraint and the second non-negativity constraint are both binding, i.e. on the assumption that $\lambda_1 > 0$, $\mu_2 > 0$, and $\mu_1 = 0$. Interpret.*
7. *Try to solve the K-T conditions on the assumption that the budget constraint and the first non-negativity constraint are both binding, i.e. on the assumption that $\lambda_1 > 0$, $\mu_1 > 0$, and $\mu_2 = 0$. Interpret.*

If I set it up correctly, the previous problem had a solution at a corner. This is, for obvious reasons, called a **corner solution**. Corner solutions are essentially always what happens in consumer demand, but we rarely draw or analyze them in intermediate microeconomics. The next two problems have corner solutions (at least some of the time).

Homework 3.1.15. *Using the K-T conditions, completely solve the problem*

$$\max_x \; u(x) \text{ s.t. } p^T x \le w, \; -x \le 0,$$

where $u(x_1, x_2) = x_1 + 2x_2$, $p \gg 0$, $w > 0$. *Letting* $(x^*(p,w), \lambda^*(p,w))$ *denote the solution to the K-T conditions, define* $v(p,w) = u(x^*(p,w))$, *and show that* $\lambda^* = \partial v/\partial w$ *at all points when* $v(\cdot,\cdot)$ *is differentiable.*

Homework 3.1.16. *Using the K-T conditions, completely solve the problem*

$$\max_x \; u(x) \text{ s.t. } p^T x \le w, \; -x \le 0,$$

where $u(x_1, x_2) = x_1 + 2\sqrt{x_2}$, $p \gg 0$, $w > 0$. *Letting* $(x^*(p,w), \lambda^*(p,w))$ *denote the solution to the K-T conditions, define* $v(p,w) = u(x^*(p,w))$, *and show that* $\lambda^* = \partial v/\partial w$.

Homework 3.1.17. *Using the K-T conditions, completely solve the problem*

$$\max_x \; u(x) \text{ s.t. } p^T x \le w, \; -x \le 0,$$

where $u(x_1, x_2) = (\frac{1}{x_1} + \frac{1}{x_2})^{-1}$, $p \gg 0$, $w > 0$. *Letting* $(x^*(p,w), \lambda^*(p,w))$ *denote the solution, define* $v(p,w) = u(x^*(p,w))$, *and show that* $\lambda^* = \partial v/\partial w$.

Homework 3.1.18. *Return to the simplest kind of production theory — one input, with level* $x \ge 0$, *and one output, with level,* $y \ge 0$. *Letting* Π *denote profits, the* Π*-maximization problem is*

$$\max_{x,y} \; \Pi(x,y) = py - wx \text{ subject to } g(x,y) := y - f(x) \le 0, \; x, y \ge 0.$$

For this problem, assume that $f(x) = \sqrt{x+1} - 1$.

1. *Write out and solve the Lagrangean function for* $y^*(p,w)$ *(the supply function),* $x^*(p,w)$ *(the input demand function), and for* $\lambda^*(p,w)$.
2. *Find the gradient of the objective function,* $\Pi(\cdot,\cdot)$, *and show how to express it as a positive linear combination of the binding constraints.*
3. *Define the profit function* $\Pi(p,w) = py^*(p,w) - wx^*(p,w)$. *Show that* $\partial \Pi/\partial p = y^*(p,w)$ *and* $\partial \Pi/\partial w = x^*(p,w)$. [*The cancellation of all of the extra terms is an implication of the envelope theorem.*]

Homework 3.1.19. *Some amount of output,* y^0, *is produced using inputs,* $x \in \mathbb{R}^2_+$. *The set of feasible* (x,y) *combinations satisfies* $y \le f(x_1, x_2)$, $x, y \ge 0$, *where (for this problem)*

$$f(x_1, x_2) = (x_1 + \sqrt{x_2})^{0.75}.$$

Assuming that the producer takes the prices $w^T = (w_1, w_2)$ *of inputs as given, the cost minimization problem is*

$$\min_{x_1,x_2} \; w_1 x_1 + w_2 x_2 \text{ subject to } f(x_1,x_2) \ge y^0, \; x_1, x_2 \ge 0.$$

To set this up as a standard maximization problem, multiply by -1:

$$\max_{x_1,x_2} \; -(w_1 x_1 + w_2 x_2) \text{ subject to } -f(x_1,x_2) \le -y^0, \; x_1, x_2 \ge 0.$$

1. *Write out and solve the Lagrangean function for* $x_1^*(w, y^0)$ *and* $x_2^*(w, y^0)$, *the conditional factor demands, and for* $\lambda^*(w, y^0)$.

2. *Find the gradient of the objective function, the gradient of the constraint function, and draw the geometric relationship between the two gradients and* λ^* *at the solution. Also show the separation of the "strictly better than" set from the feasible set.*
3. *Define the cost function* $c(w, y^0) = w^T x^*(p, y^0)$*. Show that* $\lambda^* = \partial c / \partial y^0$*.*

3.1.4. *Kuhn-Tucker Conditions Do Not Always Work.* So far, Lagrangean functions and the Kuhn-Tucker conditions have worked quite well. Further, they give some extra information in the form of the λ^*'s. Would that life were always so simple.

Homework 3.1.20. *Let* $f(x_1, x_2) = (x_1^2 + x_2^2)/2$*,* $g(x_1, x_2) = x_1 + x_2$*, and consider the problem*

$$\max_{x_1, x_2} f(x_1, x_2) \ \ s.t. \ \ g(x_1, x_2) \leq 10, \ -x_1 \leq 0, \ -x_2 \leq 0.$$

Show that $(x_1^*, x_2^*, \lambda^*) = (5, 5, 5)$ *solves the K-T conditions but does not solve the maximization problem.*

Homework 3.1.21. *Let* $f(x_1, x_2) = x_1 + x_2$*,* $g(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$*, and consider the problem*

$$\max_{x_1, x_2} f(x_1, x_2) \ \ s.t. \ \ g(x_1, x_2) \leq 4, \ -x_1 \leq 0, \ -x_2 \leq 0.$$

Show that $(x_1^*, x_2^*, \lambda^*) = (4, 4, \frac{1}{2})$ *solves the K-T conditions but does not solve the maximization problem.*

The point of the previous problems is that for the K-T conditions to characterize an optimum, we need more. The more that we are going looking for involves the $f(\cdot)$ being quasi-concave and the $g_m(\cdot)$ being quasi-convex.

3.1.5. *Saddle Points, or Why Does it Work?* In this subsection we're going to introduce saddle points for Lagrangean functions, and show that any saddle point contains a solution to the constrained optimization problem. The reverse is not generally true, a solution need not be part of a saddle point, and we need an excursion through some geometry to see when the converse is true.

I found saddle points a bit weird at first.

Put the non-negativity constraints back into $g(\cdot)$ and construct the Lagrangean,

$$L(x, \lambda) = f(x) + \lambda^T (b - g(x)).$$

We say that $(x^*, \lambda^*) \in X \times \mathbb{R}_+^M$ is a **saddle point** for $L(\cdot, \cdot)$ if

$$(\forall x \in X)(\forall \lambda \geq 0)[L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda)].$$

In words, a x^* is a maximum with respect to x for λ fixed at λ^*, and λ^* is a minimum with respect to λ for x fixed at x^*.

Notice that for x fixed at x^*, the function $L(x^*, \cdot)$ is linear in λ.

Lemma 3.1.1. *If $b - g(x^*) \geq 0$, then the following derivative conditions for the problem characterize the solution to the problem* $\min_{\lambda \geq 0} \; L(x^*, \lambda)$*:*

$$\frac{\partial L}{\partial \lambda} \geq 0, \quad \lambda \geq 0, \text{ and } \quad \lambda \cdot \frac{\partial L}{\partial \lambda} = 0.$$

These should look familiar, they are part of the K-T conditions.

Homework 3.1.22. *Prove the previous lemma.*

The next result says that saddle points give solutions. The converse, solutions are part of a saddle point, is quite a bit harder, and we will get to it later.

Theorem 3.1.1. *If (x^*, λ^*) is a saddle point for $L(\cdot, \cdot)$, then x^* solves the problem*

$$\max_{x \in X} \; f(x) \text{ subject to } g(x) \leq b.$$

Homework 3.1.23. *Prove the previous theorem.*

This means that one way to understand the problems you've worked above is that we wrote down the first order derivative conditions for a saddle point and solved them. There are the usual problems to worry about when all you look at are first order derivatives, you need to check second order derivatives, or else to use some concavity arguments. So now it's time to learn some more geometry.

3.1.6. *Convexity.* A set $X \subset \mathbb{R}^L$ is **convex** if

$$(\forall x, y \in X)(\forall \alpha \in (0,1))[\alpha x + (1 - \alpha)y \in X].$$

The geometric character of this definition should be clear to you before moving on.

Homework 3.1.24. *Let $x^T = (12, 45)$, $y^T = (0, 15)$. Graph $\alpha x + (1 - \alpha)y$ for*

$$\alpha \in \{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, 1\}.$$

Examples of convex sets: $\mathbb{R}^L$, $\mathbb{R}^L_+$, $B_{p,w}$ if X is convex, spheres, lines, line segments.

Examples of non-convex sets: donuts, dragons, curved arcs, hands.

Lemma 3.1.2. *If X and Y are convex, then so is $X \cap Y$.*

This is a special case of

Lemma 3.1.3. *If $(X_j)_{j \in J}$ is a collection of convex sets indexed by the set J, then $\bigcap_{j \in J} X_j$ is convex.*

Homework 3.1.25. *Prove the last lemma.*

Definition 3.1.1. *If $A, B \subset \mathbb{R}^L$, then the* **sum of** A **and** B *is*

$$A + B := \{c : (\exists a \in A)(\exists b \in B)[c = a + b]\}.$$

The sum is different than the union. If you look in some older math books you may find the union of two sets written as we now write the sum. This can be confusing.

Lemma 3.1.4. *If A and B are convex, then $A+B$ is convex. Give an example of a non-convex A and a convex B such that $A+B$ is not convex. Give an example of convex A and B such that $A \cup B$ is not convex.*

Optional Homework 3.1.26. *The ball around x with radius ϵ is $B(x,\epsilon) = \{x \in \mathbb{R}^L : \|x\| < \epsilon\}$. Show that $B(x,\epsilon)$ is convex. [This is not so easy at it may appear.]*

3.1.7. *Hyperplanes.* Any non-zero $p \in \mathbb{R}^L$ and constant c defines a hyperplane

$$H_{p,c} = \{x : p \cdot x = c\}$$

that divides $\mathbb{R}^L$ "evenly" in two. Specifically, letting

$$H^+_{p,c} = \{x : p \cdot x > c\} \text{ and } H^-_{p,c} = \{x : p \cdot x < c\},$$

$\mathbb{R}^L$ can be partitioned into three parts:

$$\mathbb{R}^L = H^-_{p,c} \cup H_{p,c} \cup H^+_{p,c}.$$

Optional Homework 3.1.27. *Show that for any $p \neq 0$, $H^-_{p,c}$, $H_{p,c}$, and $H^+_{p,c}$ are convex sets.*

3.1.8. *Separation.* Hyperplanes split $\mathbb{R}^L$ in half. Being in different halves is the kind of separation of sets that we are going to look at here.

Definition 3.1.2. *For $S, T \subset \mathbb{R}^L$, $S, T \neq \emptyset$. A hyperplane $H_{p,c}$* **separates** *S and T if*

$$\left[S \subset H^+_{p,c} \cup H_{p,c} \text{ and } T \subset H^-_{p,c} \cup H_{p,c}\right], \quad \text{or} \quad \left[T \subset H^+_{p,c} \cup H_{p,c} \text{ and } S \subset H^-_{p,c} \cup H_{p,c}\right].$$

The hyperplane $H_{p,c}$ **separates** *S* **and** *T* **properly** *if it separates S and T and is not that case that $S, T \subset H$.*

The hyperplane $H_{p,c}$ **separates** *S* **and** *T* **strictly** *if*

$$\left[S \subset H^+_{p,c} \text{ and } T \subset H^-_{p,c}\right], \quad \text{or} \quad \left[T \subset H^+_{p,c} \text{ and } S \subset H^-_{p,c}\right].$$

The hyperplane $H_{p,c}$ **separates** *S* **and** *T* **strongly** *if it for some $\epsilon > 0$, $H_{p,c}$ separates $S + \epsilon B(0,1)$ and $T + \epsilon B(0,1)$.*

The sets S and T are separated (properly, strictly, or strongly) if $\exists H_{p,c}$ that separates them (properly, strictly, or strongly).

It should be clear that being strongly separated implies being strictly separated, being strictly separated implies being properly separated, and that being properly separated implies being separated.

Homework 3.1.28. *Give a pair of convex sets S and T that are separated but not properly separated. Give another pair of convex sets that are properly separated but not strictly separated. Give another pair of convex sets that are strictly separated but not strongly separated.*

Another way to talk about separation is through the infima and suprema of linear functions. The infimum is "like" the minimum and the supremum is "like" the maximum.

Definition 3.1.3. *Let $S \subset \mathbb{R}$. The* **infimum** *or* **greatest lower bound (glb)** *of S is a number r such that for all $s \in S$, $r \le s$, and if $t > r$, then there exists a number $s \in S$ such that $s < t$.*

Optional Homework 3.1.29. *Give the parallel definition of the* **supremum** *or* **least upper bound** *of a set $S \subset \mathbb{R}$.*

The following is a deep and important result that we are not going to prove.

Lemma 3.1.5. *If there exists a number s' such that for all $s \in S$, $s' < s$, then S has an infimum.*

This is summarized by saying "if S has a lower bound, then it has a greatest lower bound." In a symmetric way, if S has an upper bound, then it has a least upper bound.

Optional Homework 3.1.30. *Show that non-empty S and T are separated if and only if*

$$(\exists p \neq 0)[\inf_{s \in S} p \cdot s \ge \sup_{t \in T} p \cdot t].$$

Homework 3.1.31. *Let $p = (1,1)^T$, $A = \{x \in \mathbb{R}^2_+ : p^T x \le 10\}$, $B^\circ = \{x \in \mathbb{R}^2 : \|x - (6,6)^T\| < \sqrt{2}\}$, $B = \{x \in \mathbb{R}^2 : \|x - (6,6)^T\| \le \sqrt{2}\}$, and $C = \{(6,6)^T\}$.*

1. *Find all hyperplanes that separate, properly separate, strictly separate, and strongly separate A from B.*
2. *Find all hyperplanes that separate, properly separate, strictly separate, and strongly separate A from B°.*
3. *Find all hyperplanes that separate, properly separate, strictly separate, and strongly separate A from C.*
4. *Give three pairs of sets that cannot be separated.*

A basic set of results state that disjoint convex sets can be separated, i.e. there exist hyperplanes separating disjoint convex sets. The results come in many flavors because there are many flavors of separation. To make life confusing, the different flavors all have the same name, the Separating Hyperplane Theorem. Here is one version.

Theorem 3.1.2 (Separating Hyperplane)**.** *Two convex sets S and T can be strongly separated if and only if*

$$\inf\{\|s - t\| : s \in S,\ t \in T\} > 0.$$

While this is not hard to prove if you know how, we (you) are not going to prove it here.

Homework 3.1.32. *Draw several two or three pictures demonstrating that the previous theorem is reasonable.*

Homework 3.1.33. *Give two sets S and T, only one of them non-convex, satisfying*

$$\inf\{\|s - t\| : s \in S,\ t \in T\} > 0$$

that cannot be separated, much less strongly separated.

The next version of the separating hyperplane theorem is weaker than what is available, but it is strong enough for our purposes. It requires the notion of the interior of a set.

Definition 3.1.4. *A point x in a set $X \subset \mathbb{R}^L$ is* **an interior point of** *X if $(\exists \epsilon > 0)[B(x, \epsilon) \subset X]$. The* **interior of** *$X$, $\mathsf{int}\, X$, is the set of all interior points of X. A set X is* **open** *if $X = \mathsf{int}\, X$.*

Optional Homework 3.1.34. *Prove: a line in $\mathbb{R}^2$ is not open, $B^\circ = \{x \in \mathbb{R}^2 : \|x\| < \sqrt{2}\}$ is open.*

Optional Homework 3.1.35. *Prove: X is open if and only if $(\forall x \in X)(\exists \epsilon > 0)[B(x, \epsilon) \subset X]$.*

Here is the promised other version of the separating hyperplane theorem.

Theorem 3.1.3 (Separating Hyperplane)**.** *If S and T are non-empty, convex sets, the interior of S is non-empty and does not intersect T, then S and T can be properly separated.*

Homework 3.1.36. *Draw several two or three pictures demonstrating that the previous theorem is reasonable.*

An important special case of the previous theorem involves T being a one point set right at the edge of the convex set S. The formal name for the edge is the **boundary**.

Definition 3.1.5. *A point x is a* **boundary point of** *X if*

$$(\forall \epsilon > 0)[B(x, \epsilon) \cap X \neq \emptyset \ \& \ B(x, \epsilon) \cap X^c \neq \emptyset]$$

where X^c is the complement of X. The **boundary of** *X is the set of all boundary points of X.*

Optional Homework 3.1.37. *Find the boundary points of line in $\mathbb{R}^2$ and of the set $B^\circ = \{x \in \mathbb{R}^2 : \|x\| < \sqrt{2}\}$.*

The following is called the supporting hyperplane theorem. It is a useful special case of the last separating hyperplane theorem.

Theorem 3.1.4 (Supporting Hyperplane)**.** *If S is a convex set with non-empty interior, and x is a boundary point of S, there there exists $p \neq 0$ such that for all $s \in S$, $p^T s \geq p^T x$, and for all s in the interior of S, $p^T s > p^T x$.*

Let $p \neq 0$ be given by the previous theorem and $c = p^T x$. The hyperplane $H_{p,c}$ **supports the set** S **at the point** x. This is "like" being a tangent hyperplane, and you should review of a couple of your pictures from the optimization problems above to see what is going on. Hence

Homework 3.1.38. *Draw a picture or two demonstrating what is going on in the previous theorem.*

3.1.9. *The (Quasi)-Concavity and (Quasi)-Convexity of Functions.* For this subsubsection, let X be a convex subset of $\mathbb{R}^L$ and $f : X \to \mathbb{R}$.

Definition 3.1.6. *The function f is* **concave** *if*

$$(\forall x, y \in X)(\forall \alpha \in (0, 1))[f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)].$$

The function f is **convex** *if the function $-1 \cdot f$ is concave.*

Draw some pictures.

Homework 3.1.39. *Here are some properties of concave functions. You should prove them.*

1. f *is concave if and only if the subgraph,* $sub(f) = \{(x, y) : x \in X,\ y \leq f(x)\}$, *is a convex set.*
2. *If* f_i, $i = 1, \ldots, N$ *is a collection of concave functions, then* $f(x) = \min_{i \in \{1,\ldots,N\}} f_i(x)$ *is a concave function.*
3. *If* f *is concave, then for any* $x \in X$ *the set* $WB(x) = \{y : f(y) \geq f(x)\}$ *is a convex set.*
4. *If* f *is concave and* $h : \mathbb{R} \to \mathbb{R}$ *is a monotonic increasing transformation, then for any* $x \in X$*, the set* $WB(x) = \{y : h(f(y)) \geq h(f(x))\}$ *is a convex set.*

Definition 3.1.7. *A function* $g : X \to \mathbb{R}$*,* X *convex, is* **quasi-concave** *if* $(\forall x \in X)[WB(x) = \{y : g(y) \geq g(x)\}$ *is a convex set.*

Since utility functions don't actually measure anything, asking whether or not they are concave is not meaningful. However, asking if they are quasi-concave is meaningful. Quasi-concavity is a property of upper contour sets, the sets $WB(x)$ should be read as the set that is Weakly Better than x, and these are properties of the preferences, not of any specific utility functions that represents the preferences.

Homework 3.1.40. *Which of the functions* $f(\cdot)$ *and* $g(\cdot)$ *in Homeworks* 3.1.20 *and* 3.1.21 *are concave, which convex?* [*The reason for this question is to examine what can go wrong in applying the K-T condition.*]

The function f is **strictly concave** if

$$(\forall x \neq y,\ x, y \in X)(\forall \alpha \in (0,1))[f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y)].$$

Homework 3.1.41. *Draw a picture showing the difference between a concave and a strictly concave function.*

The following ties concavity to derivatives. You will have many occasions to use this result. Even if you cannot prove it, you should have an idea of why it is likely to be true.

Theorem 3.1.5. *Suppose that* f *is twice continuously differentiable on the convex set* X*. Then,* f *is concave if and only if for all* $x^\circ \in X$*,* $D^2_x f(x^\circ)$ *is negative semi-definite, and if for all* $x^\circ inX$*,* $D^2_x f(x^\circ)$ *is negative definite, then* f *is strictly concave.*

Optional Homework 3.1.42. *Prove Theorem* 3.1.5 *when* $X = \mathbb{R}^1$.

3.1.10. *Derivative Conditions for Unconstrained Maximization with Concavity.* We are now going to suppose that the function f is continuously differentiable and that X is an open set.[12] We say that $x^* \in X$ is a **local maximizer** if

$$(\exists \epsilon > 0)[[\,\|x - x^*\| < \epsilon] \Rightarrow [f(x^*) \geq f(x)]].$$

We say that $x^* \in X$ is a **global maximizer** if

$$(\forall x \in X)[f(x^*) \geq f(x)].$$

[12] A set X is **open** if $(\forall x \in X)(\exists \epsilon > 0)[B(x, \epsilon) \subset X]$ where $B(x, \epsilon) = \{y : \|x - y\| < \epsilon\}$.

Theorem 3.1.6. *If x^* is a local maximum, then $D_x f(x^*) = 0$.*

Homework 3.1.43. *Give two examples that show that the converse to the previous theorem is not generally true.*

Lemma 3.1.6. *For a twice continuously differentiable f, if $D_x f(x^*) = 0$ and $D_x^2 f(x^*)$ is negative definite, then x^* is a local maximizer.*

The idea of the proof of this is pretty simple. Recall Taylor's theorem: for $x \simeq x^*$,

$$f(x) \simeq f(x^*) + D_x f(x^*)(x - x^*) + \tfrac{1}{2}(x - x^*)^T D_x^2 f(x^*)(x - x^*).$$

If $D_x f(x^*) = 0$, then $f(x) - f(x^*) \simeq (x - x^*)^T D_x^2 f(x^*)(x - x^*)$. If $D_x^2 f(x^*)$ is negative definite, $f(x) - f(x^*)$ must be negative for $x \neq x^*$, i.e. $f(x) - f(x^*) < 0$. But this is the same as $f(x^*) > f(x)$.

In the presence of concavity, we don't need to check second derivatives, and we get the stronger conclusion.

Theorem 3.1.7. *For a continuously differentiable, (strictly) concave f, if $D_x f(x^*) = 0$, then x^* is a (the unique) global maximizer.*

Homework 3.1.44. *Read MWG's Mathematical Appendixes* **M.E** *and bf M.J.*

It turns out that we can get the basic comparative static results for differentiable neoclassical production theory with the tools we have so far developed. This next problem walks you through this procedure. Economically, what is being shown is that (1) increasing the price of the output increases the price-taking, profit maximizing supply (supply curves have positive slope), (2) increasing the price of the output cannot decrease all of the price-taking profit maximizing factor demands, and (3) the price-taking profit-maximizing demand for a factor of production decreases in its own price (demand curves have a negative slope).

Homework 3.1.45. *Suppose that $f : \mathbb{R}^N_{++} \to \mathbb{R}^1$ is strictly concave, twice continuously differentiable, and satisfies $D_x f \gg 0$. Further suppose that for some $p^\circ > 0$ and $w^\circ \gg 0$, the (profit) maximization problem*

$$\max_{x \in \mathbb{R}^N_{++}} \Pi(x) = pf(x) - w \cdot x$$

has a strictly positive solution.

1. [*Optional*] *Using the Implicit Function theorem, show that the solution to the above problem, $x^*(p, w)$, is a differentiable function of p and w on a neighborhood of (p°, w°).*
2. *Show that the derivative conditions for the optimum, x^*, are $pD_x f(x^*) = w$ or $D_x f(x^*) = \frac{1}{p}w$, and write these out as a system of equations. The x^* are called the factor demands.*
3. *The "supply function" is defined as $y(p, w)$. Note that $D_p y(p, w) = D_x f(x^*) D_p x^*$. Write this out using summation notation.*
4. *Taking the derivative with respect to p on both sides of the equivalence $D_x f(x^*(p, w)) \equiv \frac{1}{p}w$ gives the equation $D_x^2 f(x^*) D_p x^* = -\frac{1}{p^2}w$. This implies that $D_p x^* = -\frac{1}{p^2}(D_x^2 f(x^*))^{-1} w$. Write both of these out as a system of equations.*
5. *Using the negative definiteness of $D_x^2 f$, Theorem 2.1.3, and the previous three parts of this problem, show that $D_p y(p, w) > 0$.*
6. *Using the previous part of this problem, show that it is not the case that $D_p x^* \leq 0$.*

7. *Let e_n be the unit vector in the n'th direction. Taking the derivative with respect to w_n on both sides of the equivalence $D_x f(x^*(p,w)) \equiv \frac{1}{p}w$ gives the equation $D_x^2 f(x^*) D_{w_n} x^* = \frac{1}{p}e_n$. Write this out as a system of equations.*
8. *Pre-multiply both sides of $D_{w_n} x^* = \frac{1}{p}(D_x^2 f(x^*))^{-1} e_n$ by e_n. Using Theorem 2.1.3, conclude that $\partial x_n^* / \partial w_n < 0$.*

3.1.11. *Back to Saddle Points and the Kuhn-Tucker Theorem.* Okay, we now have enough to give the partial converse to the result that if (x^*, λ^*) is a saddle point for the Lagrangean function, then x^* solves the maximization problem.

Theorem 3.1.8. *Suppose that X is an open convex set (e.g. $\mathbb{R}^L$). Suppose further that f is concave, that each g_k, $k = 1, \dots M$ is continuous and quasi-convex, and that $(\exists x^0 \in X)[g(x^0) \ll b]$. Then x^* solves the problem*

$$\max_{x \in X} f(x) \text{ subject to } g(x) \leq b.$$

if and only if there exists a vector $\lambda^* \in \mathbb{R}_+^M$ such that (x^*, λ^*) is a saddle point for the Lagrangean function

$$L(x, \lambda) = f(x) + \lambda^T (b - g(x)).$$

Proof: We know that if the multipliers exist making (x^*, λ^*) a saddle point, then x^* is a solution to the maximization problem.

To go the other direction, suppose that x^* solves the maximization problem. We will use the separating hyperplane theorem in a fundamental way. The basic trick is to look at the sets

$$A = \{(a_0, a)^T \in \mathbb{R}^{M+1} : (\exists x \in X)[a_0 \leq f(x) \ \& \ a \leq b - g(x)]\},$$

and

$$B = \{(b_0, b)^T \in \mathbb{R}^{M+1} : b_0 > f(x^*) \ \& \ b \gg 0\}.$$

These are disjoint convex subsets or $\mathbb{R}^{M+1}$ with non-empty interiors. Hence there exists $\lambda \in \mathbb{R}^{M+1}$, $\lambda \neq 0$, separating A and B. Etc.[13] ∎

This means that in the presence of concavity and quasi-convexity we need only look for saddle points. We detour here to pick up the derivative conditions that we'll need for saddle points: suppose that $h : \mathbb{R} \to \mathbb{R}$ is a differentiable convex function, and consider the problem

$$\min_{\lambda \in \mathbb{R}} h(\lambda) \text{ subject to } \lambda \geq 0.$$

It is pretty easy to see that λ^* solves this problem if and only if either

$$\lambda^* > 0 \text{ and } \frac{dh(\lambda^*)}{d\lambda} = 0,$$

or

$$\lambda^* = 0 \text{ and } \frac{dh(\lambda^*)}{d\lambda} \geq 0.$$

[13] This is an invitation to try and fill in the remaining steps. If you get stuck, consult me or Intrilligator's textbook, *Mathematical Optimization and Economic Analysis*.

These two equations are equivalent to

$$\frac{dh(\lambda^*)}{d\lambda} \geq 0, \;\; \lambda^* \geq 0, \;\; \text{and} \;\; \frac{dh(\lambda^*)}{d\lambda}\lambda^* = 0.$$

Often this is called a **complementary slackness condition**.

If g is differentiable and concave, and we are interested in solving

$$\max_{x \in \mathbb{R}} g(x) \text{ subject to } x \geq 0,$$

then the derivative conditions are either

$$x^* > 0 \text{ and } \frac{dg(x^*)}{dx} = 0,$$

or

$$x^* = 0 \text{ and } \frac{dg(x^*)}{dx} \leq 0.$$

These two equations are equivalent to

$$\frac{dg(x^*)}{dx} \leq 0, \;\; x^* \geq 0, \;\; \text{and} \;\; \frac{dg(x^*)}{dx}\lambda^* = 0.$$

Again, this is often called a **complementary slackness condition**.

The derivative conditions in the following are the **Kuhn-Tucker** conditions.

Theorem 3.1.9. *Assume that f is concave, that the g_k are quasi-convex, and all of them differentiable. The pair (x^*, λ^*) satisfies the following derivative conditions*

$$D_x L(x^*, \lambda^*) = 0,$$

and

$$D_\lambda L(x^*, \lambda^*) \geq 0, \;\; \lambda^* \geq 0, \;\; \textit{and } D_\lambda L(x^*, \lambda^*)\lambda^* = 0$$

if and only if (x^*, λ^*) is a saddle point.

Homework 3.1.46. *Give the derivative conditions explicitly using μ's as the multipliers of the non-negativity constraints. Reformulate the conditions so that the μ's disappear.*

The geometric reasons are hopefully pretty clear, at a saddle point, the gradient of the function being maximized can be expressed as a positive linear combination of the binding constraints. All that's left is to understand the calculus reasons that the multipliers give the partial derivative of the value function.

There are two calculus kinds of tools that are useful, the implicit function theorem and the envelope theorem. We take them up in turn.

3.1.12. *The Implicit Function Theorem.* An observation: typical Lagrangean equations characterizing a solution to a constrained optimization problem are of the form

$$D_x f(x) + \lambda(b - g(x)) = 0.$$

Often, we are interested in the dependence of the optimal x^* on the parameter(s) b.[14] Letting $h(x,b)$ denote $D_x(f(x) + \lambda(b - g(x)))$, we are interested in a function $x(b)$ that makes

$$h(x(b), b) \equiv 0$$

where "$\equiv$" is read as "is equivalent to" or "is identically equal to." In words, the equation $h(x,b) = 0$ implicitly defines x as a function of b.

Homework 3.1.47. *Explain when you can and when you cannot solve $h(x,b) = 0$ for x as a function of b when*

1. $x, b \in \mathbb{R}^1$, $h(x,b) = rx + sb + t$, $r, s, t \in \mathbb{R}^1$.
2. $x, b \in \mathbb{R}^1$, $h(x,b) = r(x-b)^n + t$, $r, t \in \mathbb{R}^1$, $n \geq 1$ *an integer.*
3. $x \in \mathbb{R}^N$, $b \in \mathbb{R}^M$, $h(x,b) = Rx + Sb + t$, R *an* $N \times N$ *matrix,* S *an* $N \times M$ *matrix, and* $t \in \mathbb{R}^N$.

Homework 3.1.48. *Find the Jacobian matrix for $h(x,b) = rx + sb + t$ when $x, b \in \mathbb{R}^1$, $r, s, t \in \mathbb{R}^1$. Compare the conditions of MWG's Theorem* **M.E.1** *with the conditions you found above.*

Homework 3.1.49. *Find the Jacobian matrix for $h(x,b) = Rx + Sb + t$, R an $N \times N$ matrix, S an $N \times M$ matrix, and $t \in \mathbb{R}^N$. Compare the conditions of MWG's Theorem* **M.E.1** *with the conditions you found above.*

Sometimes all we need is information about $\partial x/\partial b$ rather the whole function $x(b)$. Suppose that $h(x,b) = 0$ defines $x(b)$ implicitly, and that $x(\cdot)$ is differentiable. Then

$$h(x(b), b) \equiv 0 \text{ implies } \frac{\partial h(x(b), b)}{\partial b} = 0.$$

What we did in the last part of the above was to **totally differentiate** $h(x(b), b) \equiv 0$ **with respect to** b. The resultant equation is

$$D_x h(x(b), b) D_b x(b) + D_b h(x(b), b) = 0.$$

Provided $D_x h(x(b), b)$ is invertible, we find that

$$D_b x(b) = -[D_x h(x(b), b)]^{-1} D_b h(x(b), b),$$

which looks like more of a mess than it really is.

Homework 3.1.50. *Suppose that $h(x,b) = (x-b)^3 - 1$.*

1. *Solve for $x(b)$ implicitly defined by $h(x(b), b) \equiv 0$ and find dx/db.*
2. *Totally differentiate $h(x(b), b) \equiv 0$ and find dx/db.*

Homework 3.1.51. *Suppose that $h(x,b) = \ln(x+1) + x + b$ and that $x(b)$ is implicitly defined by $h(x(b), b) \equiv 0$. Find dx/db.*

3.1.13. *The Envelope Theorem.* You should read MWG's Mathematical Appendix **M.L**. Here's my version of the Envelope Theorem. Suppose we have a differentiable function

[14] From Webster's again, "1a: an arbitrary constant whose value characterizes a member of a system (as a family of curves); also: a quantity (as a mean or variance) that describes a statistical population".

$f : \mathbb{R}^K \times \Theta \to \mathbb{R}$, $\Theta \subset \mathbb{R}^N$, $\mathsf{int}\,\Theta \neq \emptyset$. Consider the function

$$v(\theta) = \max_{x \in \mathbb{R}^K} f(x, \theta).$$

We are interested in $\partial v/\partial\theta$. In words, we want to know how the maximized value depends on θ, we think of the vector θ as being parameters, and we think of the vector x as being under the control of a maximizer.

Technical stuff: For each θ in some neighborhood of $\theta^\circ \in \mathsf{int}\,\Theta$ suppose that a unique maximizer $x^*(\theta)$ so that locally, $x^* = x^*(\theta)$ is characterized by the FOC

$$\frac{\partial f(x^*(\theta), \theta)}{\partial x} = 0.$$

Suppose that the conditions of the implicit function hold so that locally $x^*(\cdot)$ is a differentiable function of θ. Note that $v(\theta) = f(x^*(\theta), \theta)$, so that θ has two effects on $v(\cdot)$, the direct one and the indirect one that operates through $x^*(\cdot)$. The envelope theorem says that the indirect effect does not matter.

Theorem 3.1.10 (Envelope)**.** *Under the conditions just given,*

$$\frac{\partial v(\theta^\circ)}{\partial \theta} = \frac{\partial f(x^*(\theta^\circ), \theta^\circ)}{\partial \theta}.$$

To see why, start taking derivatives and apply the FOC,

$$\frac{\partial v(\theta^\circ)}{\partial \theta} = \underbrace{\frac{\partial f(x^*(\theta^\circ), \theta^\circ)}{\partial x}}_{= 0 \text{ by the FOC}} \frac{\partial x^*(\theta^\circ)}{\partial \theta} + \frac{\partial f(x^*(\theta^\circ), \theta^\circ)}{\partial \theta}.$$

This is particularly useful if we think about Lagrangeans and the fact that they turn constrained optimization problems into unconstrained optimization problems. For example, in the utility maximization problems you had before,

$$v(p, w) = \max_{x \in \mathbb{R}^L} u(x) + \lambda^*(w - p \cdot x),$$

equivalently,

$$v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w)),$$

where λ^* was part of the saddle point. Notice that this really is the same as setting $v(p, w) = u(x(p, w))$ because $\lambda^*(w - p \cdot x(p, w)) \equiv 0$. Directly by the envelope theorem, we do not need to consider how p or w affects v through the optimal $x(\cdot, \cdot)$, but we can directly conclude that $\partial v(p, w)/\partial w = \lambda^*$, and $\partial v(p, w)/\partial p_k = \lambda^* x_k(p, w)$.

Homework 3.1.52. *Directly taking the derivative of* $v(p, w) = u(x(p, w)) + \lambda^*(w - p \cdot x(p, w))$ *with respect to* p *and* w *and using the FOC, check that the results just given are true.*

Homework 3.1.53. *Using the Envelope theorem, explain what* $\partial c/\partial y^0 = \lambda$ *in the cost minimization problems you solved earlier.*

Okay, enough of this, back to economics.

3.2. **Basic Properties of Preferences.** We take the choice set X to be $\mathbb{R}^L_+$, and preferences $\succeq$ are on X. The important properties are rationality (as above), as well as (in decreasing order of restrictiveness) strong monotonicity, monotonicity, local non-satiation.

Derived sets are the upper contour set, the indifference set, and the lower countour set. From these we can define the convexity and strict convexity of $\succeq$.

Special cases are homothetic and quasi-linear (w.r.t.) good 1 preferences.

3.3. **Utility Representations.** We are going to assume that our preferences $\succeq$ have a continuous utility function representation. This is not a strong assumption, but it does rule out some otherwise nice preferences (lexicographic which are monotonic and convex). We'll also assume that the preferences are convex and that the utility function is differentiable whenever it's convenient.

My favorite proof of the existence of a continuous utility function is a bit more mathematical than is appropriate for this course, so we'll give an outline of a proof in the simpler case where preferences $\succeq$ are monotonic. For all $x \in \mathbb{R}^L_+$, define $u(x)$ to be that number u such that $ue \sim x$ where $e = (1, \dots, 1)^T \in \mathbb{R}^L_{++}$. Because preferences are monotone, there can only be one such number u for any x, picking $\overline{u}$ so large that $\overline{u}e \succ x$ can be done because $\succeq$ is monotone, now we use the continuity assumption plus another argument.

3.4. **Utility Maximization Problems.** Suppose that $\succeq$ is represented by the continuous function u. The UMP is

$$\max_x \; u(x) \text{ subject to } p \cdot x \leq w, \; x \geq 0.$$

Detour through compactness and the existence of a solution if we have time. Otherwise just make the blanket assumption that the UMP has a non-empty set of solutions, $x(p, w)$, called the **Walrasian demand correspondence**, or the **Walrasian demand function** if $x(p, w)$ is single-valued.

Theorem 3.4.1. *If $u(\cdot)$ is a continuous representation of a locally non-satiated $\succeq$, then*
(a) *$x(p, w)$ is $hd(0)$,*
(b) *$p \cdot x(p, w) = w$, and*
(c) *if $\succeq$ is convex, then the set $x(p, w)$ is convex, if $\succeq$ is strictly convex, then $x(p, w)$ is a singleton set.*

So, the choice based theory of the previous section reappears.

Use the K-T conditions, if we are at an interior solution, that is, $x^* = x^*(p, w) \gg 0$, the K-T conditions are

$$D_x u(x^*) = \lambda^* p,$$

$$p \cdot x^* = w.$$

There is a "utils per dollar" interpretation of being at the optimum, and λ^* is the marginal utility of more w.

Suppose now that we have a boundary solution, the K-T conditions again give us a "utils per dollar" interpretation of being at the optimum, and λ^* is still the marginal utility of more w.

Define the **indirect utility function**,

$$v(p, w) = u(x(p, w)).$$

Theorem 3.4.2. *The indirect utility function is continuous, hd(0), increasing in w and non-increasing in any p_ℓ, and quasi-convex.*

3.5. **Expenditure Minimization Problems.** The EMP is

$$\min_x \; p \cdot x \text{ subject to } u(x) \geq u, \; x \geq 0.$$

The solutions to this problem are denoted by $h(p, u)$, and the expenditure function is defined as

$$e(p, u) = p \cdot h(p, u).$$

Work some examples, use the K-T conditions.

Theorem 3.5.1. *Suppose that u is a continuous function representing locally non-satiated preferences. Then $v(p, e(p, u)) = u$ and $e(p, v(p, w)) = w$.*

Theorem 3.5.2. *Suppose that u is a continuous function representing locally non-satiated preferences. Then the expenditure function is continuous, hd(1) in p, strictly increasing in u and non-decreasing in each p_ℓ, and concave.*

3.6. **A Detour Through Support Functions.** For $K \subset \mathbb{R}^L$, the **support function for** K is

$$\mu_K(p) = \inf\{p \cdot x : x \in K\}.$$

The "inf" in the previous is also known as the "greatest lower bound" (glb), and is not too much different than "min". For a set of numbers $S \subset \mathbb{R}$, inf S is defined as that number t such that (i) for all $s \in S$, $t \leq s$, and (ii) for all $\epsilon > 0$, there is an $s \in S$ such that $s < t + \epsilon$. It is a true fact, with the mathematical model of continuous quantities that we presently use, that if S is bounded below, then inf S exists.

With $K_u = \{x \in \mathbb{R}^L_+ : u(x) \geq u\}$, $\mu_{K_u}(p) = e(p, u)$, so the expenditure function is an example of a support function. In the theory of the firm, K_y will the the set of possible input combinations that produce an amount y and w will be the vector of prices of the inputs. In this case $\mu_{K_y}(w) = c(w, y)$ is the cost of producing y at prices w. Also in the theory of the firm, with X being the set of possible input-output vectors (inputs negative, outputs positive), and p the prices, $-\Pi(p) = \mu_{-X}(p)$ gives the profit function. In other words, support functions have already shown up, and will continue to later.

The basic properties of $\mu_K(\cdot)$ are:

1. $\mu_K(\cdot)$ is homogenous of degree 1,
2. $\mu_K(\cdot)$ is concave,
3. if (some technical stuff about solutions existing is satisfied and) $\mu_K(\cdot)$ is differentiable at $\overline{p}$, then $D_p\mu_K(\overline{p}) = \overline{x}$ where $\overline{x}$ solves the problem

$$\min_{x \in K} p \cdot x.$$

Here (loosely) are the arguments:

1. $(tp) \cdot x = t(p \cdot x)$.
2. Pick p, p', $\alpha \in [0, 1]$, let $p'' = \alpha p + (1-\alpha)p'$, suppose that x'' solves $\min_{x \in K} p'' \cdot x$, note that

$$\mu_K(p'') = p''x'' = (\alpha p + (1-\alpha)p')x'' = \alpha p x'' + (1-\alpha)p'x'',$$

and that $px'' \geq \mu_K(p)$, $p'x'' \geq \mu_K(p')$. Combining,

$$\mu_K(p'') \geq \alpha \mu_K(p) + (1-\alpha)\mu_K(p').$$

3. Define $\psi(p) = p\overline{x} - \mu_K(p)$, note that $\psi(p) \geq 0$, $\psi(\overline{p}) = 0$, so that $D_p\psi(\overline{p}) = 0$, implying that $\overline{x} = D_p\mu_K(\overline{p})$.

3.7. **Relations Between the Creatures.** Note that the first two properties of support functions imply that $e(\cdot, u)$ is hd(1) and concave. The third property is used next.

Theorem 3.7.1. *Suppose that u is a continuous function representing locally non-satiated preferences, and $e(p, u)$ is differentiable at $\overline{p}$. Then $h(\overline{p}, u) = D_p e(\overline{p}, u)$.*

The envelope theorem proof is pretty easy too.

Theorem 3.7.2. *Suppose that u is a continuous function representing locally non-satiated preferences and $e(\cdot, u)$ is twice continuously differentiable. Then*
(a) *$D_p h(p, u)$ is equal to the symmetric, negative semidefinite matrix $D_p^2 e(p, u)$,*
(b) *$D_p h(p, u)p = 0$.*

Why: The first is because the matrix of second derivatives of a concave function is negative semidefinite, the second is because $h(\cdot, u)$ is hd(0).

Implications: First, because $D_p h(p, u)p = 0$, we know that

$$pD_p^2 e(p, u)p = 0,$$

so that the matrix of second derivatives is negative semi-definite but not negative definite. If you think about the fact that $e(\cdot, u)$ is hd(1), then you know that its graph is a straightline along rays from the origin, so in this direction, the second derivative is 0. That's the geometry behind the algebra here. Second, if we change prices by a (small) vector dp, the change in the Hicksian demands is $dh = D_p h(p, u)dp$ so that

$$dp \cdot dh = dpD_p h(p, u)dp \leq 0.$$

In other words, we have a Hicskian Law of Demand. Draw pictures of moving along the boundaries of upper contour sets to get some intuition.

Goods k and ℓ are **gross substitutes** if $\partial h_k / \partial p_\ell \geq 0$. Negative semi-definiteness implies that every good has at least one gross substitute.

Theorem 3.7.3. *Suppose that u is a continuous function representing locally non-satiated preferences, and that the associated expenditure function is twice continuously differentiable. Then, evaluated at the point $u = v(p, w)$ so that $w = e(p, u)$,*

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w)x(p, w)^T.$$

To prove this, take the derivative of $h(p, u) \equiv x(p, e(p, u))$ on both sides w.r.t. p, getting

$$D_p h(p, u) = D_p x(p, e(p, u)) + D_w x(p, e(p, u)) D_p e(p, u).$$

Because the derivative of the expenditure function are the Hicksian demands, we get

$$D_p h(p, u) = D_p x(p, e(p, u)) + D_w x(p, e(p, u)) h(p, u)^T.$$

At $u = v(p, w)$, $h(p, u) = x(p, w)$ and $w = e(p, u)$, substituting these in, we get

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^T.$$

Since $h(p, u) = D_p e(p, u)$, this implies that the Slutsky matrix, S,

$$S = D_p x(p, w) + D_w x(p, w) x(p, w)^T,$$

is symmetric and negative semi-definite. The following converse is very important, but proving it would take us on a rather long detour. So we'll just state it.

Theorem 3.7.4. *If $x(p, w) \in \mathbb{R}^L_+$ is differentiable, hd(0), satisfies Walras' law, and $D_p x(p, w) + D_w x(p, w) x(p, w)^T$ is negative semi-definite, then there exists a locally non-satiated utility function $u(\cdot)$ such that $x(p, w)$ are the demand functions from the associated UMP.*

Recall that WARP implies that S is negative semi-definite. Thus, preference maximization implies just a bit more than WARP alone.

Theorem 3.7.5. *Suppose that u is a continuous function representing locally non-satiated preferences. Suppose also that $v(p, w)$ is differentiable at $(\overline{p}, \overline{w})$, Then*

$$x(\overline{p}, \overline{w}) = \frac{-1}{\partial v(\overline{p}, \overline{w})/\partial w} D_p v(\overline{p}, \overline{w}).$$

This last result is known as Roy's identity. It means that it is much easier to get demand functions from indirect utility functions, we just take the correct derivatives. In particular, this can be useful when making separability assumptions in systems of many goods.

3.8. **SARP.** Let us return for a bit to the choice framework so that $x(p, w)$ is not the preference based demand function of this section, just any old demand function satisfying Walras' law and WARP. Recall the definition of revealed prefered. Define $x(p^1, w^1)$ to be **directly or indirectly revealed prefered to** $x(p^N, w^N)$ if there is a list of distinct vectors

$$x(p^1, w^1), \ldots, x(p^N, w^N)$$

such that $x(p^n, w^n)$ is revealed prefered to $x(p^{n+1}, w^{n+1})$, $1 \leq n \leq N - 1$.

The choice based demand function $x(p, w)$ satisfies SARP if $x(p, w)$ being directly or indirectly prefered to $x(p', w')$ implies that it is not that case that $x(p', w')$ is ever strictly directly or indirectly prefered to $x(p, w)$.

Note that if $x(p, w)$ is derived from preference maximization, then it must satisfy SARP (modulo picky technical details like hd(0) and Walras' Law). The following is the converse.

Theorem 3.8.1. *If the Walrasian demand function $x(p, w)$ satisfies SARP, then there is a rational preference relation $\succeq$ such that*

$$(\forall (p, w) \gg 0)(\forall y \neq x(p, w),\ y \in B_{p,w})[x(p, w) \succ y].$$

The proof is rather complicated. The easiest one uses the Axiom of Choice. The following also has a rather complicated proof.

Theorem 3.8.2. *If $x(p, w)$ (satisfies hd(0), Walras' Law and) has a symmetric, negative semi-definite Slutsky matrix, then $x(p, w)$ can be rationalized by a preference relation.*

Summary of Demand Theory in Chapter

> Preference based demand theory reduces to choice based theory satisfying SARP, and both are equivalent to the Slutsky matrix being symmetric, and negative semi-definite.

3.9. **Welfare Analysis.** Here is a simple kind of question about the welfare of a consumer: suppose they start with wealth w facing prices p^0, and then prices change to p^1 with wealth unchanged. Does the consumer with locally non-satiated preferences $\succeq$ like the change? Let $x^0 = x(p^0, w)$.

A simple observation: If $(p^1 - p^0)x^0 < 0$, then the consumer strictly likes the change.

This is easy to see, $(p^1 - p^0)x^0 < 0$ and local non-satiation imply that $p^1 x^0 < w$, and in turn this implies that the consumer strictly prefers what they can consume at p^1 with wealth w.

The last argument did not use any knowledge of the specific preferences. If we knew $v(\cdot, \cdot)$, we could just check if

$$v(p^0, w) > v(p^1, w) \text{ or } v(p^1, w) > v(p^0, w).$$

However, observing v is rather difficult — for example, we have no idea what units it is measured in.

At this point, it is worth summarizing the results of what is called (for pretty misleading reasons) duality theory. These say that (modulo some picky technical details), knowing $u(\cdot)$ is the same as knowing $v(\cdot, \cdot)$ is the same as knowing $e(\cdot, \cdot)$ is the same as knowing $h(\cdot, \cdot)$ is the same as knowing $x(\cdot, \cdot)$. In theory at least, knowing any one of these is equivalent to knowing all of them. In principle we can observe and therefore estimate $x(\cdot, \cdot)$, and this means that, in principle we can find $v(\cdot, \cdot)$, and from there do the welfare comparison.

From above, we know that any $x(\cdot, \cdot)$ satisfying hd(0) and Walras' Law and having a symmetric, negative semi-definite Slutsky matrix comes from some $u(\cdot)$, from which we can in principle derived $v(\cdot, \cdot)$. However, finding a $v(\cdot, \cdot)$ when given $x(\cdot, \cdot)$ can be challenging.[15] The classical welfare analyses provide a combination of clever and devious devices that allow us to go from $x(\cdot, \cdot)$ to the appropriate indirect utility comparison without solving the challenging problem.

Clever devices

Define the Equivalent Variation, EV, as the solution to the equation

$$v(p^0, w + EV) = v(p^1, w).$$

[15]Like climbing Mt. Everest in a swimsuit is challenging.

If $EV > 0$, then $v(p^0, w) < v(p^1, w)$, if $EV > 0$, then $v(p^0, w) > v(p^1, w)$. Thus, we need only compare the money quantity EV to 0. Note that

$$EV = e(p^0, u^1) - e(p^0, u^0),$$

just plug it in and check (recalling that locally non-satiated preferences are a background assumption). Now,

$$e(p^0, u^0) = w = e(p^1, u^1),$$

so that

$$EV = e(p^0, u^1) - e(p^1, u^1).$$

Thus, to find EV, we need only take a path integral of the derivative of $e(\cdot, u^1)$. But the derivative is just the Hicksian demand function, so EV can be found by looking at the areas under Hicksian demand functions.

Devious devices

Now, we've observed and estimate $x(\cdot, w)$, and want to get the area under some of the the $h_\ell(\cdot, u^1)$ curves. If these two function were equal, it would be really easy. So now let's do something devious — let's name the equality of the demand functions something innocuous, say, calling it "negligible wealth effects". Then we're done, we can do the welfare comparison simply by looking at the area under the correct demand curves, and anyone who doesn't like what we're doing is worrying about something negligible. Rhetoric, unchallenged, can be grand. If not being devious, then we could use $D_p h = D_p x + D_w x\, x^T$ and do a harder set of integrals. Our own Dan Slesnick is one of the world's leading experts on how to do this in practice.

3.10. **Some Broader Methodological Issues.** Let us take an overview of what we have done — a choice based theory satisfying SARP ends up being (modulo some picky technical details) the same as a theory of preference maximization. In other words, we take preferences as a given and work out the implications of the assumption that consumers pick their most prefered affordable bundle, or, equivalently, we take consumers' choice behavior as given and assume that it satisfies SARP. Absolutely unexamined in this approach is the question of where the preferences or choice behavior comes from. The question is very interesting even if unexamined. Consider the following snippet of the *Encyclopedia Britannica*'s discussion of Hobbes and Spinoza:

> The first of these contrasts with Hobbes is Spinoza's attitude toward natural desires. As has been noted, Hobbes took self-interested desire for pleasure as an unchangeable fact about human nature and proceeded to build a moral and political system to cope with it. Spinoza did just the opposite. He saw natural desires as a form of bondage. We do not choose to have them of our own will. Our will cannot be free if it is subject to forces outside itself. Thus our real interests lie not in satisfying these desires but in transforming them by the application of reason. Spinoza thus stands in opposition not only to Hobbes but also to the position later to be taken by Hume, for Spinoza saw reason not as the slave of the passions but as their master.

Hobbes' position is the one adopted by most economists. Spinoza's approach, the transformation of desires, leads one then to questions such as, "How do we transform desires?" "To what do we transform desires?" and "What systematic influences are there on these transformations?" These kinds of questions take economics in a very different direction than we have seen, towards analyses of the formation of preferences. Such questions lead to broader considerations of history and culture. Though I believe that John Kenneth Galbraith's argument (in *The Affluent Society*) that preferences are formed by the advertising industry to be overstated (Michael Schudson's *Advertising: The Uneasy Persuasion* is the best book I've seen on advertising and should be consulted on this), it is certainly going in the direction of answering these questions. Neo-classical economics is limited in this direction because the foundations are built so as to ignore these issues.

There is a difference between being limited and being useless. The type of analysis we have been studying should be understood as being limited to situations where changes in preferences or choice behavior are not the important aspect. In the longer run, the market for milk is heavily influenced by the dairy industry's lobby and its success at getting subsidies, the advertisement, and subsidies of research touting the virtues of protein and calcium. These longer run effects are hard to talk about using fixed preferences. However, in the shorter run, fixed preferences are a reasonable approximation, and it is difficult to talk about specific price and income effects with a theory large enough to contain analyses of the formation of markets. In other kinds of markets, say products aimed at minorities in a racist society, the more interesting questions are those involving perceptions, and a fixed preference approach is not the correct tool.

3.11. Homeworks.

Due date: Tuesday, Oct. 12.
From MWG: 3.B.1, 3 – 3.C.1, 6 – 3.D.4, 5, 6 – 3.E.1, 2, 3, 8 – 3.G.15 – 3.I.1, 2, 6.

Optional Homework 3.11.1. *A continuously differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *is concave if and only if* $(\forall x \in \mathbb{R})[f''(x) \leq 0]$.

Homework 3.11.2. *Suppose that preferences* $\succeq$ *on* $\mathbb{R}^2_+$ *can be represented by the utility function* $u(x_1, x_2) = \min\{x_1, x_2\}$.

1. *Find*
   (a) $x(p, w)$,
   (b) *the income and price elasticities of consumption,*
   (c) *the indirect utility function,*
   (d) *the Hicksian demand function,*
   (e) *the expenditure function,*
   (f) *the Slutsky substitution matrix.*
2. *Check that*
   (a) *the indirect utility function is* $hd(0)$ *in* (p, w),
   (b) *the indirect utility function is strictly increasing in* w,
   (c) *the indirect utility function is strictly decreasing in* p_ℓ,
   (d) *the indirect utility function is quasi-convex,*
   (e) *the Hicksian demand function is* $hd(0)$ *in* p,

(f) *the Hicksian demand function is strictly increasing in* u*,*
(g) *the Hicksian demand function is the derivative of the expenditure function,*
(h) *the expenditure function is* $hd(1)$ *in* p*,*
(i) *the expenditure function is strictly increasing in* u*,*
(j) *the expenditure function is strictly increasing in* p*,*
(k) *the expenditure function is concave in* p*,*
(l) *the following equalities hold*

$$h(p,u) = x(p, e(p,u)), \quad \text{and} \quad x(p,w) = h(p, v(p,w)),$$

(m) *Roy's identity, and*
(n) *the symmetry of the Slutsky matrix.*

Homework 3.11.3. *Repeat the previous problem with the utility function* $u(x_1, x_2) = x_1 + \sqrt{x_2}$*. Be sure to give a complete treatment of the corner solutions.*

Homework 3.11.4. *In an* L*-commodity world, a consumer's Walrasian demand function is*

$$x_k(p,w) = \frac{w}{\sum_{\ell=1}^{L} p_\ell} \quad \text{for} \quad k = 1, \ldots, L.$$

1. *Give two different utility functions for which the above demand function solves the consumer's utility maximization problem (UMP).*
2. *Give two different indirect utility functions with the property that applying Roy's identity gives the above demand function, and in each case check that Roy's identity does give the above demand function.*
3. *Give two different Hicksian demand functions consistent with the above demand function.*
4. *Give two different expenditure functions consistent with the above demand function.*
5. *Give the Slutsky substitution matrix for this demand function in the case* $L = 2$*.*

Homework 3.11.5. *In comparing the "true value" of income in period* 1 *versus period* 0*, a commonly used deflator is the Laspeyres price index,*

$$\frac{p^1 \cdot q^0}{p^0 \cdot q^0},$$

where p^t *is the price vector at time* t *and* q^t *is the quantity chosen at time* t*,* $t = 0, 1$*.*

1. *Show if you consume* q^0 *in period* 0*, that this measure of inflation overstates, at least weakly, any harm done by inflation.* [*Hint: This needs only a simple argument using the logic of the expenditure function.*]
2. *A consumer's preferences can be represented by the utility function*

$$u_1(x,y) = \min\{x, y\},$$

The prices faced by the consumer in period 0 *are* $p^0 = (0.9, 1.1)$*, and in period* 1 *they are* $p^1 = (1.3, 1.1)$*. The income of the consumer in period* 0 *is* 20*. With* q^0 *defined as the consumer's period* 0 *bundle, to what extent does the Laspeyres price index overstate the harmful effects of inflation?*

3. *Repeat the previous question when the consumer's preferences can be represented by*

$$u_2(x, y) = x + y.$$

Homework 3.11.6. *Suppose that a consumer has preferences that can be represented by the Stone-Geary utility function*

$$u(x_1, x_2) = (x_1 - b_1)^{\alpha_1} \cdot (x_2 - b_2)^{\alpha_2}.$$

[For any numbers r and α, we define r^α to be $|r|^\alpha \cdot \mathsf{sgn}(r)$ where $\mathsf{sgn}(r) = +1$ if $r \geq 0$ and $\mathsf{sgn}(r) = -1$ if $r < 0$.]

1. *Find weak conditions on the b_i and the α_i that guarantee that the preferences are monotonic on all of $\mathbb{R}^2_+$. What about strictly monotonic on $\mathbb{R}^2_{++}$? Explain.*

   For the rest of this problem, assume that the conditions giving strict monotonicity hold.

2. *Why is there no loss in generality in assuming that $\sum_i \alpha_i = 1$?*
3. *Supposing that the solutions to the consumer's maximization problem are interior, explicitly show that the demand curves for this consumer are of the form*

$$p_i x_i^*(p_i, p_j, m) = \beta_{i,m} m + \beta_{i,i} p_i + \beta_{i,j} p_j$$

   where $m > 0$ is the consumer's income, $p_i, p_j > 0$ are the prices of the two goods, and the three β's are constants.

4. *Give the demand functions* **not** *assuming that the solutions are interior.*
5. *Combining the previous two problems, draw the Engel curves.*

Dates: Oct. 12, 17, & 19.
Material: MWG, Ch. 4 except sections F (to be covered later) and H (to be ignored). Also, B. Zorina Khan's "Married Women's Property Laws and Female Commercial Activity: Evidence from United States Patent Records, 1790-1895," *Journal of Economic History*, **56**(2), 356-388 (1996).

4.1. Homeworks.

Due date: Tuesday Oct. 24.
From MWG: Ch. 4: B.1, 2; C.3; D.1, 2, 3, and some problems in the notes below.

4.2. Introduction. While the behavior of any given individual is very hard to predict, there are systematic influences that lead to behavioral regularities amongst large numbers. Given the modern prevalence of statistical reasoning and numerical analyses, it can be hard to see how novel an idea this was.[16] This section concerns aggregate demand behavior. The results are mostly negative — they are of the form "aggregate behavior will be like individual behavior if and only if X holds," and you would never in a million years believe X. Despite the theoretical couter arguments to aggregation, and despite decades of empirical research showing that assumptions that aggregate behavior in no way resembles individual behavior, many economists continue to assume that it does. The *faute de mieux* argument advanced is quite strong — it is easy to through stones at a theory, the criticism only counts intellectually if it leads toward an improvement. The criticism only counts to the profession if it is accepted. Acceptance and intellectual merit are sometimes different.

Aggregate demand for a group of I consumers is simply

$$x(p, w_1, \dots, w_I) = \sum_i x_i(p, w_i).$$

There are 3 basic questions we will look at:

1. When does $x(p, w_1, \dots, w_I) = f(p, \sum_i w_i)$?
2. When does $x(p, w_1, \dots, w_I)$ satisfy WARP or the law of demand?
3. When do welfare measures (areas under demand curves above) have (any) welfare significance?

Roughly speaking, the answers are

1. Never.
2. Never.
3. Never.

That's okay, this means that part of our study will be short. Unfortunately, if one sticks with the strong objections to supposing that demand functions/preferences can be aggregated, then one is left not being able to say very much that is interesting in economics. For example, in our game theory section, we will look at competition between firms on the

[16] Emile Durkheim's *Suicide*, published in 1897, is often credited as the first really convincing use of numbers to draw conclusions about society, e.g. anomie making suicide more frequent though not being a predictor of any individual suicide.

assumption that there are predictable reactions to their changes in (say) prices. In other words, we will assume that there is a demand function. Otherwise we have no way to talk about what firms do. We're between a rock and a hard place. You need to come to your own peace with these issues.

We will end with a discussion of Khan's article, looking this time mostly at the Nash bargaining solution to the problem of aggregating the preferences of two people in a marriage into a household preference relation.

4.3. **Aggregate Demand as a Function of Aggregate Wealth?**

Fix $w = (w_1, \ldots, w_I)$. If

$$x(p, w_1, \ldots, w_I) = f(p, \textstyle\sum_i w_i),$$

then $x(p, w_1, \ldots, w_I)$ must be constant for all changes $dw = (dw_1, \ldots, dw_I)$ such that $\sum_i dw_i = 0$. That is, for all small dw,

$$\sum_i \frac{\partial x_{\ell,i}(p, w_i)}{\partial w_i} dw_i = 0.$$

But this can be true at w if and only if

$$(\forall i, j \in I) \left[\frac{\partial x_{\ell,i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell,j}(p, w_j)}{\partial w_j} \right].$$

Further, we want this to hold at all of the w and p. **So, if we believe that all consumers have parallel Engel expansion paths, then aggregate demand is a function of aggregate wealth.** Nope, that's not credible.

An equivalent condition is that there exists a function $b(p)$ such that each i has indirect utility function

$$v_i(p, w_i) = a_i(p) + b(p) w_i.$$

By restricting the dw to lie in (small) subspaces, one could substantially loosen the requirement of parallel Engel expansion paths. For example, if the wealth of each i is a function of p and $w = \sum_j w_j$, then

$$x(p, w_1, \ldots, w_I) = \sum_i x_i(p, w_i(p, w)) = f(p, w)$$

automatically. **So, if we believe that individual wealth is a function of price and aggregate wealth, then we believe that aggregate demand is a function of aggregate wealth.** Put it that way, and it sounds rather silly.

4.4. **The Weak Axiom for Aggregate Demand?**

Well, $x(p, w_1, \ldots, w_I)$ is continuous, hd(0), and

$$p \cdot x(p, w_1, \ldots, w_I) = \sum_i p \cdot x_i(p, w_i) = \sum_i w_i$$

provided only that the individual $x_i(p, w_i)$ are continuous, hd(0), and satisfy Walras' law.

Suppose that we suspend our disbelief for an instant, a process you are familiar with if you read fiction, and assume that individual wealth is a function of price and aggregate wealth. Let us go even further, and assume that $w_i(p, w) = \alpha_i w$ where each $\alpha_i \geq 0$ and

$\sum_i \alpha_i = 1$. Then it is clear that

$$x(p, w_1, \ldots, w_I) = \sum_i x_i(p, \alpha_i w) = x(p, w).$$

Even after we have so strongly suspended disbelief, $x(p, w)$ does not satisfy WARP.

Looking at the example (4.C.1, p. 110), we can see that income effects are at work. If, keeping our assumption that individual wealth is a not a function of price but is a linear function of aggregate wealth, we are also willing to assume either that all consumers are identical and that I is huge or that all consumers satisfy the Uncompensated Law of Demand, then $x(p, w)$ will satisfy WARP. Aggregation is clearly a potentially important issue. However, I refuse to spend anymore time on this part of it.

4.5. **Existence of a Representative Consumer?**

A special case of individual wealth being a function of price and aggregate wealth arises if some benevolent force aggregates individual utilities into a social utility using a concave, monotonic Bergson-Samuelson utility function $W(u_1, \ldots, u_I)$ where u_i is the utility of consumer i. In this context, it would serve us well to remember that utilities do not actually measure anything. Utility functions represent preferences. In other words, you might well argue that the aggregation just given does not mean anything.[17] Putting the carping aside, let us suppose that at each p and w, something solves the problem

$$\max_{w_1, \cdots, w_I} W(v_1(p, w_1), \ldots, v_I(p, w_I)) \text{ subject to } \sum_i w_i \leq w$$

where v_i is i's indirect utility function. It is rather surprising to me, but true, that if $w_i(p, w)$ are the solutions to the above problem, then the value function

$$v(p, w) = W(v_1(p, w_1(p, w)), \ldots, v_I(p, w_I(p, w)))$$

is a perfectly good indirect utility function, and the demand functions derived from it (using Roy's identity) can be interpreted as the demand functions of a representative consumer.

Surprise does not a convincing argument make. We can now say that if we believe that there is some force that acts as if it were maximizing a Bergson-Samuelson social welfare function, then we can treat demand as if there is a representative consumer.

During your time here, you will see many models using a representative consumer for the entire demand side of an economy. In these models there are comparative statics results and results about what policies are optimal. These can be really deep and impressive results about economic models. They seem to me to be based on nothing. However, the people who build and analyze these models are clearly working on and concerned about very important economic questions.

In the second semester of Micro, you will see a result about general equilibrium models that implies that (almost) every one of the representative consumer based results can be reversed provided only that there are a reasonable (and small) number of types of consumer.

4.6. **Household Preferences.** Khan shows that increases in patent activity be women followed changes in the laws that gave women legal control over the rewards of their own

[17] I often think about Alice's discussion with the Caterpillar at this point.

inventive labors. There is a theory of household behavior that says, roughly, "Treat a household as if it were an individual consumer." In the language of this section, the theory aggregates two or more preference relations into one household preference relation. One of the arguments Khan advances is that changing who in a marriage has legal control over the returns to inventiveness cannot make any difference if the aggregation holds. Since changes in the laws do change behavior, aggregation does not hold.

A possible response is to drop the individual optimization assumption once we are analyzing two or more people. The resultant theory of joint decisions allows a form of aggregation of pairs of utility functions in which changes in the laws can change behavior. It is due to John Nash and is called, in his honor, the "Nash bargaining solution."

4.7. **Nash's bargaining solution.** Let X be the set of options available to a household. A point $x \in X$ may specify an allocation of the rights and duties to the household members. Let $u_i(x)$ be i's utility to the option x, $i = 1, 2$. Let $V = \{(u_1(x), u_2(x)) : x \in X\} \subset \mathbb{R}^2$ be the set of possible utility levels V. Let e be a point in $\mathbb{R}^2$. For $v \in V$, let $L_i(v)$ be the line $L_i(v) = \{v + \lambda e_i : \lambda \in \mathbb{R}\}$, e_i the unit vector in the i'th direction.

Definition 4.7.1. *A bargaining situation (V, e) is a set $V \subset \mathbb{R}^2$ and a point e satisfying*

1. *V is closed,*
2. *V is convex,*
3. *$V = V + \mathbb{R}^2_-$, and*
4. *for all $v \in V$, $L_1(v) \not\subset V$, and $L_2(v) \not\subset V$.*
5. *e is a point in the interior of V*

Lemma 4.7.1. *If $V \subset \mathbb{R}^2$ satisfies the first three assumptions, then there exists $v' \in V$ and $L_i(v') \not\subset V$, iff for all $v \in V$, $L_i(v) \not\subset V$.*

Homework 4.7.1. *Prove this lemma.*

The interpretation of $e = (e_1, e_2)$ is that e_i is i's reservation utility level, the utility they would get by breaking off the bargaining. This gives a lower bound to what i must get out of the bargaining situation in order to keep them in it. By assuming that e is in the interior of V, we are assuming that there is something to bargain about.

Definition 4.7.2. *The Nash bargaining solution is the utility allocation that solves*

$$\max\ (v_1 - e_1) \cdot (v_2 - e_2) \ \ \textit{subject to} \ \ (v_1, v_2) \in V,\ v \geq e.$$

Equivalently,

$$\max_{x \in X}\ (u_1(x) - e_1)(u_2(x) - e_2) \ \textit{subject to} \ \ (u_1(x), u_2(x)) \geq e.$$

It is worthwhile drawing a couple of pictures to see what happens as you move e around. Note that the solution is invariant to affine positive rescaling of the players' utilities, that is, x^* solves

$$\max_{x \in X}\ (u_1(x) - e_1)(u_2(x) - e_2) \text{ subject to } \ (u_1(x), u_2(x)) \geq e$$

if and only if for all $a_1, a_2 > 0$ and all b_1, b_2, x^* solves

$$\max_{x \in X}\ ((a_1 u_1(x) + b_1) - (a_1 e_1 + b_1))((a_2 u_2(x) + b_2) - (a_2 e_2 + b_2)) \text{ subject to}$$

$$((a_1u_1(x) + b_1), (a_2u_2(x) + b_2) \geq (a_1e_1 + b_2, a_2e_2 + b_2).$$

Homework 4.7.2. *Let* $s^*(e) = (s_1^*(e_1, e_2), s_2^*(e_1, e_2))$ *be the Nash bargaining solution, i.e. solves*

$$\max(v_1 - e_1) \cdot (v_2 - e_2) \quad \textit{subject to} \quad (v_1, v_2) \in V.$$

Suppose also that

$$V = \{(v_1, v_2) : f(v_1, v_2) \leq 0\}$$

where f *is a differentiable, convex function with* $\partial f / \partial v_i > 0$.

1. *Where possible, find whether the following partial derivatives are positive or negative:*
$$\frac{\partial s_1^*}{\partial e_1}, \quad \frac{\partial s_1^*}{\partial e_2}, \quad \frac{\partial s_2^*}{\partial e_1}, \quad \frac{\partial s_2^*}{\partial e_2}.$$
2. *Where possible, find whether the following partial derivatives are positive or negative:*
$$\frac{\partial^2 s_1^*}{\partial e_1^2}, \quad \frac{\partial^2 s_1^*}{\partial e_1 \partial e_2}, \quad \frac{\partial^2 s_2^*}{\partial e_2}.$$
3. *Consider the following variant of the Nash maximization problem,*
$$\max\ ((av_1 + b) - (ae_1 + b)) \cdot (v_2 - e_2) \quad \textit{subject to} \quad (v_1, v_2) \in V$$
where $a > 0$. *Show that the solution to this problem is* $(as_1^* + b, s_2^*)$ *where* (s_1^*, s_2^*) *is the Nash bargaining solution we started with. In other words, show that the Nash bargaining solution is independent of affine rescalings. (You might want to avoid using calculus arguments for this problem.)*

It is remarkable that this solution is the only one that satisfies some rather innocuous-looking axioms – here's one version of the axioms.

Definition 4.7.3. *A* **bargaining solution** *is a mapping* $(V, e) \mapsto s(V, e)$, $s \in V$, $s \geq e$. *A solution* $(V, e) \mapsto s(V, e)$ *is* **efficient** *if for all* (V, e), *if there is no* $v' \in V$ *such that* $v' > s$.

A positive affine rescaling of $\mathbb{R}^2$ is a mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $Ax = Mx + b$ where $M = \left(\begin{smallmatrix} m_1 & 0 \\ 0 & m_2 \end{smallmatrix}\right)$, $m_1, m_2 > 0$ and $b \in \mathbb{R}^2$.

Here are some reasonable looking axioms for efficient bargaining solutions:

1. Affine rescaling: For all positive affine rescalings A and for all (V, e), $s(AV, Ae) = A(s(V, e))$.
2. Midpoint axiom: If $V = \{(u_1, u_2) : u_1 + u_2 \leq 1\}$ and $e = (0, 0)$, then $s(V, e) = (1/2, 1/2)$.
3. Independence of irrelevant alternatives axiom: If $s(V, e) \in V' \subseteq V$, then $s(V', e) = s(V, e)$.

Affine rescaling is easy enough to accept. The midpoint axiom is one possible formulation of the idea that bargaining powers are equal. The independence axiom makes sense if we are used to thinking in terms of constrained maximization.

Theorem 4.7.1 (Nash)**.** *There is only one efficient bargaining solution that satisfies affine rescaling, the midpoint axiom, and independence of irrelevant alternatives, and it is the* *solution to the problem*

$$\max\ (v_1 - e_1) \cdot (v_2 - e_2) \quad \textit{subject to} \quad (v_1, v_2) \in V.$$

Homework 4.7.3. *Prove this theorem.*[18]

Back to property rights, the household problem when facing a set of options X is now modeled as

$$\max_{x \in X,\ u_i(x) \geq e_i,\ i=1,2} (u_1(x) - e_1)(u_2(x) - e_2).$$

In effect, $w(x) = (u_1(x) - e_1)(u_2(x) - e_2)$ is the household utility function, and the constraints are $u_i(x) \geq e_i$. Khan argues that changing the property laws does not change X. Therefore, changes in the property laws can only affect the optimal behavior in the above problem if they change the e_i. This may be a reasonable way to understand the legal changes – they gave women a better set of outside options, which is captured by increasing the womens' reservation utility level in their bargaining game.

4.8. **The Kalai-Smorodinsky Bargaining Solution.** Rarely one to let good enough alone, I'd like to look at a different but still reasonable set of axioms. This one leads to the Kalai-Smorodinsky bargaining solution. It is a direct assault on the reasonableness of the independence of irrelevant alternatives axiom. One observation is that increasing j's maximal possible happiness from a bargaining situation might have some affect on how bargaining is carried out.

For a bargaining problem, (V, e), let ∂V denote the (upper) boundary of V, and let $\bar{u}_i^V = \max\{u_i : (u_i, e_i) \in V\}$. Geometrically, the Kalai-Smorodinsky bargaining solution when $e = (0, 0)$ is

$$s^{KS}(V, (0,0)) = \lambda^*(\bar{u}_1^V, \bar{u}_2^V) \text{ where } \lambda^* = \text{argmax}\{\lambda : \lambda \geq 0,\ \lambda(\bar{u}_1^V, \bar{u}_2^V) \in V\}.$$

When $e \neq (0, 0)$, one sets $s^{KS}(V, e) = e + s^{KS}(V, (0, 0))$.

1. Affine rescaling: The solution should be independent of affine rescalings of the utilities, that is, $s(AV, Ae) = A(s(V, e))$ for all positive affine rescalings A.
2. Box: If $V = \{(u_1, u_2) :\ u_i \leq u_i^\circ\}$, then $s(V, e) = (u_1^\circ, u_2^\circ)$.
3. Proportional increases: If $s(V, e) \in \partial V'$ and $(\bar{u}_1^V - e_1, \bar{u}_2^V - e_2)$ is proportional to $(\bar{u}_1^{V'} - e_1, \bar{u}_2^{V'} - e_2)$, then $s(V, e) = s(V', e)$.

Theorem 4.8.1 (Kalai, Smorodinsky)**.** *There is only one efficient bargaining solution that satisfies affine rescaling, the box axiom, and the proportional increases axiom, and it is the Kalai-Smorodinsky solution described above.*

Homework 4.8.1. *Prove this theorem.*

Homework 4.8.2. *Let s^{KS} denote the Kalai-Smorodinsky bargaining solution and s^N the Nash bargaining solution.*

1. *Show that s^{KS} satisfies the midpoint axiom and s^N satisfies the box axiom.*

[18]There are many places to find this theorem proved if you want to look, a good recent one is Nejat Anbarci's paper, "Simple Characterizations of the Nash and Kalai/Smorodinsky Solutions," *Theory and Decision* **45**, n3 (December 1998): 255-61. It also covers the next bargaining solution that we're going to look at.

2. *Give two* (V, e) *where* $s^{KS}(V, e) \neq s^N(V, e)$.
3. *Find whether or not* $s_i^{KS}(V, e)$ *is increasing or decreasing in* e_j, $i, j \in \{1, 2\}$.

Nash's bargaining solution let us "explain" the effect of changes in property laws as increases in womens' reservation utility levels. A corresponding "explanation" for the Kalai-Smorodinsky solution might be that allowing people to claim the rewards for their own efforts makes their maximal possible happiness higher. This might not affect the Nash solution if the Nash solution did not involve picking an $x \in X$ with patenting activity being done by the female partner. However, it might well affect the K-S solution in these circumstances.

5.1. Homeworks.

Dates: Oct. 24, 26, & 31.
Due date: Tuesday Nov. 7.
From MWG: Ch. 5: B.1, 2, 3, 6; C.1, 2, 5, 6, 9, 10; D.1, 2, 4; E.1, 3, 5; F.1, and some homeworks in the notes below.

5.2. The Basic Idea.

The basic theory of the firm is as simple-minded as the basic preference based theory of the consumer. The consumer maximizes utility subject to budget constraints, the firm maximizes profit subject to technological constraints. The rest is details.

Omitted from our study: Who owns and who manages a firm? Why? What implications do the answers have for the performance of firms? For example, how do owner-managers' behaviors differ from professional managers? How are firms organized? What are the constraints on organizational form imposed by strategic behavior? How do firms and, more generally, organizations go about trying to organize the congruence of self-interest and firm interest? In these problems, how and by who is firm self-interest determined? Actualized? Why are firms organized?

All of these are very important questions. Over the last several years, they are even the primary interesting questions for a largish part of the industrial organization literature. However, before we turn you loose on the development of new theories, we insist that you understand the old ones. Therefore, we are presently going to build the standard, neoclassical "black box" theory of the firm.

5.3. An Example.

Suppose that

$$Y = \{(0,0), (-4,12), (-9,17), (-7,13)\} \subset \mathbb{R}^2$$

represents the set of technologically possible options for a firm. Negative components in a vector $y \in Y$ correspond to inputs, positive to outputs. The assumption that $(0,0) \in Y$ corresponds to the firm having the option of shutting down.

The profit maximization problem for the firm facing prices p is

$$\max\ p \cdot y \text{ subject to } y \in Y.$$

Let $y(p)$ denote the solution to this problem, and $\pi(p) = p \cdot y(p)$ the maximized value. The negative components of y are the demand functions for input, the positive components the supply function for outputs, and $\pi(\cdot)$ is the profit function.

Homework 5.3.1. *The technologically feasible input-output pairs for a profit-maximizing, price-taking firm is given by*

$$Y = \{(0,0), (-4,12), (-9,17), (-7,13)\} \subset \mathbb{R}^2.$$

1. *Find the firm's supply and demand function $y(p)$.*
2. *Find the firm's profit function, $\pi_Y(p)$.*

3. *We say that a technology Y' is larger than a technology Y'' if $Y'' \subset Y'$. Find the largest technology Y' in $\mathbb{R}^2$ having the same profit function that Y has, $\pi_Y(p) = \pi_{Y'}(p)$ for all $p \gg 0$.*

Note that the demand for good 1 is (weakly) decreasing in the price of good 1, the supply of good 2 is (weakly) increasing in the price of good 2, and the profit function is convex.

Note that, so far as $y(p)$ is concerned, there is no loss in replacing Y by $Y + \mathbb{R}^2_-$ in this problem provided that $p \gg 0$. Also, insofar as $\pi(\cdot)$ is concerned, there is no loss in replacing Y by $\text{co}\,(Y) + \mathbb{R}^2_-$. One can interpret convex combinations $\alpha y^1 + (1-\alpha)y^2$, of points in $y^1, y^2 \in Y$ as representing using point y^1 α of the time and y^2 the rest of the time.

You may be used to the firm's problem being stated as

$$\max\ py - w \cdot x \text{ subject to } y = f(x).$$

Let $Y = \{(-x, y) : y - f(x) \leq 0,\ x \geq 0\}$, and use the price vector $p' = (w, p)$. The problem

$$\max\ p' \cdot y' \text{ subject to } y' \in Y'$$

is exactly the same as the one you are used to.

5.4. **Properties of Technologies.** Usually we assume that $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$. Until we start making assumptions (such as differentiability) on F, this is no loss of generality, simply set $F(y) = 0$ if $y \in S$ and $F(y) = 1$ if $y \notin S$ and we have $S = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.

Some properties that Y should have include:

1. $Y \neq \emptyset$, non-triviality.
2. Y is closed, i.e. y^n a sequence in Y and $y^n \to y$ imply that $y \in Y$.
3. $Y \cap \mathbb{R}^L_+ \subset \{0\}$, no free lunch.
4. $0 \in Y$, shutdown is possible.
5. $Y + \mathbb{R}^L_- \subset Y$, free disposal.
6. $y \in Y \Rightarrow -y \notin Y$, irreversibility.
7. $(\forall y \in Y)(\forall \alpha \in [0,1])[\alpha y \in Y]$, this is called non-increasing returns to scale.
8. $(\forall y \in Y)(\forall \alpha \geq 1)[\alpha y \in Y]$, this is called non-decreasing returns to scale.
9. $(\forall y \in Y)(\forall \alpha \geq 0)[\alpha y \in Y]$, this is called constant returns to scale. Geometrically, the name for this condition is "Y is a cone".
10. $Y + Y \subset Y$, additivity.
11. Y is a convex set, convexity of the technology.
12. Y is a convex cone.

Non-triviality, closedness, no free lunch, irreversibility are always assumed (unless perhaps to show how weird the math gets if they are not assumed). In "long run" analyses, shutdown is usually assumed. Convexity is also usually assumed, but as we have already seen, this may not be an assumption with a lot of bite. What kind of returns to scale are assumed is a matter of contention. A useful result to bear in mind is:

Lemma 5.4.1. *Y is additive and has non-increasing returns to scale if and only if Y is a convex cone.*

Some people take constant returns to scale as a self-obvious statement, as an axiom if you will. Perhaps this is right. It seems to me that if I had the powers of a major diety (not some small potatoes diety), then I could create a second, alternate universe in which I duplicated a given technology exactly. Thus, $y \in Y$ leads to $2y \in Y$. If I ran the second universe only α of the time, then I could in principle get $(1+\alpha)y \in Y$. If I can create one duplicate universe, why not many?

However, it's not clear to me that this argument is anymore than wishful thinking — it is a statement of the form "if I can do the following impossible thing, then for all $y \in Y$ and for all $\alpha \geq 0$, $\alpha y \in Y$." While logically true, the argument doesn't seem to have much more going for it. I am going to present two counter-arguments, one involving physics, the other involving the empirical observation that decreasing returns to scale always set in.

1. Physics – if I "increase an input pipe by a factor of 2," what have I done? Have I increased the diameter of the pipe by 2, in which case I've increased its capacity by 4, or have I increased its capacity by 2, in which case I've increased the volume of (say) metal in the pipe by $(\sqrt{2})^3$, or do I increase the metal in the pipe by 2, in which case the capacity of the valve goes up by $(2^{1/3})^2$. Note that these kinds of calculations are based on capacity at a given pressure, while real pipes deliver a whole schedule of capacities, in math, a pipe is a function from pressure to capacity. Also, the simple minded calculations done above (correctly? who knows?) are based on what are called smooth or laminar flows, usually not a good assumption. When there is turbulence in the pipe, things are much more complicated, and nobody knows the exact formula relating cross-section to capacity at given pressures. All of this means that increasing y to αy is rather trickier than it might appear, even when $\alpha = 2$.
2. Empirics – in situation after situation, we find that there are decreasing returns to scale. Why hold to the position that returns to scale must (axiomatically) be constant in the face of overwhelming empirical evidence to the contrary? One argument says, "Suppose there is some unmeasured input. Observing decreasing returns to scale in the measured y is consistent with constant returns to scale with the unmeasured input fixed." Usually the unmeasured input is thought of as "entrepreneurial talent." This is a rather hard concept to define, but I think that there is something there to be captured (i.e. entrepreneurial talent is not Gertrue Stein's Oakland). Anyhow, we have the following:

Lemma 5.4.2. *For any $Y \subset \mathbb{R}^L$, the set*

$$Y' = \{y' \in \mathbb{R}^{L+1} : (\exists y \in Y)(\exists \alpha \geq 0)[y' = \alpha(y, -1)]$$

is a cone, and is convex if Y is convex.

Exercise: Draw a picture of this construction.

5.5. **Profit Maximization and Cost Minimization.** The profit maximization problem is

$$\max p \cdot y \text{ subject to } y \in Y,$$

equivalently,

$$\max p \cdot y \text{ subject to } F(y) \leq 0.$$

We denote by $y(p)$ the argmax of this problem, the negative components of $y(p)$ are the demands for inputs, the positive components are the supply. We denote by $\pi(p)$ the function $p \cdot y(p)$. If at prices p, profits are unbounded above, we write $\pi(p) = +\infty$.

Already there is an important observation:

Lemma 5.5.1. *If Y exhibits nondecreasing rts, then either $\pi(p) \leq 0$, or $\pi(p) = +\infty$.*

Proof: Suppose that there exists $y \in Y$ such that $p \cdot y = r > 0$. By nondecreasing rts, for all $\alpha > 1$, $\alpha y \in Y$, which implies that maximal profits must be at least αr for any α, so set α larger and larger. ∎

Now, in general, we wouldn't want to assume that the transformation function F is differentiable in all goods. Suppose that you have a firm that produces ice cubes, y_1, using water, y_2, a freezer, y_3, and electricity, y_4. You neither use nor produce balsa wood, y_5, in the process. Since this is only one output, we can use a production function, $y_1 = f(z_2, z_3, z_4, z_5)$, $z_\ell \geq 0$, $\ell = 2, \ldots, 5$. Let us suppose that the production function f is differentiable, and that $\partial f / \partial z_5 \equiv 0$. The corresponding Y is the set

$$Y = \{(q, -z_2, -z_3, -z_4, -z_5) : q - f(z_2, z_3, z_4, z_5) \leq 0, \ (z_2, z_3, z_4, z_5) \geq 0\}.$$

The only possible F has some bad (non-differentiable) behavior along rays of the form $(r, 1, 1, 1, z_5)$, $r > 0$, as z_5 in the neighborhood of 0. So, we won't often be assuming that F is differentiable, unless we have implicitly or explicitly limited ourselves to the goods that enter into the firm's possibility set.

Suppose that we have a technology Y. Suppose that y^* is a point in the boundary of Y. Then the support functions of Y at y^* is

$$S_Y(y^*) = \{x \geq 0 : (\forall y \in Y)[x \cdot y \leq x \cdot y^*]\}.$$

If Y is convex, then such support functions exist by the supporting hyperplane theorem and a little bit of argument provided we assume that there is no free lunch and . It is now elementary to see that if $y^* \in y(p)$, then the FOC are

$$p \in S_Y(y^*).$$

If we restrict attention to dimensions ℓ for which Y is smooth, we have

$$p_\ell = \lambda \frac{\partial F}{\partial y_\ell}.$$

So when $L = 2$, we have a very simple-minded (read intermediate micro) result, that the slope of the trnasformation frontier at the optimum must equal the negative of the price ratio.

For a single output production function firm, the FOC for the problem

$$\max pf(z) - w \cdot z$$

are

$$pD_z f \leq w, \ z \geq 0, \ [pD_z f - w]z = 0.$$

If Y is convex or f is concave, the FOC are also sufficient.

Here is the catch-all result for profit functions and supply/demand functions:

Theorem 5.5.1. *Suppose Y is closed and satisfies free disposal.*

1. *π is $hd(1)$.*
2. *π is a convex function.*
3. *If Y is convex, then $Y = \{y \in \mathbb{R}^L : (\forall p \gg 0)[p \cdot y \leq \pi(p)]\}$.*
4. *y is $hd(0)$.*
5. *If Y is convex, then so is $y(p)$.*
6. *(Hotelling's Lemma) If $y(\bar{p})$ is a single point at $\bar{p}$, then $\pi(\cdot)$ is differentiable at $\bar{p}$, and $D_p\pi(\bar{p}) = y(\bar{p})$.*
7. *If $y(\cdot)$ is a differentiable function at $\bar{p}$, the $D_py(\bar{p}) = D^2_p\pi(\bar{p})$ is a symmetric, positive semi-definite matrix with $D_py(\bar{p})\bar{p} = 0$.*

The law of supply is contained in the last of these items. Note that

$$dy = D_py(p)dp,$$

so that

$$dpdy = dpD_py(p)dp \geq 0.$$

Go through supply and demand relations for single price changes. The discrete change version of the law of supply is

$$(\forall p, p')(\forall y \in y(p), y' \in y(p'))[(p - p')(y - y') \geq 0].$$

This follows directly from the definition of $y(p)$. In a previous homework problem, repeated directly below, we worked very hard with much stronger assumptions to derive the same results we just found. Examples such as this have effectively killed the habit of extensively teaching microeconomics with derivatives.[19]

Homework 3.1.45. *Suppose that $f : \mathbb{R}^N_{++} \to \mathbb{R}^1$ is strictly concave, twice continuously differentiable, and satisfies $D_xf \gg 0$. Further suppose that for some $p^\circ > 0$ and $w^\circ \gg 0$, the (profit) maximization problem*

$$\max_{x \in \mathbb{R}^N_{++}} \Pi(x) = pf(x) - w \cdot x$$

has a strictly positive solution.

1. *[Optional] Using the Implicit Function theorem, show that the solution to the above problem, $x^*(p, w)$, is a differentiable function of p and w on a neighborhood of (p°, w°).*
2. *Show that the derivative conditions for the optimum, x^*, are $pD_xf(x^*) = w$ or $D_xf(x^*) = \frac{1}{p}w$, and write these out as a system of equations. The x^* are called the factor demands.*
3. *The "supply function" is defined as $y(p, w)$. Note that $D_py(p, w) = D_xf(x^*)D_px^*$. Write this out using summation notation.*
4. *Taking the derivative with respect to p on both sides of the equivalence $D_xf(x^*(p, w)) \equiv \frac{1}{p}w$ gives the equation $D^2_xf(x^*)D_px^* = -\frac{1}{p^2}w$. This implies that $D_px^* = -\frac{1}{p^2}(D^2_xf(x^*))^{-1}w$. Write both of these out as a system of equations.*

[19]In other words, we have advanced beyond Paul Samuelson's dissertation.

5. *Using the negative definiteness of* $D^2_x f$, *Theorem* 2.1.3, *and the previous three parts of this problem, show that* $D_p y(p, w) > 0$.
6. *Using the previous part of this problem, show that it is not the case that* $D_p x^* \leq 0$.
7. *Let* e_n *be the unit vector in the* n*'th direction. Taking the derivative with respect to* w_n *on both sides of the equivalence* $D_x f(x^*(p, w)) \equiv \frac{1}{p} w$ *gives the equation* $D^2_x f(x^*) D_{w_n} x^* = \frac{1}{p} e_n$. *Write this out as a system of equations.*
8. *Pre-multiply both sides of* $D_{w_n} x^* = \frac{1}{p}(D^2_x f(x^*))^{-1} e_n$ *by* e_n. *Using Theorem* 2.1.3, *conclude that* $\partial x^*_n / \partial w_n < 0$.

The relation between profit maximization and cost minimization is simple – if the profit maximization problem has a solution, then the solution is cost minimizing. The cost minimization problem may have a solution even when the profit maximization problem doesn't.

For the single output case, the cost minimization problem is

$$c(w, q) = \min\ w \cdot z \text{ s.t. } f(z) \geq q.$$

The solutions to this problem are $z(w, q)$, and are called the **conditional factor demands**.

Here is the catch-all result for cost functions and conditional factor demand functions, there are only small changes from what we know about expenditure functions in preference maximization:

Theorem 5.5.2. *Suppose* Y *is a single output technology, is closed and satisfies free disposal. Let* $c(w, q)$ *denote the cost function and* $z(w, q)$ *the conditional factor demand correspondence.*

1. $c(\cdot, \cdot)$ *is* $hd(1)$ *in* w, *nondecreasing in* q.
2. $c(\cdot, \cdot)$ *is concave function of* w.
3. *If input requirements sets are convex for every* q, *then*
$$Y = \{(-z, q) : w \cdot z \geq c(w, q) \text{ for all } w \gg 0\}.$$
4. $z(\cdot, \cdot)$ *is* $hd(0)$ *in* w.
5. *If input requirement sets are convex, then so is* z.
6. *(Shepard's Lemma)* $D_w c(w, q) = z(w, q)$.
7. $D_w z = D^2_w c$ *is symmetric, negative semidefinite, and* $D_w z(\bar{w}, q)\bar{w} = 0$.
8. *If* f *is* $hd(1)$, *then* c *and* z *are* $hd(1)$ *in* q.
9. *If* f *is concave, then for all* w, c *is convex in* q

5.6. **Geometry of Cost and Supply in the Single-Output Case.** Go through the various examples.

5.7. **Externalities and Aggregation.** Intuitively, a production externality between two firms involves the activity level(s) of one firm affecting the output or necessary input levels of another. There are no externalities when there are no such interactions.

Definition 5.7.1. *If the combined technology of firms* 1 *and* 2 *is* $Y = Y_1 + Y_2$, *then there are no externalities.*

If $y_1 = (-x_1, q_1)$ (resp. $y_2 = (-x_2, q_2)$) where $x_1 \geq 0$ (resp. $x_2 \geq 0$ is firm 1's (resp. firm 2's) vector of inputs and $q_1 \geq 0$ (resp. $q_2 \geq 0$) is firm 1's vector of outputs, (resp. firm 2's) vector of outputs, then $y = y_1 + y_2$ being technologically feasible is a direct way to capture lack of interaction between firms. If it is possible to jointly produce $q = q_1 + q_2$ with fewer inputs than $x = x_1 + x_2$, then there is a positive externality. If it requires more than the $x = x_1 + x_2$ inputs to jointly produce $q = q_1 + q_2$, then there is a positive externality.

Lemma 5.7.1 (Perfect aggregation)**.** *If firms $j = 1, \ldots, J$ have no externalities, then for all $p > 0$, $y(p) = \sum_{j=1}^{J} y_j(p)$.*

The aggregation fails if there are externalities. We think that externalities are the rule rather than the exception. Indeed, a large part of the arguments about the historical development of economies is based on the idea that there are huge positive externalities between firms.

Anyhow, since cost functions characterize technologies, one way to capture externalities is through cost functions. One way to do this is to suppose that for at least one of two firms, i, j,

$$\frac{\partial c_i(w, q_i, q_j}{\partial q_j} \neq 0.$$

The firms' profit maximization problems are

$$\max_{q_i} p_i q_i - c_i(w, q_i, q_j).$$

Assuming that each c_i is differentiable and convex in q_i, the FOC characterize the solutions. They are

$$p_i = D_{q_i} c_i(w, q_i, q_j).$$

Let $q^* = (q_i^*, q_j^*)$ denote the solution set to these equations.

Suppose that one were to operate the firms jointly. The joint profit maximization problem is

$$\max_{q_i, q_j} p_i q_i + p_j q_j - (c_i(w, q_i, q_j) + c_j(w, q_i, q_j)).$$

Since the sum of concave functions is concave, the FOC characterize the solutions. They are

$$p_i = D_{q_i} c_i(w, q_i, q_j) + D_{q_i} c_j(w, q_i, q_j),$$

$$p_j = D_{q_j} c_i(w, q_i, q_j) + D_{q_j} c_j(w, q_i, q_j).$$

At least one of the terms $D_{q_j} c_i$ and $D_{q_i} c_j$ is not equal to 0. Therefore, no $q \in q^*$ can solve these two equations. Therefore, the joint profit maximizing profit must be strictly higher than then sum of the individual profit maximizing profits.

Dates: Nov. 2, 7, 9, 14, & 16.

6.1. **Homeworks.**

Due date: Tuesday, Nov. 21.
From MWG: Ch. 6: B.1, 2; C.1, 2, 4(a),(b), 13, 15, 16; D.2; E. 1; F.2, and homework problems in the notes below.

6.2. **On Probability Spaces and Random Variables.** The basic model that we use for randomness is due to Kolmogorov. A probability space is a non-empty set, Ω, a collection $\mathcal{F}$ of subsets of Ω, and a probability $P : \mathcal{F} \to [0, 1]$. Sets $E \in \mathcal{F}$ are called events. These are the sets that we need to be able to assign probabilities to. In our present development, and in almost all that you will see unless you do research on ambiguity, we are going to assume that

1. $E_1, E_2 \in \mathcal{F}$ implies that $E_1 \cup E_2 \in \mathcal{F}$ and $E_1 \cap E_2 \in \mathcal{F}$,
2. $\Omega \in \mathcal{F}$,
3. $E \in \mathcal{F}$ implies that $\Omega \backslash E \in \mathcal{F}$,
4. $P(\Omega) = 1$,
5. for all $E_1, E_2 \in \mathcal{F}$, $E_1 \cap E_2 = \emptyset$ implies $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

A random variable taking values in a set C is a **measurable** function $X : \Omega \to C$. The idea is that we do not observe which $\omega \in \Omega$ happened, but we do observe the value $X(\omega)$. To explain what measurability is about, we need the following: for $B \subset C$, $P(X^{-1}(B))$ should be the probability that the random variable X ends up in B. Further, for this to be sensible, we must assume that for all $B \subset C$ that we care about, $X^{-1}(B) \in \mathcal{F}$. When this happens, we say that X is **measurable**, or that X is a **C-valued random variable**, or just an rv.

Homework 6.2.1. *When C is finite and $X : \Omega \to C$, then*

$$(\forall B \in 2^C)[X^{-1}(B) \in \mathcal{F}] \Leftrightarrow (\forall c \in C)[X^{-1}(c) \in \mathcal{F}].$$

When Ω is finite and $C \subset \mathbb{R}^N$, the expectation of an rv X is

$$E\,X = \sum_{c \in C} c \cdot P(X^{-1}(c)).$$

We can rearrange the summation so that

$$E\,X = \sum_{c \in C} c \cdot P(X^{-1}(c)) = \sum_{\omega \in \Omega} X(\omega) P(\{\omega\}).$$

When Ω is finite (and in many situations where Ω is infinite), we might as well assume that for each $\omega \in \Omega$, $\{\omega\} = \bigcap\{E : \omega \in E \in \mathcal{F}\}$. In other words, we might as well assume that the smallest events are of the form $\{\omega\}$. This implies that all (bigger) events are the unions of these smallest events. If we observe $X = c$, we conclude that the true ω belongs to $X^{-1}(c)$. Thus, a fully revealing random variable would be one that is one-to-one rather than many-to-one.

C-valued random variables induce distributions on C. The distributions are called **image laws** or **induced distributions**. Specifically, any random variable X gives rise to the image law P_X on C defined by $P_X(B) = P(X^{-1}(B))$.

Let ι denote the identity function from C to C (that is, $\iota : C \to C$ satisfies $\iota(c) = c$ for all $c \in C$). Substituting integral signs for summations, the equalities for $E\,X$ given just above are

$$E\,X = \int_C c\,dP_X(c) = \int_\Omega X(\omega)\,dP(\omega).$$

This is just the change-of-variable formula from calculus.

Again, let us keep with finite Ω, the math to do all this for infinite Ω's is fun, and you should be exposed to it before you get out of graduate school, but we won't do it here. If X and Y are rv's taking values in $\mathbb{R}^N$, then $f(\omega) = E(Y\big|X)(\omega)$ is a function of ω, i.e. another random variable. Specifically, for any c such that $P(X^{-1}(c)) > 0$ and for all ω such $X(\omega) = c$,

$$f(\omega) = \int_\Omega Y(\omega)\,dQ(\omega) \text{ where } Q(B) = \frac{P(B \cap X^{-1}(c))}{P(X^{-1}(c))}.$$

In words, condition P on the set $\{\omega : X(\omega) = c\}$ and integrate Y using the resulting conditional distribution. Notice that $f(\omega) = f(\omega')$ if $X(\omega) = X(\omega')$.

Homework 6.2.2. *For all of this problem,* $\Omega = \{1a, 1b, 2a, 2b, 3a, 3b, \ldots, 10a, 10b\}$, $\mathcal{F} = 2^\Omega$, *and for each* $\omega \in \Omega$, $P(\omega) = \frac{1}{20}$. *For* $n \in \{1, \ldots, 10\}$, *the random variables* X, Y, Z *are defined by*

$$X(n_a) = X(n_b) = n,\ Y(n_a) = Y(n_b) = \begin{cases} 1 & \text{if } n \leq 5 \\ 0 & \text{if } n > 5 \end{cases},$$

and $Z(1a) = Z(1b) = Z(10a) = Z(10b) = 0$, $Z(n_a) = -1$, *and* $Z(n_b) = +1$ *for* $2 \leq n \leq 9$.

1. *Find* $E\,X$, $E\,X^2$, *and the variance of* X.
2. *Find* $E\,Y$, $E\,Y^2$, *and the variance of* Y.
3. *Find* $E\,Z$, $E\,Z^2$, *and the variance of* Z.
4. *Find the function* $E(X\big|Y)$ *and verify that* $E\,X$ *is equal to* $E\,E(X\big|Y)$.
5. *Find the function* $E(Y\big|X)$ *and verify that* $E\,Y$ *is equal to* $E\,E(Y\big|X)$.
6. *Find the function* $E(X\big|Z)$ *and verify that* $E\,X$ *is equal to* $E\,E(X\big|Z)$.
7. *Show that the function* $E(Z\big|X)$ *is constant and equal to* 0.
8. *Let* R *be the rv* $X + Z$. *Give the image law of* R. *Argue that people who dislike risk would prefer* X *to* R.

Homework 6.2.3. *Show that for all* $\mathbb{R}^N$*-valued random variables* X *and* Y *on a finite* Ω, $E\,E(X\big|Y) = E\,X$.

The lessons and definitions from this section work for more Ω, we will sweep many details under the rug.

6.3. **Lotteries.**

IMPORTANT: We assume that our preferences over random variables depend only on their image laws. The set of image laws is called the set of lotteries.

With a finite set of consequences $C = \{1, \ldots, N\}$, the set of lotteries is

$$\mathcal{L} = \Delta(C) = \{P \in \mathbb{R}^N_+ : \sum_{i=1}^{N} P_i = 1\}.$$

We assume that there is a rational preference ordering, $\succeq$, over $\mathcal{L}$. Compound lotteries are

$$(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K),$$

with the interpretation that there is an independent probability α_k event that you will get lottery L_k. Such a compound lottery reduces to the lottery

$$\sum_k \alpha_k L_k.$$

When the set of consequences, C, is a subset of $\mathbb{R}$, the set of lotteries $\Delta(C)$, is a bit more complicated. We will identify $\Delta(\mathbb{R})$ with the set of cdf's — a cdf is a function $F : \mathbb{R} \to [0, 1]$ such that

1. $x < y \Rightarrow F(x) \leq F(y)$,
2. $\lim_{x \uparrow \infty} F(x) = 1$, $\lim_{x \downarrow -\infty} F(x) = 0$,
3. for all x, $\lim_{\epsilon \downarrow 0} F(x + \epsilon) = F(x)$.

For those of you who have seen the development, countably additive Borel measures on $\mathbb{R}$ can be identified with their cdf's, and the $\mathcal{F}$ being used is the (universal completion of) the smallest σ-field containing the open sets.

On the assumption that $C \subset \mathbb{R}$ and that $c_1 < c_2$ implies that c_2 is prefered to c_1, we can talk about when one lottery is clearly better than another. This is called **stochastic dominance**.

6.4. **Stochastic Dominance.** There are two kinds of stochastic dominance that we're going to discuss. Since we're not very imaginative, we'll call them First Order Stochastic Dominance (FOSD) and Second Order Stochastic Dominance (SOSD).

6.4.1. *First Order Stochastic Dominance.* Consider two distributions, F and G.

Definition 6.4.1. *F* **first order stochastic dominates** *G, written $F \succeq_{FOSD} G$, if for all $x \in C$, $F(x) \leq G(x)$.*

It seems sensible to restrict our attention to preferences $\succeq$ on $\mathcal{L}$ that satisfy $F \succeq_{FOSD} G$ implies $F \succeq G$. Pick any number x, the probability that the random variable with cdf G is less than or equal to x is larger than the corresponding probability for the random variable with cdf F. This means that G is F shifted to the left, towards the bad stuff. No-one should enjoy that kind of shift.

Theorem 6.4.1. *$F \succeq_{FOSD} G$ if and only if for all bounded, non-decreasing $u : C \to \mathbb{R}$, $\int u\, dF \geq \int u\, dG$.*

So, let's see what this means when the distributions have finite support or carrier – a **carrier** of a probability distribution P is a set E with the property that $P(E) = 1$. An important implication of the next problem is that even in the case of 3 consequences, $\succeq_{FOSD}$

is not a complete ordering of $\mathcal{L}$. The last part of the problem is the beginnings of our thinking about second order stochastic dominance.

Homework 6.4.1. *Suppose that $C = \{10, 20, 30\}$, and that the lottery Q is given by $Q(10) = Q(20) = Q(30) = 1/3$.*

1. *Draw Q in the 2-dimensional simplex $\mathcal{L}$.*
2. *Draw the set of all lotteries have the same expectation as Q.*
3. *Draw the set S of all lotteries P having the property that for all $x \in \mathbb{R}$, $P(-\infty, x] \leq Q(-\infty, x]$. Every $P \in S$ is arguably better than Q, after all, every $P \in S$ puts lower probability on sets of outcomes "x or less." We assume that everyone prefers P's in S to Q.*
4. *Draw the set T of all lotteries P having the property that for all $x \in \mathbb{R}$, $Q(-\infty, x] \leq P(-\infty, x]$. Every $P \in T$ is arguably worse than Q, after all, every $P \in S$ puts higher probability on sets of outcomes "x or less." We assume that everyone prefers Q to P's in T.*
5. *Let $\mathcal{ND} = \{u \in \mathbb{R}^C : u_{10} \leq u_{20} \leq u_{30})\}$ be the set of non-decreasing, real-valued functions on C. Show directly (i.e. don't simply cite the Theorem 6.4.1) that the set S' of all lotteries P having the property that for all $u \in \mathcal{ND}$, $u \cdot P \leq u \cdot Q$ satisfies $S' = S$.*
6. *Suppose that $\succeq$ satisfies*
   (a) $F \succeq_{FOSD} G \Rightarrow F \succeq G$,
   (b) $\succeq$ *has indifference curves in $\Delta(C)$.*
   Identify the region of $\Delta(C)$ in which the indifference curve through Q must lie.
7. *Draw the set of all lotteries that can be arrived at from Q by taking mass away from $c = 20$ and spreading it over 10 and/or 30. [Even along the line of points having the same expectation as Q, some are clearly better than Q, some are clearly worse.]*

6.4.2. *Second Order Stochastic Dominance.* SOSD is useful in comparing cdfs having the same mean.

Definition 6.4.2. *For F and G having the same mean, F* **second order stochastic dominates** *G, written $F \succeq_{SOSD} G$, if for any non-decreasing, concave u, $\int u\,dF \geq \int u\,dG$.*

One obvious way to make someone who dislikes risk, as risk averter, worse off while keeping the mean the same is to scoop some mass out of the middle of the distribution and put it on the ends. For F, G let P_F and P_G denote the corresponding probabilities.

Definition 6.4.3. *G differs from F by an (a, b)* **spread** *if for any $(a', b') \subset (a, b)$, $P_G(a', b') \leq P_F(a', b')$, and for any $(c', d') \subset (-\infty, a) \cup (b, +\infty)$, $P_G(c', d') \geq P_F(c', d')$. An (a, b) spread is* **mean preserving** *if F and G have the same mean.*

Important: From here on, all spreads are, by assumption, mean preserving spreads unless I explicitly say otherwise.

Theorem 6.4.2. *If G differs from F by a (mean preserving) (a, b) spread, then $F \succeq_{SOSD} G$.*

The converse isn't true, but something close to it is true.

Homework 6.4.2. *Let Q be the probability $(1/3, 1/3, 1/3)$ on the carrier $C = \{10, 20, 30\}$. Both algebraically and as a subset of the 2-dimensional simplex, give*

1. *The set of P that are mean preserving spreads of Q.*
2. *The set of P such that Q is a mean preserving spreads of P.*
3. *Suppose that $\succeq$ satisfies*
   (a) $F \succeq_{FOSD} G \Rightarrow F \succeq G$,
   (b) $F \succeq_{SOSD} G \Rightarrow F \succeq G$, *and*
   (c) $\succeq$ *has indifference curves in* $\Delta(C)$.
   Identify the region of $\Delta(C)$ in which the indifference curve through Q must lie.

To talk about the general relation between SOSD and mean preserving spreads, we need a detour.

6.4.3. *Detour Through the Weak Convergence of Distributions.*

Definition 6.4.4. *A sequence F_n of cdf's* **converges weakly** *to the cdf F if for all continuity points, x, of F, $F_n(x) \to F(x)$.*

Weak convergence is the kind of convergence discussed in the central limit theorem.

The following is simple but useful: for any random variable $X \geq 0$ with cdf F,

$$E\,X = \int_{[0,\infty)} x\,dF(x) \geq \int_{[t,\infty)} x\,dF(x) \geq \int_{[t,\infty)} t\,dF(x) = tP(X \geq t).$$

Rearranging,

$$P(X \geq t) \leq \tfrac{1}{t} E\,X.$$

Homework 6.4.3. *Let X_n be a sequence of i.i.d. random variables with mean μ and variance $\sigma^2 > 0$. Let S_n be the random variable $\frac{1}{n}\sum_{i=1}^{n}(X_n - \mu)$, and let F_n be the cdf of S_n. Proving any inequality you use, show that F_n converges weakly to the cdf*

$$F(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

but that $F^n(0)$ need not converge to $F(0)$.

The part of the central limit theorem that delivers Gaussian limits is not the only part. The following shows that there are other distributions that are the limits of sums of very small independent random variables.

Homework 6.4.4. *Let X_n the sum of n i.i.d. random variables, $Y_{n,k}$, $k = 1, \ldots, n$, with $P(Y_{n,k} = 1) = \lambda/n$, $P(Y_{n,k} = 0) = 1 - \lambda/n$. Let F_n be the cdf of X_n. Find the weak limit of the sequence F_n.*

6.4.4. *SOSD Redux.* Suppose that X, Y, Z are $\mathbb{R}$-valued random variables, that $E(Y\big|X) = 0$, and that $Z = X + Y$. It seems pretty clear that $X \succeq_{SOSD} Z$ in this case.

The basic theoretical result on $\succeq_{SOSD}$ is

Theorem 6.4.3. *Let X_F and X_G be a pair of random variables having cdfs F and G, and carried by the set $[a, b]$. The following three statements are equivalent:*

1. *$F \succeq_{SOSD} G$,*
2. *$G = \lim_n F_n$ where for all n, F_n is carried on $[a, b]$, and F_{n+1} is a mean preserving spread of F_n,*
3. *there exists a rondom variable X_H with bounded carrier satisfying $X_G = X_F + X_H$ and $E(X_H \big| X_F) = 0$.*

When F and G do not have bounded carriers, the first two parts of this theorem are not equivalent. Alfred Müller's example shows that it is possible to find a sequence F^n where (1) each F^n has mean 0, (2) each F^{n+1} is a mean preserving spread of F^n, and (3) the weak limit of the sequence F^n is point mass on 0. In other words, it is possible to find a sequence of distributions becoming more and more risky but converging to absolutely no risk at all.

6.5. **The Independence Assumption on $\succeq$.** For rational preference orderings, $\succeq$, there are two additional assumptions that we will make, one technical (continuity), and one substantive (independence). We will then exam the conditions under which the substantive assumption is consistent with both respecting FOSD, $F \succeq_{FOSD} G \Rightarrow F \succeq G$, and respecting SOSD, $F \succeq_{SOSD} G \Rightarrow F \succeq G$.

Throughout, we will use the following piece of **notation**: Rather than writing $\alpha L + (1 - \alpha)L'$, I will write $L\alpha L'$ to be read "L with weight α, L' with the rest".

Continuity: $\forall L, L', L''$ the following sets are closed:

$$\{\alpha \in [0, 1] : L\alpha L' \succeq L''\} \text{ and } \{\alpha \in [0, 1] : L'' \succeq L\alpha L'\}.$$

Indepedence: $(\forall L, L', L'')(\forall \alpha \in (0, 1))[L \succeq L']$ iff $[L\alpha L' \succeq L\alpha L'']$.

Lemma 6.5.1. *The indifference curves of a preference ordering satisfying independence are parallel straight lines.*

A utility function U on $\mathcal{L}$ has the **expected utility form** if it is linear, that is, if it is of the form

$$U(L) = L \cdot u$$

for some $u \in \mathbb{R}^N$.

The crucial result for finite C is

Theorem 6.5.1. *$\succeq$ satisfies independence and continuity iff it can be represented by a utility function U having the expected utility form.*

Note that we have introduced cardinality, the size differences between the u_i in the vector u mean something. For example, let x, y, r, s be four elements of C, and suppose that $u_x - u_y > u_r - u_s$. We would usually say that there is no meaning to saying that may preferences for x over y is larger than my preference for r over s. This is because such a statement would not survive monotonic transformations. This may be an overly formal understanding of utility representations, but it's the one we've worked with until now. Notice however that

$$[u_x - u_y > u_r - u_s] \Leftrightarrow [u_x + u_s > u_r + u_y] \Leftrightarrow [\frac{1}{2}u_x + \frac{1}{2}u_s > \frac{1}{2}u_r + \frac{1}{2}u_y].$$

In other words, $u_x - u_y > u_r - u_s$ implies something definite about preferences over lotteries.

Recall that $F(x)$ is the probability that the lottery delivers a monetary payoff less than or equal to x. We let $X \sim F$ be a random variable with the cdf F. If C is a bounded subset of $\mathbb{R}$ (indeed, if it's a subset of a compact Hausdorff space if you want to get fancy), then modulo one stronger kind of continuity assumption, we have

Theorem 6.5.2. *If $\succeq$ on $\mathcal{F}$ satisfies (a stronger form of continuity) and the independence axiom, there there exists a continuous $u : C \to \mathbb{R}$ such that*

$$F \succeq F' \text{ iff } \int_C u(x)\, dF(x) \geq \int_C u(x)\, dF'(x).$$

We now study the implications of the independence axiom by looking at the case $C \subset \mathbb{R}$.

6.6. **Applications to Monetary Lotteries.** Here $C \subset \mathbb{R}$, the distributions are given by cdf's, $F : \mathbb{R} \to [0, 1]$ that are non-decreasing, continuous from the right, and satisfy

$$\lim_{x\to\infty} F(x) = 1 - \lim_{x\to-\infty} F(x) = 0.$$

Let $\mathcal{L}$ denote the set of cdf's. To make some technical stuff easier, were going to assume that there exists an $M > 0$ such that $C \subset [-M, +M]$.

We're going to open with a pair of very quick discussions, first of expected utility maximizing preferences that respect FOSD, of expected utility maximizing preferences that demonstrate risk aversion. The applications will look to the behavior of risk averse, expected utility maximizers whose preference orderings respect FOSD.

Theorem 6.6.1. *If $\succeq$ has an expected utility representation with utility function $u(\cdot)$, then $\succeq$ respects FOSD iff $u(\cdot)$ is non-decreasing.*

Definition 6.6.1. *$\succeq$ is* **risk averse** *if $\forall F \in \mathcal{L}$ with $E\,X = \int x\, dF(x)$ finite, $E\,X$ for sure is at least weakly prefered to X.*

Homework 6.6.1. *Show that X is a mean preserving spread of the (degenerate) rv that delivers $E\,X$ for sure. In other words, show that any preference ordering that respects SOSD is risk averse. [This problem really is trivial, it's here to make sure you see what's happening.]*

Theorem 6.6.2. *If $\succeq$ has an expected utility representation with utility function $u(\cdot)$, then $\succeq$ is risk averse iff $u(\cdot)$ is concave.*

To prove this we use Jensen's inequality.

Theorem 6.6.3 (Jensen's inequality)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is concave, then $f(E\,X) \geq E\,f(X)$.*

Going back to the definition of SOSD, we conclude that, amongst all preferences $\succeq$ that have an expected utility representation, respecting SOSD is equivalent to risk aversion.

The following pair of definitions and the will help in several of the applications.

Definition 6.6.2. *Given a Bernoulli utility function $u(\cdot)$, the* **certainty equivalent** *of a lottery X is denoted by $c(X, u)$ and defined as the number satisfying*

$$u(c(X, u)) = E\,u(X).$$

The **probability premium at** x **and** ϵ *is the number* $\Pi(x,\epsilon,u)$ *with the property that*

$$u(x) = (\tfrac{1}{2} + \Pi(x,\epsilon,u))u(x+\epsilon) + (\tfrac{1}{2} - \Pi(x,\epsilon,u))u(x-\epsilon).$$

Theorem 6.6.4. $u(\cdot)$ *is concave iff* $(\forall X)[c(X,u)] \leq E\,X$ *iff* $(\forall x, \epsilon > 0)[\Pi(x,\epsilon,u) \geq 0]$.

6.6.1. *Risk averse demand for insurance: I.* Your initial wealth is w. You face a probability π of a loss of size D, can buy a policy paying α in case of a loss at a price αq (so that q is the premium). Solve the problem

$$\max_{\alpha \geq 0}\ f(\alpha) = (1-\pi)u(w - \alpha q) + \pi u(w - \alpha q - D + \alpha).$$

Look at the FOC (sufficient because f is concave), note that when $q = \pi$, $\alpha^* = D$. Usually insurance companies do not sell full coverage insurance policies. Another way this is sometimes said is that insurance companies require that you be partially self-insured.

6.6.2. *Risk averse demand for insurance: II.* Let $X \geq 0$ denote the random loss you may suffer. You can ensure that you only suffer the fraction $\beta \in [0,1]$ of your loss at a price $(1-\beta)p$. (In other words, to suffer only 10% of your loss, you pay 90% of the price p.) This means that after insuring yourself of only suffering β of your loss, your random position is given by $w - (\beta X + (1-\beta)p)$. Solve the problem

$$\max_{\beta \in [0,1]}\ f(\beta) = Eu(w - (\beta X + (1-\beta)p)).$$

Again, we can look at FOC because $f(\cdot)$ is concave. Note that actuarially fair insurance in this case has $p = E\,X$. In this case, note that

$$(\forall \beta)[E\,(w - (\beta X + (1-\beta)p)) = w - \beta E\,(X + E\,X)) - E\,X = w - E\,X.$$

Now, the definition of risk aversion means that $\beta^* = 0$ is one of the optima here. Again, no self-insurance is the optimum, but insurance companies do not sell such policies.

Homework 6.6.2. *A risk averse expected utility maximizer has an initial wealth of* $w = 10{,}000$ *and a von Neumann-Morgenstern utility function* $u(x) = 200\sqrt{x}$. *They face a random loss* $X \geq 0$ *with the distribution* $\pi = P(X = 1{,}000) = 1/20$ *and* $(1-\pi) = P(X = 0) = 19/20$. *The consumer can ensure that they only suffer the fraction* $\beta \in [0,1]$ *of their loss at a price* $(1-\beta)p$. *(E.g., to suffer only* $\beta = 15\%$ *of their loss, they pay* $(1-\beta) = 85\%$ *of the price* p.*) Formulate and solve the consumer's insurance demand problem as a function of* p, $p \geq E\,X$, *and evaluate when* $p = E\,X$.

The following is yet another way to think about modeling demand and supply of insurance.

Homework 6.6.3. *Your present wealth is* W. *You are an expected utility maximizer with a strictly concave von Neumann-Morgenstern utility function* $u(\cdot)$ *defined on wealth. You may suffer one of three losses, corresponding to states* $s \in \{0, L, L'\}$ *with* $0 < L < L' < W$ *and probabilities* P_s *which are strictly positive (and sum to 1). Your initial, random wealth is given by*

$$X = (W, P_0; W - L, P_L; W - L', P_{L'}),$$

and your certainty equivalent is $c = c(X,u)$.

You are going to sign a contract with a risk neutral insurance company. This contract specifies your wealth, $w(s)$*, in each of the three states,* $s \in \{0, L, L'\}$*.*

For you to be willing to sign the contract, it must be the case that

$$\sum_s u(w(s))P(s) \geq E\,u(X).$$

For the risk neutral insurance company to be willing to sign the contract, it must be the case that

$$\sum_s w(s)P(s) \leq E\,X.$$

For $0 \leq \alpha \leq 1$*, consider the social utility function for the contract* $w = w(\cdot)$

$$V_w^\alpha = \alpha \underbrace{\sum_s u(w(s))P(s)}_{\textit{consumer's } E\,u} + (1-\alpha) \underbrace{\sum_s -w(s)P(s)}_{\textit{insurer's } E\,u}.$$

1. *When* $\alpha = 1$*, solve the problem* $\max_w V_w^\alpha$ *subject to the constraints that both you and the insurance company are willing to sign the contract.*
2. *When* $\alpha = 0$*, solve the problem* $\max_w V_w^\alpha$ *subject to the constraints that both you and the insurance company are willing to sign the contract.*
3. *Characterize the set of solutions to the problem* $\max_w V_w^\alpha$ *for* $0 < \alpha < 1$*.*

6.6.3. *Demand for risky assets.* Suppose that your initial wealth is w and you can invest $\alpha \leq w$ in a risky (limited liability) asset with random per unit return $Z \geq 0$, $E\,Z > 1$. The assumption that $Z \geq 0$ means that we are considering what is called a **limited liability stock**. Historically, the development of limited liability stocks was crucial to the development of the modern stock market in England.

Solve the problem

$$\max_{\alpha+\beta=w} E\,u(\alpha Z + \beta).$$

This is equivalent to

$$\max_{0\leq\alpha\leq w} \varphi(\alpha) = E\,u(w + \alpha(Z-1)).$$

Note that φ is concave, and that

$$\varphi' = \frac{\partial\varphi}{\partial\alpha} \int u(w + \alpha(z-1))\,dP(z) =$$

$$\int \frac{\partial u(w + \alpha(z-1)}{\partial\alpha}\,dP(z) = \int u'(w + \alpha(z-1))(z-1)\,dP(z)$$

provided we can interchange differentiation and integration. This gives

$$\varphi'(0) = \int u'(w)(z-1)\,dP(z) = u'(w) \int (z-1)\,dP(z) > 0,$$

implying that $\alpha^* > 0$.

The constraint $\alpha \leq w$ may be binding. This corresponds to not allowing a person to borrow money to invest.

This problem is often formulated with $\alpha \in [0,1]$ representing the proportion of wealth that is put into the risky asset. This would be of the form

$$\max_{0\le\alpha\le 1} \varphi(\alpha) = E\,u((1-\alpha)w + \alpha w Z) = E\,u(w + \alpha w(Z-1)).$$

6.6.4. *Portfolio choice theory.* Suppose that your initial wealth is w and you can invest α_k in $Z_k \ge 0$, $E\,Z_k \ge 1$ risky assets. For notational simplicity, assume $Z_1 \equiv 1$ (this represents holding assets as money). Solve the problem

$$\max_{\alpha_k \ge 0,\ \sum_k \alpha_k = w} \varphi(\alpha_1, \dots, \alpha_K) = E\,u(\sum_k \alpha_k Z_k).$$

Again, the FOC are sufficient here because $\varphi(\cdot)$ is concave. Here the assumption that $\alpha_k \ge 0$ can be thought of as there being no short sales.

6.6.5. *Portfolio choice theory when you also face losses.* You have w, face losses $X \ge 0$, can invest in a risky (limited liability) stock with return $Z \ge 0$, $E\,Z > 1$. Even if insurance is actuarially fair, you may want to self-insure when X and Z are not independent.

If $Z = z$ and $X = x$, and $\alpha \in [0,1]$ of your wealth is in the risky asset while you have bought insurance that means that you suffer only $\beta \in [0,1]$ of the loss at a price $(1-\beta)p$, then your wealth is

$$w + \alpha w(z-1) - (\beta x + (1-\beta)p).$$

This means that the expected utility maximizer solves the problem

$$\max E\,u(w + \alpha w(Z-1) - (\beta X + (1-\beta)p)) \text{ s.t. } \alpha, \beta \in [0,1].$$

Homework 6.6.4. *Supposing that $u' > 0$, $u'' < 0$, $E\,Z > 1$, X and Z are independent, and $p = E\,X$, show that the solution to the previous problem involves $\alpha^* > 0$ and $\beta^* = 0$. Intuitively, what kind of conditions on the interdependence of X and Z might make $\beta^* > 0$? Can you give a formal result?*

6.7. **Some More Comments on Insurance Markets.** Insurance markets are fascinating in this context because they are, most often, not pure markets. A pure market for insurance would involve insurance companies calculating odds on the basis of collected information, and then setting prices in some kind of a competitive fashion. But we see much more than this, insurance companies exhibit a variety of strategic behaviors. Insurance companies

1. help write building and fire safety codes,
2. require compliance with 3'rd party regulation (such as maritime classification societies),
3. audit,
4. perform background checks on employees when bonding is sought,
5. inspect,
6. supervise,
7. change rates,
8. set rates on the basis of cumulative loss experience,
9. write policies with deductibles.

In all of these cases, they are trying to manage what Heimer calls reactive risk.[20]

The basic idea is quite simple — after an insurance policy has been written and accepted, the motivations of the policyholder are changed. If someone has bought fire insurance, it is no longer quite so pressing to have regular, expensive, time-consuming fire equipment checks and fire drills. The problem for the insurance companies is that this means that the original odds no longer hold. This is because the policyholder reacts to the changed situation they find themselves in. A frequently-used response to this kind of problem is to require that (say) warehouses meet safety regulations on a regular basis, and if they do not, the policy is void. This is strategic behavior designed to counter the reactive element in the risk faced by the insurance company.

In a similar vein, being insured for the value of a ship and its cargo changes the incentives of the shipowner, maritime insurance has a fairly strict set of criteria for when it will pay for a loss:

1. Policyholders must have met the requirements of the classification society;
2. vessels must have been seaworthy;
3. vessels must have followed specified routes;
4. policyholders and their agents must have "sued and labored" to try to reduce the losses to the insurer.[21]

These kinds of policies are aimed at making the policyholder behave as the insurer would have behaved (given the chance). These activities are and should be part of the study of markets, but they are not the kinds of supply and demand decisions we generally discuss in the theory of markets so far.

We would expect a variety regularities out of these observations on strategic behavior, regularities that an economist would hope to capture through comparative statics:

1. When the policyholder has less control over the actions that lead to loss, as in a large corporation, we would expect less strategic behavior, especially if strategic behavior is costly.
2. More generally, when the cost of observing the actions taken by the policyholder go up, we would expect less strategic behavior.
3. When there are economies of scale in defining strategic behaviors, we would expect third party determination of the actions the insurer requires policyholder to carry out (e.g. uniform fire codes and insurance contracts just require meeting the pre-defined fire code, this saves on case by case negotiation about what exact rules should be followed in each building, and also has the effect of blunting some of the pressure on insurance agents to cut corners in their requirements in order to make a sale).
4. When there are economies of scale in monitoring preventive actions taken by the policyholder, we would expect third parties to be responsible for making sure that the preventive actions are taken (e.g. fire marshals inspect buildings for violations, not insurance companies).

---

[20] *Reactive Risk and Rational Action*, Berkeley, University of California Press (1985).

[21] This from Heimer (1985, p. 202). One of my favorite examples is the pre-telegraph/radio rule that the insurance company would not pay for a lost ship if too many of the crew survived.

6.8. **Comparing Degrees of Risk Aversion.** Shift and scale the concave Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$ so that they have the same slope at x. The one with the larger (in absolute value) second derivative at x is the one that is locally more risk averse. This is one motivation for $r_A(x) := -u''(x)/u'(x)$.

Another way to get at r_A is to note that the second derivative of $\Pi(x, \epsilon, u)$ w.r.t. ϵ evaluated at $\epsilon = 0$ is $r_A(x)/4$.

The utility function $u(x) = -e^{-ax}$, $a > 0$, is interesting in this regard.

Theorem 6.8.1. *The following are equivalent:*

1. $(\forall x)[r_A(x, u_2) \geq r_A(x, u_a)]$.
2. u_2 *is a concave increasing transformation of* u_1.
3. $(\forall X)[c(X, u_2) \leq c(X, u_1)]$.
4. $(\forall x, \epsilon)[\Pi(x, \epsilon, u_2) \geq \Pi(x, \epsilon, u_1)]$.
5. $(\forall \overline{x}, X)[E\, u_2(X) \geq u_2(\overline{x}) \Rightarrow E\, u_1(X) \geq u_1(\overline{x})]$.

6.9. **A Social Choice Application.** There are different kinds of errors that social systems can make, and any system involving humans will make mistakes. Next semester you will look at social choice issues more formally, but that theory does not explicitly think about probabilities. Here's an introduction to issues you might think about.

Homework 6.9.1. *The police arrest a man and accuse him of a crime. Given the police department's record, there is a prior probability* ρ, $0 < \rho < 1$, *that the man is guilty,* $\omega = g$, *and a* $(1 - \rho)$ *probability that the man is innocent,* $\omega = i$. *The man will be tried in front of a jury of* M *people. These* M *people will cast random, stochastically independent votes,* $V_m = G$ *for guilty and* $V_m = I$ *for innocent,* $m = 1, \ldots, M$ *with probabilities*

$$P(V_m = G \big| \omega = i) = p, \quad P(V_m = G \big| \omega = g) = q, \quad 0 < p < \frac{1}{2} < q < 1.$$

Suppose that social utility depends on the innocence or guilt of the defendant, $\omega = i, g$, *and the jury's decision,* $V = I, G$, *and*

$$\underbrace{0 = u(V = G \ \&\ \omega = i)}_{\textit{worst mistake}} < \underbrace{u(V = I \ \&\ \omega = g) = r}_{\textit{mistake}} <$$
$$\underbrace{u(V = G \ \&\ \omega = g) = u(V = I \ \&\ \omega = i) = 1}_{\textit{correct decision}}.$$

The number r *measures which kind of mistake you think is worse — r close to* 0 *means that you'd be willing to run a substantial risk of sending innocent people to jail so as to avoid letting the guilty go free very often;* r *close to* 1 *means that you'd be willing to run the risk of letting the guilty go free in order to avoid locking up the innocent. It is easy to believe that racist societies, such as this one, would use, or act as if they use, different* r*'s for different groups.*

1. *Consider the unanimity rule for the jury, "Convict only if all jurors return a guilty vote," i.e.* $V = G$ *if* $V_1 = V_2 = \cdots = V_M = G$, *and* $V = I$ *otherwise. What are*

   $P(V = G \ \&\ \omega = g)$, $P(V = I \ \&\ \omega = g)$, $P(V = G \ \&\ \omega = i)$, *and* $P(V = I \ \&\ \omega = i)$*?*

2. *Consider again the unanimity rule for the jury and treat $M \geq 1$ as a continuous variable. If juries are costless, set up and solve for the optimal M^*, being careful about the boundary and the second order conditions. [Reminder: for $x > 0$, $x^M = e^{M \ln x}$ so that $dx^M/dM = (\ln x) \cdot x^M$.]*
3. *Assuming that the optimal M^* in the previous problem is strictly greater than 1 (an interior solution), find whether or not $\partial M^*/\partial \rho$ is positive or negative and interpret.*

6.10. **Four Questions for the Class.** There is an urn in front of you that has 90 balls in it. 30 of them are yellow, the rest are either blue or green, and you cannot see into the urn to see how many blues and greens there are. You have to choose between three tickets, a yellow ticket, a blue ticket, and a green ticket. A ball will be drawn at random from the urn. If the color of the ball matches the color of your ticket, you will get \$100. Otherwise you will get nothing. Question #1: What are your preferences over the tickets?

There is a machine that will narrow down your time of death to within ± 3 months. Question #2: Are you willing to pay to use the machine? or are you willing to pay to avoid using the machine?

Question #3: In this class, are the answers to questions #1 and #2 stochastically independent?

Question #4 (the Allais paradox): Suppose that $C = \{2,500,000,\ 500,000,\ 0\}$, that $L_1 = (0, 1, 0)$, $L_1' = (10/11, 0, 1/11)$, that $L_2 = (0, 0.11, 0.89)$, and $L_2' = (0.10, 0, 0.90)$. How many people have the preference pattern $L_1 \succ L_1'$ and $L_2' \succ L_2$? Carefully draw the simplex here and look at the parallel lines joining the choices. The independence axiom fails here.

6.11. **Some Homeworks (from Previous Comprehensive Exams).** The following problems have appeared on previous comprehensive exams. They're appropriate here.

Homework 6.11.1. *Mary, through hard work and concentration on her education, had managed to become the largest sheep farmer in the county. But all was not well. Every month Mary faced a 50% chance that Peter, on the lam from the next county for illegal confinement of his ex-wife, would steal some of her sheep. Denote the value of her total flock as w, and the value of the potential loss as L, and assume that Mary is a expected utility maximizer with a Bernoulli utility function*

$$u(w) = \ln w.$$

1. *Assume that Mary can buy insurance against theft for a price of p per dollar of insurance. Find Mary's demand for insurance. At what price will Mary choose to fully insure?*
2. *Assume that the price of insurance is set by a profit maximizing monopolist who knows Mary's demand function and the true probability of loss. Assume also that the only cost to the monopolist is the insurance payout. Find the profit maximizing linear price (i.e. single price per unit) for insurance. Can the monopolist do better than charging a linear price?*

Homework 6.11.2. *Consider a consumer who must allocate her wealth between consumption in two periods, 1 and 2. Assume that the consumer has preferences on consumption* *streams (c_1, c_2) represented by the utility function*

$$U(c_1, c_2) = u(c_1) + u(c_2)$$

where

$$u(c_i) = \frac{c_i^{1-a}}{1-a},$$

$0 < a < 1$. *Suppose further that she has wealth W at the start of period 1, and receives no other income, so all of her period 2 consumption is supported by saving in period 1, and expects to pay a share t of her savings at the start of period 2 in taxes. Finally, suppose that the tax rate on savings is set by the government at the start of period 2 at the time it is levied, and is uncertain at the time of the saving decision in period 1.*

1. *Assume that tax rates are determined according to the density $f(t)$, and carefully write down the consumer's lifetime utility maximization problem.*
2. *Assume that t will take on a value of $1/2$ or 0 with equal probability. Find the optimal choice of consumption in period 1. Does this increase or decrease with an increase in the parameter a? Explain.*

Homework 6.11.3. *A strictly risk averse expected utility maximizer with von Neumann-Morgenstern utility function $u(\cdot)$ is contemplating the purchase of insurance from a risk neutral insurance company. At present, her random wealth is $80,000 - L$ where $0 \leq L \leq 80,000$ is a random variable with expectation μ_L and non-trivial cdf $F(\cdot)$. Contracts are of the form of a premium, $P \geq 0$, and a share α, $0 \leq \alpha \leq 1$, of the loss that will be made good by the insurance company. Thus, accepting a contract (P, α) gives the expected utility maximizer the random income $80,000 - P - (1-\alpha)L$.*

Note that the first two problems make no assumptions about $u(\cdot)$ or $F(\cdot)$ not mentioned above.

1. *Show that amongst the contracts $(\alpha\mu_L, \alpha)$, the potential insurance purchaser strictly prefers higher to lower α.*
2. *Characterize the set $\mathcal{E}$ of efficient contracts (P, α) acceptable to both the potential purchaser and the insurance company. Explain your work.*
3. *Suppose that L is equal to $50,000$ with probability 0.2, is equal to 0 with probability 0.8, and that $u(x) = \sqrt{x}$. Find the contract in $\mathcal{E}$ that the potential purchaser most prefers. Find the contract in $\mathcal{E}$ that the insurance company most prefers. Explain.*

Homework 6.11.4. *A strictly risk averse person with wealth W and a twice continuously differentiable von Neumann-Morgenstern utility function $u(\cdot)$ depending on income spends $a \geq 0$ on loss prevention. After spending a, their random income is $Y = X - a$ where Y is their net income, and X is their gross income. X is a random variable whose distribution, R_a, is a convex combination of two distributions μ and Q_a, $R_a = cQ_a + (1-c)\mu$, $c \in [0, 1]$. Here, μ is an arbitrary distribution on the non-negative reals that does not depend on a, and Q_a is a two-point distribution with probability $P(a)$ at $W - L$ and $1 - P(a)$ at W. We interpret $c \in [0, 1]$ as the consumer's level of control — if $c = 0$, then a makes no difference in the distribution, as c rises to 1, the control over the distribution increases.*

If $f(\cdot)$ is a function, then

$$E\, f(Y) = \int f(x-a)\, dR_a(x) = c \int f(x-a)\, dQ_a(x) + (1-c) \int f(x-a)\, d\mu(x),$$

and $\int f(x-a)\, dQ_a(x) = P(a) f(W-L-a) + (1-P(a)) f(W-a)$.

The function $P(\cdot)$ is twice continuously differentiable on $\mathbb{R}_{++}$, and satisfies

$$P(0) = p_0 \leq 1,\ \ P'(a) < 0,\ \ P''(a) > 0,\ \ \lim_{a \downarrow 0} P'(a) = -\infty.$$

1. *Write out the person's expected utility and their expected income as a function of a.*
2. *Give the FOC for expected utility maximization and verify that the SOC for maximization hold.*
3. *What can you say about the optimal choice of action as a function of their control, c?*

7. Game Theory

Dates: Nov. 21, 28, 30, Dec. 4, 6.

Game theory is a branch of decision theory in which we take a great deal of care to analyze how other people's actions affect decision makers's optimization problems. The optimization by the agents will always be expected utility maximization. The options available to the people being modeled will vary, as will the effects on themselves and on others. An equilibrium is a vector of strategies, one for each person involved, with the property that, given that the others are using their part of the vector, each player is doing the best they can for themselves. If this property were violated, then at least one person would have an incentive to change their actions. At an equilibrium, or, as economists say it, "in equilibrium", no-one has any incentive to change their plans.

It is important to note that the game theory does not assume that people do not care about what happens to others. However, economists have a tendency to put that lack of care in as an additional assumption. This is clearly not sensible in general, but it may be sensible for many kinds of market interactions — an insurance agent cares far more about the extent to which your strategic reactions cost the insurance company than the extent to which they affect your personal happiness.

A basic division is between static games, those in which actions are taken all at once or all in "one shot," and dynamic games, those in which actions are taken over time with varying degrees of information about what others have done.

7.1. **Homeworks.**

Due date: Monday Dec. 11, 2000.
From MWG: **Ch. 7**: 7.E.1. **Ch. 8**: 8.B.1, 8.B.3, 8.B.5(a), 8.B.6 – 8.C.1 – 8.D.1, 8.D.2, 8.D.4, 8.D.5 – 8.E.1, 8.E.3 – 8.F.2. **Ch. 9**: 9.B.1, 9.B.2, 9.B.3, 9.B.9, 9.B.11, either 9.C.4 or 9.C.7, and some problems in the notes below.

7.2. **Static Games.**

We begin with notation, then turn to a dynamic kind of motivation for the equilibria of static games, then to a variety of examples.

7.2.1. *Generalities and Notation.* A game is defined by $\Gamma = (S_i, u_i)_{i \in I}$ where I is a finite set (of people), S_i is a non-empty set (of strategies for agent i), $S = \times_{i \in I} S_i$ is the set of vectors of strategies, and $u_i : S \to \mathbb{R}$ is a bounded von Neumann-Morgenstern utility function.

The interpretation is that S_i represents what person $i \in I$ can do, S represents the set of all possible combinations of choices by all people involved, and $u_i(s)$ is i's utility if $s \in S$ is chosen. In principle, some distribution over S may arise if people pick at random, or in response to randomness not in the model. This is a possibility contained in the following definition.

Definition 7.2.1. *Let $\mathcal{G}$ be a set of games. A* **solution concept for** $\mathcal{G}$ *is a mapping $\Gamma \mapsto S(\Gamma) \subset \Delta(S)$.*

The interpretation is that $S(\Gamma)$ is what we believe will happen in the game Γ. The idea of a point-to-set mapping has a special name, a correspondence. We've seen correspondences before, think e.g. of the demand set.

Desired properties for the correspondence S would have to include

1. $S \neq \emptyset$, it's embarassing not to be able to say that anything is going to happen,
2. $\#S(\Gamma) = 1$, that is, when we can say that the following single thing will happen, and
3. $S(\Gamma)$ depends on the specification of Γ in a "sensible" fashion.

Be careful of the word "sensible," it covers a multitude of sins.

There are two crucial pieces of notation.

First, for

$$s = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_I) \in S$$

and $t_i \in S_i$,

$$s\backslash t_i = (s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_I) \in S.$$

In words, $s\backslash t_i$ replaces the i'th component of s with t_i.

The second piece of notation is for the best response correspondence. For $s \in S$,

$$Br_i(s) = \{t_i \in S_i : u_i(s\backslash t_i) \geq u_i(s\backslash S_i)\},$$

in words, $Br_i(s)$ is the set of i's best utility maximizing "responses" to s. Note that for all t_i, t_i', $Br_i(s\backslash t_i) = Br_i(s\backslash t_i')$, so that writing the best response to s carries some redundant information. That's fine, it's convenient redundancy.

7.2.2. *Cournot-Nash Competition and a Dynamic.* We're going to tell a story that goes back to the previous century, told first by Auguste Cournot. It leads to a dynamic justification for equilibria in static games.

There are two firms, $I = \{1, 2\}$, producing non-negative quantities of the same good, $S_i = [0, +\infty)$, at a cost $C_i(q_i) = c \cdot q_i$, and revenues are given by $R_i(q_i, q_j) = q_i(a - b(q_i + q_j))$, so that utility (profit) functions are

$$\pi_i(q_i, q_j) = (a - b(q_i + q_j))q_i - cq_i.$$

The best responses are

$$Br_i(q_i, q_j) = \{\max\{\frac{a-c}{2b} - \frac{1}{2}q_j, 0\}\}.$$

(Note the lack of dependence of $Br_i(\cdot)$ on q_i.)

Consider the dynamic

$$q_i^{t+1} = Br_i(q_j^t),\ q_j^{t+1} = q_j^t \ \text{ for } t \text{ even},$$

$$q_j^{t+1} = Br_i(q_i^t),\ q_i^{t+1} = q_i^t \ \text{ for } t \text{ odd}.$$

Here in odd periods, i looks at what j did previously and best responds to it, in even periods, this is reversed. You should check that from any starting point, the best response dynamic converges to the unique intersection of the two best response curves. Call intersection $q^* = (q_i^*, q_j^*)$. The stationary point of the best response dynamics satisfies

$$(\forall i \in I)[q_i^* \in Br_i(q^*)]\}.$$

This property means that if the two firms are playing their part of the vector q^*, then neither has any incentive to change their action. This is the property that we are going to extract from the example.

7.2.3. *Nash Equilibrium.* The basic definition is due to Nash: $s^* \in S = \times_{i \in I} S_i$ is a (Nash) equilibrium if

$$(\forall i \in I)(\forall t_i \in S_i)[u_i(s^*) \geq u_i(s^* \backslash t_i)], \text{ equivalently, } (\forall i \in I)[s_i^* \in Br_i(s^*)].$$

Back to generalized abstract nonsense, let $Eq(\Gamma)$ denote the set of Nash equilibria, if any, of the game Γ. We are suggesting that $Eq(\cdot)$ is a good solution concept.

A useful reformulation is that s^* is an equilibrium if and only if

$$(\forall i \in I)[s_i^* \text{ solves } \max_{t_i \in S_i} u_i(s^* \backslash t_i)].$$

In other words, an equilibrium is a vector of strategies with the property that, if every person i believes that person $j \neq i$ is playing s_j^*, then i can do no better than playing s_i^*. We have seen utility maximization many times before. What is crucially different here is that i's preferences depend on s_j, $j \neq i$, and that everybody is working on the same assumptions about s^*.

We now turn to some examples and look at what they teach us about Nash equilibrium.

7.3. **Some Examples.** The stuff above has implications for the stories we tell each other as economists. A good way to understand the implications is to look at their implications for some of the stories.

7.3.1. *Prisoners' Dilemma.* Here $I = \{1, 2\}$, $S_i = \{\text{Squeal}_i, \text{Silent}_i\}$, and the payoffs are given by

	Squeal_2	Silent_2
Squeal_1	$(-B+r, -B+r)$	$(-b+r, -B)$
Silent_1	$(-B, -b+r)$	$(-b, -b)$

where $B > b \geq r > 0$ and $-B + r < -b$. The convention has 1's options being the rows, 2's the columns, payoffs (x, y) mean "utility of x to 1, utility of y to 2." The claim is that

$$Eq(\Gamma) = \{\{\text{Squeal}_1, \text{Squeal}_2\},$$

which is sort of sad if $B \gg b, r$ because of the vector inequality

$$u(Eq(\Gamma)) = (-B + r, -B + r) \ll (-b, -b).$$

This game is **dominance solvable**.

Definition 7.3.1. *The strategy τ_i* **strictly dominates** *t_i for i if*

$$(\forall \sigma)[u_i(\sigma \backslash \tau_i) > u_i(\sigma \backslash t_i)].$$

The definition of a Nash equilibrium implies that if s^* is an equilibrium, then for all $i \in I$, s_i^* is not strictly dominated. For both players, Squeal strictly dominates Silent. Therefore the only equilibrium involves both Squealing.

The story (so far): Two criminals have been caught, but it is after they have destroyed the evidence of serious wrongdoing. Without further evidence, the prosecuting attorney can charge them both for an offense carrying a term of $b > 0$ years. However, if the prosecuting attorney gets either prisoner to give evidence on the other (Squeal), they will get a term of $B > b$ years. The prosecuting attorney makes a deal with the judge to reduce any term given to a prisoner who squeals by an amount r, $b \geq r > 0$, $B - b > r$ (equivalent to $-b > -B + r$). If $B = 20$, $b = r = 1$, then the game is

	Squeal	Silent
Squeal	$(-19, -19)$	$(0, -20)$
Silent	$(-20, 0)$	$(-1, -1)$

Note that $(-1, -1) \gg (-19, -19)$.

7.3.2. *Joint Optimality Goes out the Window.* The following is an important lesson:

THROW OUT ANY NOTION OF JOINT OPTIMALITY OF EQUILIBRIA IN GAME THEORY MODELS.

In the previous game, the unique equilibrium payoffs were $(-19, -19)$. Also available were the payoffs $(-1, -1)$. Both players are worse off in equilibrium. The following is meant to drive home the lack of relation between Nash equilibrium and joint optimality.

Lemma 7.3.1. *Let $u : X \times Y \to \mathbb{R}$ be a bounded and measurable function. Let $\Delta(X)$ denote the set of probability measures on X, $\Delta(Y)$ the probability measures on Y. Then for all $\nu \in \Delta(Y)$, and all $\mu, \mu' \in \Delta(X)$,*

$$\int u(x, y)\, d\mu(x)\, d\nu(y) \geq \int u(x, y)\, d\mu'(x)\, d\nu(y)$$

if and only if

$$\int [r \cdot u(x, y) + f(y)]\, d\mu(x)\, d\nu(y) \geq \int [r \cdot u(x, y) + f(y)]\, d\mu'(x)\, d\nu(y)$$

for all $r > 0$ and all ν-integrable functions f.

We will use the Lemma 7.3.1 with X being player i's choice, Y being the vector of choices of players $j \neq i$, and ν being the distribution over what $j \neq i$ is doing.

In words, for each i, no matter what the other people are doing, adding or substracting something that depends on their choices does not change i's preferences, therefore does not change i's best response set. Best response sets are what define Nash equilibria.

In the last game,

	Squeal	Silent
Squeal	$(-19, -19)$	$(0, -20)$
Silent	$(-20, 0)$	$(-1, -1)$

let us add 20 to i's payoff if j plays Squeal. This yields the game

	Squeal	Silent
Squeal	$(+1, +1)$	$(0, 0)$
Silent	$(0, 0)$	$(-1, -1)$

Note that the unique equilibrium must still be both Squealing, but now we think that it's a pretty good outcome.

7.3.3. *Rational Pigs.* This is a 'game' in which each of two pigs, one big and one little, has two actions. Little pig is player 1, Big pig player 2, the convention has 1's options being the rows, 2's the columns, payoffs (x, y) mean "x to 1, y to 2." The story is of two pigs in a long room, a lever at one end controls the output of food at the other end, the Big pig can push the Little pig out of the way and take all the food if they are both at the food output together, the two pigs are equally fast getting across the room, and during the time that the Big pig crosses the room, the Little pig can eat α of the food. The game is represented by

	Push	Wait
Push	$(-c, b-c)$	$(-c, b)$
Wait	$(\alpha b, (1-\alpha)b - c)$	$(0, 0)$

where $b, c > 0$, $0 < \alpha < 1$, $(1-\alpha)b - c > 0$. Think of b as the benefit of eating, c as the cost of pushing the lever and crossing the room.

For the Little pig, Waiting strictly dominates Pushing. Reduce the game by eliminating this strictly dominated strategy. This gives

	Push	Wait
Wait	$(\alpha b, (1-\alpha)b - c)$	$(0, 0)$

In this reduced game, Pushing strictly dominates Waiting for the Big pig. Further reduce the game by eliminating this strictly dominated strategy. This gives

	Push
Wait	$(\alpha b, (1-\alpha)b - c)$

We have just found the only Nash equilibrium of this game by **iteratively** deleting strictly dominated strategies.

Definition 7.3.2. *If there is a unique strategy $s \in S$ that survives iterative deletion of strictly dominated strategies, then the game Γ is* **dominance solvable***.*

Lemma 7.3.2. *If Γ is dominance solvable, then $Eq(\Gamma) = \{s\}$ where s is the unique strategy that survives iterative deletion of strictly dominated strategies.*

For dominance solvable games, Nash equilibrium seems like a pretty good bet — even pigs get it right.

7.3.4. *A Coordination Game.* The first coordination game is called Battle of the Partners. The story is of two partners who are either going to the (loud) Dance club or to a (quiet) romantic evening Picnic on Friday after work. Unfortunately, they work at different ends of town and their cell phones have broken down so they cannot talk about which they are going to do. Each faces the decision of whether to drive to the Dance club or to the Picnic spot not knowing what the other is going to do. The Dance club and the Picnic spot are so far apart from each other that they will not have a second chance to coordinate on this Friday night. The payoff matrix is given by

	Dance	Picnic
Dance	$(F + B, B)$	(F, F)
Picnic	$(0, 0)$	$(B, F + B)$

where $B > F > 0$. The idea is that the two derive utility B from Being together, and utility F from their favorite activity, and that utilities are additive.[22]

There are two perfectly good Nash equilibria here, and an equilibrium that neither like where the two pick at random. Find them.

7.3.5. *Another Coordination Game.* The second coordination game is called Chicken. It is played by adolescent males in cultures that value macho. One variant involves two young men running at each other along a slippery, wet pier holding boogie boards.[23] At a pre-determined spots they jump onto their boards and steer towards each other. The pier is narrow, there is only space for one to get through, either they both duck, getting cold and wet and laughed at by their putative friends, they both try to go through, resulting in concussions and a week's stay in the hospital, or one ducks and the other goes through, proving something.

One assignment of payoffs to this story is

	Duck	Thru
Duck	$(0, 0)$	$(-5, 10)$
Thru	$(10, -5)$	$(-6, -6)$

Find the three equilibria.

7.3.6. *All I Remember from Graduate Macro.* An extended version of coordination reasoning is in Keynes' version of Bernard Mandeville's Fable of the Bees. Here are two versions of it. Both of them will involve games with an infinite number of players and neither is dependent on this particular technical device.

Fable of the Bees #1: Each person ω in the set of people Ω chooses an action $a_\omega \in \mathbb{R}_+$ to solve

$$\max_{a_\omega \in \mathbb{R}_+} u_\omega(a_\omega, \overline{a}) - c_\omega a_\omega$$

where $c_\omega > 0$, u_ω is monotonic in both arguments, $\overline{a} = \int_\Omega a_\omega \, d\mu(\omega)$ for some (non-atomic) probability μ on Ω. We assume that there is a unique solution $a^*_\omega(\overline{a})$ exists and increases with $\overline{a}$, as it would if, for example,

$$\frac{\partial^2 u_\omega(\cdot, \cdot)}{\partial a_\omega \partial \overline{a}} > 0.$$

We also assume that the mapping $\omega \mapsto a^*_\omega$ is measurable (as it would be if the mapping $\omega \mapsto u_\omega$ is measurable and μ is a complete probability measure). Define

$$\alpha(\overline{a}) = \int_\Omega a^*_\omega(\overline{a}) \, d\mu(\omega).$$

Any $\overline{a}$ such that $\alpha(\overline{a}) = \overline{a}$ is an equilibrium aggregate level of activity. Note that $\alpha(\cdot)$ is increasing, in the differentiable case,

$$\frac{d\alpha(a)}{d\overline{a}} = \int_\Omega \frac{\partial a^*_\omega}{\partial \overline{a}} \, d\mu(\omega) > 0.$$

[22]We can tell the story with $F \geq B > 0$, but it's not as romantic.
[23]So who's a surfer boy?

This suggests that it is possible, and it is, to arrange matters so that there are many different equilibria.[24] At any equilibrium, each person is choosing their own (unique) strict best response to the actions of others. Further, the equilibria with higher $\overline{a}$'s are strictly prefered by all to equilibria with lower $\overline{a}$'s.

Fable of the Bees #2: With the same set of people as above, each person $\omega \in \Omega$ picks present demand, d_ω, and savings for future demand to maximize

$$u_\omega(d_\omega, (1+r)s_\omega) \text{ subject to } d_\omega + s_\omega = m_\omega,\ d_\omega, s_\omega \geq 0,$$

where

$$r = r(\overline{d}),\ r'(d) > 0,\ \overline{d} = \int_\Omega d_\omega \, d\mu(\omega),$$

and the mapping $\omega \mapsto u_\omega(\cdot, \cdot)$ is measurable. In other words, the more people spend now, the higher the level of economic activity, $\overline{d}$, which leads to a higher return on capital, $r(\overline{d})$, which means more to spend next period for each unit saved. For any given $\overline{d}$, denote by $(d^*_\omega, s^*\omega)$ the solution to the problem

$$\max u_\omega(d_\omega, (1 + r(\overline{d}))s_\omega) \text{ subject to } d_\omega + s_\omega = m_\omega,\ d_\omega, s_\omega \geq 0.$$

For many reasonable specifications of $u_\omega(\cdot, \cdot)$, $d^*\omega(\cdot)$ is increasing in $\overline{d}$, for even more specifications,

$$\delta(\overline{d}) := \int_\Omega d^*_\omega \, d\mu(\omega)$$

is increasing in $\overline{d}$. Any $\overline{d}$ such that $\delta(\overline{d}) = \overline{d}$ is an equilibrium aggregate level of demand activity, and it can be arranged that there are many equilibria. Equilibria with higher $\overline{d}$'s are strictly prefered to equilibria with lower $\overline{d}$'s. An alternate version of this has r fixed but has each m_ω being an increasing function of $\overline{d}$, and, presuming the two consumption goods are normal, the same basic story goes through.

These stories can be re-told as stories about people not internalizing the external effects that their own actions take.

7.3.7. *Matching Pennies.* This is the game with the matrix representation

	H	T
H	$(+1, -1)$	$(-1, +1)$
T	$(-1, +1)$	$(+1, -1)$

Try to draw out best response dynamics here, you get rather dizzy. Since we're after an equilibrium, we're after mutual best responses. Think about playing against Sherlock Holmes as a way to understand finding the mixed strategy equilibrium — at an equilibrium, your opponent in this game will be best responding to whatever you do, Sherlock can figure out what you're going to do by subtle clues involving minute particles of thread and barking dogs that are silent in the night. If you're playing Sherlock in this game, it looks like you've lost your money. Except that you could flip the coin somewhere that Sherlock cannot see,

[24]Perhaps the easiest example is to take $u_\omega(a_\omega, \overline{a}) - c_\omega a_\omega = 2\overline{a}a_\omega - ca_\omega$ for all ω, that is, a population of identical individuals. In this case there are two equilibria, $\overline{a} = 0$ and $\overline{a} = c^2$.

and then not look at the result so that Sherlock has no clue as to whether it's Heads or Tails.

More formally, let α be the probability that 1 plays H, β the probability that 2 plays H, find and graph the best response correspondences. The unique Nash equilibrium is where these two correspondences meet. It requires that both players play H with probability $\frac{1}{2}$. This is called an equilibrium in **mixed** or **randomized strategies**. It is the only equilibrium for this game. Therefore, if we are to have equilibrium existence in all finite games, we must allow for mixed strategy equilibria.

7.4. 0-**Sum Games.** Matching Pennies is a 2×2 example of a general class of games, the 0-sum games. These are games with $I = 2$ and

$$u_1(s) + u_2(s) \equiv 0.$$

This means that the only way for one person to win is for the other one to lose. One person maximizing their own utility is equivalent to them minimizing the other person's utility. Mutual gains, from trade or from coordination or from any other source(s), can not appear in this class of games. This class of games provides another in which Nash equilibria are easy to believe in, and this is the point of Homework 7.4.2. As background, some generally useful concepts are the **maximin** and the **minimax** payoff levels, $\underline{w}$ and $\underline{v}$ respectively. These are defined for any game $(S_i, u_i)_{i \in I}$, whether or not it is 0-sum, by

$$\varphi(\sigma_i) := \min_{\sigma_{-i} \in \Delta_{-i}} u_i(\sigma_i, \sigma_{-i}), \ \underline{w}_i := \max_{\sigma_i \in \Delta_i} \varphi(\sigma_i),$$

and

$$\psi(\sigma_{-i}) := \max_{\sigma_i \in \Delta_i} u_i(\sigma_i, \sigma_{-i}), \ \underline{v}_i := \min_{\sigma_{-i} \in \Delta_{-i}} \psi(\sigma_{-i}).$$

To understand these, suppose

1. everybody else, i.e. $-i$, know that you are playing σ_i, and decide to make you so unhappy as is possible, you will receive $\varphi(\sigma_i)$. If you know that they will know your choice of mixed strategy and then try to make you unhappy, then the best that you can do is to solve the problem $\max_{\sigma_i \in \Delta_i} \varphi(\sigma_i)$, giving you your **maximin** payoffs, $\underline{w}_i$.
2. you know what everyone else is going to do, i.e. you know σ_{-i}, and try to maximize your own utility, giving you $\psi(\sigma_{-i})$. Knowing this, other people try to make you as unhappy as possible by solving the problem $\min_{\sigma_{-i} \in \Delta_{-i}} \psi(\sigma_{-i})$, giving you your **minimax** payoffs, $\underline{v}_i$.

It seems clear (perhaps) that you're in a better situation when you know what people are doing when they are only interested in hurting you. It can (easily if you've had the background) be shown that $\varphi(\cdot)$ and $\psi(\cdot)$ are continuous functions. Given that they are continuous functions defined on their compact domains, they have optima. This is useful in the first part of the next problem, where you should start with $(\sigma_i^*, \sigma_{-i}^*)$ solving the optimization maxmin problem so that $u_i(\sigma_i^*, \sigma_{-i}^*) = \underline{w}_i$.

Homework 7.4.1. *Prove that* $\underline{v}_i \geq \underline{w}_i$ *for any game, then calculate* $\underline{v}_i$ *and* $\underline{w}_i$ *for the following games.*

	L	R
U	$(+1,-1)$	$(-1,+1)$
D	$(-1,+1)$	$(+1,-1)$

	L	R
U	$(-19,-19)$	$(-20,0)$
D	$(0,-20)$	$(-1,-1)$

	L	R
U	$(5,3)$	$(0,0)$
D	$(0,0)$	$(3,5)$

Homework 7.4.2 (The Minimax Theorem)**.** *Suppose that* $\Gamma = (S_i, u_i)_{i=1,2}$ *is a* 0 *sum game and that* (σ_1^*, σ_2^*) *is a Nash equilibrium for* Γ*.*

1. *Using the Nash equilibrium and the* 0*-sum conditions, show that* $\underline{w}_i \geq u_i(\sigma_1^*, \sigma_2^*) \geq \underline{v}_i$ *so that (by the previous homework problem)* $\underline{w}_i = u_i(\sigma_1^*, \sigma_2^*) = \underline{v}_i$*.*
2. *Suppose that* (σ_1', σ_2') *is also a Nash equilibrium for* Γ*. Show that* (σ_1', σ_2^*) *and* (σ_1^*, σ_2') *are also Nash equilibria.*

The next result is Nash's existence theorem. It says that for any finite game there is an equilibrium. Therefore, the previous homework problem shows that for any 0-sum game, $\underline{v}_i = \underline{w}_i$, which is the famous Minimax Theorem.

7.5. **Equilibrium Existence for Finite Games.** We just saw that we need mixed strategies if we are to have any hope of $Eq(\Gamma) \neq \emptyset$. So let's be formal about what we've already done informally above.

7.5.1. *Mixed Strategies.* We say that Γ is finite if each S_i is finite. For finite S_i, let $\Delta_i = \Delta(S_i)$ be the set of distributions over S_i (the simplex again). For finite games, can extend u_i to $\Delta := \times_{i \in I} \Delta_i$, where Δ_i is the set of distributions over S_i. We do it by identifying every vector $\sigma \in \Delta$ with the product probability distribution over S with marginals σ_i.
Example: $I = \{1, 2\}$, $S_i = \{L_i, R_i\}$, $\sigma_1 = (1/3, 2/3)$, $\sigma_2 = (3/4, 1/4)$, then we identify $\sigma = (\sigma_1, \sigma_2)$ with the following probability distribution over S:

	L_2	R_2
L_1	$3/12$	$1/12$
R_2	$6/12$	$2/12$

Not all distributions on S are the products of independent probabilities, consider for example

	L_2	R_2
L_1	$9/12$	$1/12$
R_2	$0/12$	$2/12$

For another example,

	L_2	R_2
L_1	$1/2$	0
R_2	0	$1/2$

These last two distributions are not the product of their marginals. We extend u_i to points $\sigma \in \Delta$ with the definition

$$u_i(\sigma) = \sum_{s=(s_1,\dots,s_I)\in S} u_i(s) \Pi_{i \in I} \sigma_i(s_i).$$

Again, this is just

$$\int_S u_i(s)\, d\sigma(s)$$

where σ is the product probability having marginals σ_i.

We will not distinguish in our notation between a person using a mixed strategy that plays s_i with probability 1 and the person playing s_i. However, a mixed strategy that puts probability 1 on some one element of S_i has a special name, it is called a **pure strategy**.

The best response notation above extends immediately to mixed strategies,

$$Br_i(\sigma) = \{\tau_i \in \Delta_i : u_i(\sigma\backslash\tau_i) \geq u_i(\sigma\backslash\Delta_i)\}.$$

Lemma 7.5.1. *For all σ, the mapping $\tau_i \mapsto u_i(\sigma\backslash\tau_i)$ from Δ_i to $\mathbb{R}$ is linear.*

Homework 7.5.1. *Let $Br_i^P(\sigma)$ denote the set of pure strategy best response to σ. Show that for finite Γ, $Br_i(\sigma) = \Delta(Br_i^P)$.*

7.5.2. *Nash's Existence Theorem.* Being repititious here, the basic definition is due to Nash: $\sigma^* \in \Delta$ is a (Nash) equilibrium if

$$(\forall i \in I)(\forall \sigma_i \in S_i)[u_i(\sigma^*) \geq u_i(s^*\backslash\sigma_i)].$$

Equivalently, $\sigma^* \in \Delta$ is a (Nash) equilibrium if

$$(\forall i \in I)[\sigma_i^* \in Br_i(\sigma^*)].$$

Theorem 7.5.1 (Nash)**.** *Every finite game has an equilibrium.*

To prove this we need Brouwer's fixed point theorem, which we will <u>**not**</u> prove.

Theorem 7.5.2 (Brouwer)**.** *If K is a non-empty, compact convex subset of $\mathbb{R}^n$ and $f : K \to K$ is continuous, then there exists $x^* \in K$ such that $f(x^*) = x^*$.*

The x^* in the previous theorem is a **fixed point of** f.

Homework 7.5.2. *Brouwer's fixed point theorem requires non-emptiness to avoid triviality, and three substantive conditions: compactness, convexity and continuity, in order to guarantee a fixed point. Find three non-trivial examples without fixed points, each example violating exactly one of the conditions.*

Nash's Second Existence Proof: Let $\sigma = (\sigma_i)_{i\in I} \in \Delta$. Following Nash, we are going to define a continuous mapping, τ, from Δ to Δ with the property that its fixed points are the equilibria of Γ.

For any $\sigma = (\sigma_i)_{i\in I}$ and $s_i \in S_i$, define

$$r_{s_i}(\sigma) = \sigma_i(s_i) + \max\{u_i(\sigma\backslash s_i) - u_i(\sigma), 0\}.$$

Note that $r_{s_i}(\cdot)$ is continuous.

Now define $\tau_i(\sigma)$ by

$$\tau_i(\sigma)(s_i) = \frac{r_{s_i}(\sigma)}{\sum_{t_i\in S_i} r_{t_i}(\sigma)}.$$

Note that $\tau_i(\sigma) \in \Delta_i$ and that the mappings $\tau_i(\cdot)$ is continuous.

Now define $\tau(\sigma)$ by

$$\tau(\sigma) = (\tau_1(\sigma), \dots, \tau_I(\sigma)).$$

The mapping $\tau : \Delta \to \Delta$ is continuous. By Brouwer's fixed point theorem, the mapping τ has a fixed point, σ^*, such that $\tau(\sigma^*) = \sigma^*$.

The last step in the proof is to show that any fixed point of τ is an equilibrium of Γ. Indeed, the result is a bit stronger,

Lemma σ^* is an equilibrium if and only if σ^* is a fixed point of τ.

Proof of Lemma. Suppose that $\sigma^* \in Eq(\Gamma)$. We need to show that $\tau(\sigma^*) = \sigma^*$. Because σ^* is an equilibrium,

$$(\forall i \in I)(\forall s_i \in S_i)[r_{s_i}(\sigma^*) = \sigma_i^*(s_i) + \max\{u_i(\sigma^* \backslash s_i) - u_i(\sigma^*), 0\},$$

and the maximum term is equal to 0 because $u_i(\sigma^* \backslash s_i) \leq u_i(\sigma^*)$. This means that

$$(\forall i \in I)(\forall s_i \in S_i)[\tau_i(\sigma^*)(s_i) = \sigma_i^*(s_i)],$$

that is, $\tau(\sigma^*) = \sigma^*$.

Now suppose that $\tau(\sigma^*) = \sigma^*$. We need to show that $\sigma^* \in Eq(\Gamma)$.

Step 1: Because σ^* is a fixed point, there is no $i \in I$ and $t_i \in S_i$ with the properties that $\sigma_i^*(t_i) = 0$ and $u_i(\sigma^* \backslash t_i) > u_i(\sigma^*)$ (look at the definition of $r_{t_i}(\cdot)$). Thus, for any fixed point, σ^*, of $\tau(\cdot)$, if $\sigma_i^*(s_i) > 0$, we must have $u_i(\sigma^* \backslash s_i) \leq u_i(\sigma^*)$.

Step 2: For each $i \in I$, and let s_i' be a solution to the problem

$$\min_{s_i \in S_i} \{u_i(\sigma^* \backslash s_i) : \text{ subject to } \sigma_i^*(s_i) > 0\}.$$

Because S_i is finite, this problem has at least one solution. We are going to show that any solution s_i' satisfies

$$u_i(\sigma^* \backslash s_i') = u_i(\sigma^*).$$

Since this means that each σ_i^* puts positive mass only on best responses, i.e. that σ^* is an equilibrium.

By Step 1, we know that

$$u_i(\sigma^* \backslash s_i') \leq u_i(\sigma^*).$$

The proof will be complete once we show that $u_i(\sigma^* \backslash s_i') < u_i(\sigma^*)$ implies that σ^* is not a fixed point. Suppose then, that $u_i(\sigma^* \backslash s_i') < u_i(\sigma^*)$. Then $r_{s_i'}(\sigma^*) = \sigma_i^*(s_i')$ and for some t_i with $\sigma_i^*(t_i) > 0$, $r_{t_i}(\sigma^*) > \sigma_i^*(t_i)$. This implies that $\tau_i(\sigma^*)(s_i') < \sigma_i^*(s_i')$, which contradicts σ^* being a fixed point.

This completes the proof of the Lemma, hence of Nash's existence theorem. (Yeah!)

Corollary: *For all finite games Γ, $Eq(\Gamma)$ is a closed set.*

Proof #1: $Eq(\Gamma)$ is the set of zeroes of the continuous function $f(\sigma) = \tau(\sigma) - \sigma$ in the closed set Δ.

Proof #2: $\sigma^* \in Eq(\Gamma)$ if and only if σ^* satisfies the following inequalities for each $i \in I$ and each $s_i \in S_i$:

$$u_i(\sigma^*) \geq u_i(\sigma^* \backslash s_i).$$

This is a finite collection of polynomial inequalities, so we not only conclude that $Eq(\Gamma)$ is closed, we conclude that it is semi-algebraic.

7.5.3. *All We Will Say About Upper Hemicontinuity.* One of the basic properties of the set of Nash equilibria is called upper hemicontinuity (with a name like that, it had better be a nice property). Models are at best approximations to reality. We would be upset if a tiny change in the model gave rise to a huge change in the set of predictions. However, let $r^n \downarrow 0$ and consider the game with payoffs r^n times the payoffs of Matching Pennies. For every r^n, there is exactly one equilibrium, the $(\frac{1}{2}, \frac{1}{2})$ for each player. In the limit, when $r = 0$, any play at all is an equilibrium. So there is no hope to have the equilibrium set move continuously as a function of parameters of the game. However, observe that $(\frac{1}{2}, \frac{1}{2})$ is still an equilibrium of the limit game. Roughly speaking, the limit of equilibria is again an equilibrium. This is the **upper hemicontinuity theorem**. One version of it is that the graph from parameters to equilibria is closed.

Define $E \subset \mathbb{R}^{S \cdot I} \times \Delta$ by

$$E = \{(u, \sigma) : \sigma \in Eq(\Gamma(u))\}.$$

The point-to-set mapping $u \mapsto Eq(\Gamma(u))$ is called the **equilibrium correspondence**. Its graph is the set E just given. Note that $(u, \sigma) \in E$ if and only if for each $i \in I$ and each $s_i \in S_i$ we have

$$u_i(\sigma) \geq u_i(\sigma \backslash s_i).$$

This is a finite collection of polynomial inequalities in u and σ, so we not only conclude that E is closed, we conclude that it is semi-algebraic.

7.6. **Extensive and Normal Form Representations of Games.** In the games above, we talked as if the players were simultaneously choosing s_i's in the S_i's. We are now going to talk about games in which the obvious stories involve players moving in some order, and having different amounts of information. However, we are going to fit these more complicated games into the simultaneous choice framework. What is a bit surprising is how good the fit is. We will first work through the following description of a game as an extensive form game, that is, in a form that highlights the dynamic nature of the choices and the informational difference between the two players (though the informational differences will be larger and more important in games we will look at later). We will then turn around and look at the game as a normal form game, that is, in a form that highlights the simultaneous choice aspect of the game.

Before we start, we need the definition of a weakly dominated strategy: $\sigma_i' \in \Delta(S_i)$ weakly dominates t_i if

$$(\forall \sigma)[u_i(\sigma \backslash \sigma_i') \geq u_i(\sigma \backslash t_i)], \text{ and}$$

$$(\exists \sigma)[u_i(\sigma \backslash \sigma_i') > u_i(\sigma \backslash t_i)].$$

We (economists) generally don't believe that weakly dominated strategies are chosen, at least not when we're telling stories using game theory. We sometimes simply delete the weakly dominated strategies from the game then look at the new, reduced game. In the new, reduced game, there may be strategies that are weakly dominated even though they were not weakly dominated before we reduced the game. We sometimes then do another round of deletion. In the new, reduced[2] game, ... You see where this is going, this is called the **iterated deletion of weakly dominated strategies**.

Example 7.6.1. *There are two players, $I = \{1, 2\}$. Player 1 moves first, choosing one of the actions in the set $\{L, M, R\}$. If player 1 picks L, the game is over with payoffs $\binom{x}{20}$. If player 1 does not pick L, the players play a "Battle of the Partners" game. Specifically, if player 1 picks M or R, then it is player 2's move, choosing one of the actions in the set $\{m, r\}$. Player 2 makes this pick without knowing whether 1 picked M or R. If 1 picks M and 2 picks m, the payoffs are $\binom{3}{5}$, if 1 picks R and 2 picks r, the payoffs are $\binom{5}{3}$, otherwise the payoffs are $\binom{0}{0}$.*

Things to do:

1. *Give an extensive form representation of this game.*
2. *Give a strategic (or normal) form representation of this game.*
3. *As a function of x, give the Nash equilibria of this game. It is very important to note that in thinking about equilibria, we must think about 2's choice even if 1 is playing L, otherwise we could not judge whether or not 1's choice is sensible.*
4. *As a function of x, give the Nash equilibria of this game that survive iterated deletion of weakly dominated strategies.*

The formal description of a game is on p. 227 of MWG. Take a careful look at it. Here is my version of the pieces, it is very very close to MWG's. As an example, let us refer to Figure 1,

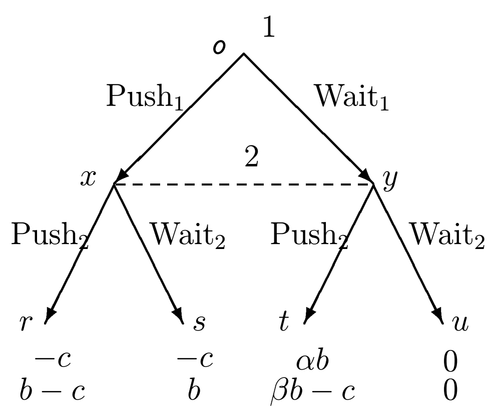


Figure 1

1. Nodes: The set of nodes for this game is $\mathfrak{X} = \{o, x, y, r, s, t, u\}$.
2. Arrows: The possibility of moving from one node to another node is represented by arrows in the diagram. There are at least two ways to represent these arrows. First, as a set of ordered pairs with the first one being the head of the arrow and the second being the tail, that is,

$$K = \{(o, x), (o, y), (x, r), (x, s), (y, t), (y, u)\}$$

where $K \subset \mathfrak{X} \times \mathfrak{X}$ is a partial ordering. (In this geometric a context, K is called a tree with a root, i.e. you cannot go in cycles, no node is its own predecessor, and only one node has no predecessor.)

3. Immediate predecessors: The arrows can also be represented by a function $p : \mathfrak{X} \to \{\mathfrak{X}, \emptyset\}$ that maps each node in $\mathfrak{X}$ to its **immediate predecessor**,
$$p(o) = \emptyset,\ p(x) = p(y) = o,\ p(r) = p(s) = x,\ p(t) = p(u) = y.$$
4. The origin: By assumption, there is only one node with no predecessor, it is called the root or the origin, here denoted by o.
5. Immediate successors: The **immediate successors of a node** x is $s(x) = p^{-1}(x)$.
6. Predecessors: Iteratively applying $p(\cdot)$ gives the set of all predecessors, e.g. $p(r) = x$, $p(p(x)) = p(o) = \emptyset$, so the set of all predecessors of the node r is $\{x, o\}$.
7. Successors: Iteratively applying $s(\cdot)$ gives the set of all successors.
8. Terminal nodes and decision nodes: The **terminal nodes** are $T = \{x \in \mathfrak{X} : s(x) = \emptyset\}$, in this game, $T = \{r, s, t, u\}$. All other nodes, $\mathfrak{X} \backslash T$, are called **decision nodes**.
9. Who plays where: The player partition P is a partition of $\mathfrak{X} \backslash T$ into $I + 1$ sets, $P_0, P_1, \dots, P_I$. At $x \in P_i$, player i is the one to choose an action. Player 0 is Chance or Nature and does not appear in this game, though she will appear often below. In this game $P_1 = \{o\}$ and $P_2 = \{x, y\}$.
10. Information partition: It matters very much what one knows when one makes a choice, and this is the function of the information partition. To each P_i, there is a partition U_i of P_i, the elements of U_i are called the **information sets** of player i. In this game $U_1 = \{P_1\}$ and $U_2 = \{P_2\}$, in later games it will not be so trivial. The idea is that player i cannot distinguish between points in an information set in U_i.

    In this game, player 1's information set is very simple, they know o and choose either Push_1 or Wait_1, while player 2's information set shows that they cannot distinguish between x and y. In the picture, this is denoted by the dashed lines connecting nodes x and y.

    The partitions U_i must satisfy two conditions,
    (a) For every terminal node z, the set of predecessors of z intersect any information set at most once. [This condition is repeated as the first condition in the Perfect Recall assumption below.]
    (b) All nodes in an information set must have the same number of successors.
11. Actions: The last two conditions are related to the choice of actions available to the players. In the game above, player 1 has two actions at the information set $H = \{o\} \in U_1$, the set of available actions, $A(H)$, is $\{\text{Push}_1, \text{Wait}_1\}$. At the information set $H = \{x, y\} \in U_2$, $A(H)$ is the set $\{\text{Push}_2, \text{Wait}_2\}$. The implications of 2 choosing a particular action depend on where in $\{x, y\}$ the player actually is — if at node x Push_2 and Wait_2 lead to terminal nodes r or s, if at y, they lead to t or u.

    For any decision node x, let $H(x)$ denote the information set containing x and let $A(x)$ be a set of actions available at x. At decision nodes x, we assume that if $H(x) = H(x')$, then $A(x) = A(x')$ so that $A(H)$ is well-defined for any information set H. Further, we assume that at every decision node x, there is a one-to-one correspondence between elements of $A(x)$ and $s(x)$, the immediate successors of x. The interpretation is that at $H(x) \in U_i$, player i chooses some $a \in A(H(x))$, and this leads to the corresponding node in $s(x)$.

Now we can explain the conditions (a) and (b) on the information partitions.

(a) If this condition is violated, then not only does player 1 not remember what he's chosen in the past, s/he may not remember having chosen.

(b) If two nodes, $x \neq x'$, in an information set $H \in U_i$ had different numbers of successors, then i's decision problem at the two nodes differ. Since we want to assume that when people choose, they choose from a known set of options, they would need to know whether are at x or at x'. But $x, x' \in H \in U_i$ represents i not being able to distinguish the two.

12. Perfect recall: We assume that the players never forget something they once knew — if they observe an action by someone else or take an action themselves, they will never arrive at a later point in the game at which they do not know all the consequences of the observed or taken action.

Formally, two conditions must hold, the first was given above:

(a) If x and x' are in the same information set for a player i, $H(x) = H(x')$, then x is neither a predecessor nor a succesor of x'.

(b) If $x \neq x'$ are in the same information set for a player i, $H(x) = H(x')$, x'' is a predecessor of x belong to one of i's information sets, and a'' is the action at $H(x'')$ that leads (eventually) to x, then there must be a predecessor, y, to x' that belongs to $H(x'')$ such that a'' is the action taken at y on the way to x'. (If i cannot distinguish between $x \neq x'$, then it had better not be the case that i had information distinguishing x and x' at any predecessor of x at which i chose an action.) The left hand game in Figure 2 violates this condition if we take $x = x_1$, $x' = x_2$, the right hand game violates this condition if we take $x \in \{y_1, y_2\}$ and $x' \in \{y_3, y_4\}$.

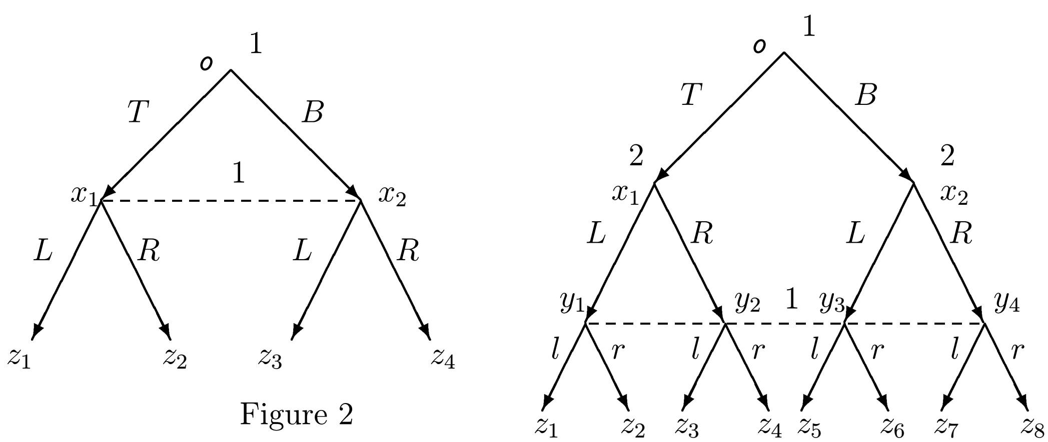


Figure 2

13. Strategies: A **pure strategy** for i is a mapping from the $H \in U_i$ to the associated $A(H)$. The H's in U_i are the possible contingencies that may arise, the places at which i may need to choose an action. A strategy for i is a complete contingent plan, a list of what will be done at each and every contingency, that is, at each possible H in U_i. A **mixed strategy** is a probability distribution over pure strategies. A **behavioral strategy** is a special case of a mixed strategy, it is a mapping from the $H \in U_i$ to

$\Delta(A(H))$ with the interpretation that the random moves at each H are independent. Kuhn's theorem says that we can perform any and all analysis of equilibria using behavioral strategies.

14. Outcomes: Given a vector $s = (s_i)_{i\in I} \in S = \times_{i\in I} S_i$ of pure strategies, there is a unique terminal node, $\mathbb{O}(s)$ that will arise, that is, $\mathbb{O} : S \to T$. The outcome associated with a mixed strategy σ is denoted $\mathbb{O}(\sigma)$ and is the image of σ under the mapping $\mathbb{O}$.
15. Utilities: Associated with every terminal node z is a utility $u(z)$. These are given as vectors at the terminal nodes in the picture.
16. Equilibrium: A vector σ^* is an equilibrium if

$$(\forall i \in I)(\forall s_i \in S_i)[u_i(\mathbb{O}(\sigma^*)) \geq u_i(\mathbb{O}(\sigma^* \backslash s_i))].$$

7.7. **Conditional Beliefs and Choice Under Uncertainty.** Sometimes, you only have partial information when you make a choice. From a decision theory point of view, making your choice after you get your partial information is equivalent to making up your mind ahead of time what you will do after each and every possible piece of partial information you may receive. The set up is as follows:

1. there is a utility function $u(s, \theta)$ depending on $s \in S$ and $\theta \in \Theta$, S and Θ finite sets (to avoid any complications in the math),
2. a partition $\mathcal{H}$ of Θ,
3. a story – $\theta \in \Theta$ is drawn according to a probability P, the person is told $h(\theta)$, the element of the partition $\mathcal{H}$ that contains the true θ, and knowing $h(\theta)$, picks $s \in S$.

There are two ways to maximize expected utility here, a complete contingent plan saying what will be pick after each h, and the "I'll figure it out when I get there" approach. The first one solves the problem

$$\max_{\sigma \in S^{\mathcal{H}}} \int_{\Theta} u(\sigma(\theta), \theta)\, dP(\theta).$$

Let $\sigma^*(h)$ be the solution to this problem.

The second approach emphasizes the beliefs you have when you are given partial information. We will use $\beta = \beta(\cdot|h)$ for the beliefs, and use Bayes' Law to find them. For $A \subset \Theta$ and $h \in \mathcal{H}$ such that $P(h) > 0$,

$$\beta(A|h) := \frac{P(A \cap h)}{P(h)}.$$

If $P(h) = 0$, define $\beta(\cdot|h)$ to be any distribution at all over $h \subset \Theta$. The second approach solves the collection of problems,

$$\text{for } h \in \mathcal{H},\ \max_{s \in S} \int_{\Theta} u(s, \theta)\, d\beta(\theta|h).$$

Let $s^*(h)$ denote the solution at h. The lemma says that the two approaches are the same on a set having probability 1.

Lemma 7.7.1 (Bridge crossing). *A complete contingent plan σ^* solves*

$$\max_{\sigma \in S^{\mathcal{H}}} \int_{\Theta} u(\sigma(\theta), \theta)\, dP(\theta)$$

if and only if for all h such that $P(h) > 0$, $\sigma^*(h) = s^*(h)$.

The proof is really really easy. One way to see what is going on is to notice that

$$\int_\Theta u(\sigma(\theta), \theta)\, dP(\theta) = \sum_{\theta \in \Theta} P(\theta) \cdot u(\sigma(\theta), \theta) = \sum_{h \in \mathcal{H}} P(h) \cdot \sum_{\theta \in h} u(\sigma(\theta), \theta) \beta(\theta|h)$$

for the simple reason that $P(\theta) = \beta(\theta|h) \cdot P(h)$.

It is important to notice what this lemma does not say. When $P(h) = 0$, $s^*(h)$ just needs to respond to some belief $\beta \in \Delta(h)$. When $P(h) = 0$, $\sigma^*(h)$ has no constraints whatsoever. So, on the set of h's having $P(h) = 0$, the set of possible values for $\sigma^*(h)$ can be larger than the set of possible values for $s^*(h)$. In game theory, this is a very important difference. The $s^*(h)$ are parts of **perfect Bayesian equilibria (PBE)**.

Definition: If an equilibrium (in behavioral strategies) σ^* satisfies $\sigma_i^*(h) = s_i^*(h)$ at all information sets h at which i moves, then we call σ^* a perfect Bayesian equilibrium.

In thinking about equilibria in games, we can imagine that i uses the "I'll cross the bridge when I get to it" approach, but for $j \neq i$ to figure out what is optimal for them, they need to have some idea of what i will do. In games where i will make a choice and then make a later choice, i needs to figure out the way ahead too. However, so far as optimality is concerned, i's choice at h that are reached with probability 0 are not constrained. This is the distinction between equilibria and subgame perfect equilibria. We will look at this first in a game called Atomic Handgrenades, sometimes known as the Chain Store Paradox, then in a Stackelberg game.

7.8. **Atomic Handgrenades.** You are approached by a fellow carrying a U.S. DOD certified atomic handgrenade. He says, "Give me 20$ or else I will explode this small thermonuclear device." Your choice is whether or not to give him the 20$. One (not the only) representation of this situation has your two strategies being "Give 20$" and "Don't Give 20$". The fellow's strategies are "Blow up hand-held thermo-nuclear device if not given 20$" and "Don't blow up hand-held thermo-nuclear device if not given 20$". These are strategies that have conditional clauses in them. The extensive form representation of the game is (where $-y \ll -20$):

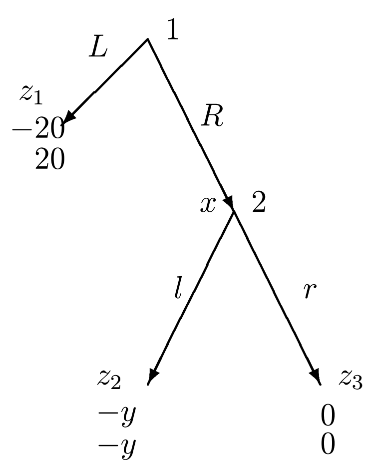


Figure 3: Atomic Handgrenades

Now, if you play the strategy "give," then $P(h) = 0$ where h is the information set representing what happens after you don't give the fellow \$20. By the commentary after the Bridge Crossing Lemma, $\sigma^*(h)$ can be arbitrary, including $\sigma^*(h)$ specifying blowing up the world, well, the neighborhood anyway. On the other hand, $s^*(h)$ must best respond to some beliefs $\beta \in \Delta(h)$. Since h contains only one point, there is only one β, hence only one optimal decision.

7.9. **Stackelberg competition.** Recall Cournot competition above, two firms, demand curve $P = 1 - Q$ and marginal cost c, $1 > c \geq 0$.[25] Now suppose that one player gets to move first, setting q_1 irrevocably. Then firm 2 moves, having seen q_1.

There are two ways to look at this: firm 1 has a disadvantage, after all, 2 sees what they are going to do then gets to figure out their best response; firm 1 has an advantage, after all, they get to choose first putting 2 on the defensive. The following is a very good exercise to keep in mind.

Homework 7.9.1. *This problem has several games with two players, each of whom have two actions. Thus there are 4 payoffs to be specified for each of the players. Consider the following 3 variants:*
(A) Player 1 moves first, player 2 sees 1's move and then moves;
(B) Player 2 moves first, player 1 sees 2's move and then moves;
(C) Players 1 and 2 pick their action with no knowledge of the choice of the other players.

1. *If possible, give games in which both players prefer (A) to (B). If it is not possible, prove it. Switching the names of the two players gives an example in which both prefer (B) to (A).*
2. *If possible, give games in which both players prefer to move second. If it is not possible, prove it.*
3. *If possible, give games in which both players prefer to move first. If it is not possible, prove it.*
4. *If possible, give a game in which one of the players prefers (C) to moving second. If it is not possible, prove it.*
5. *If possible, give a game in which one of the players prefers (C) to moving first. If it is not possible, prove it.*

In any case, there is a first mover advantage in the Stackelberg game provided we are looking at PBE's. However, there is not if we are only looking at Nash equilibria.

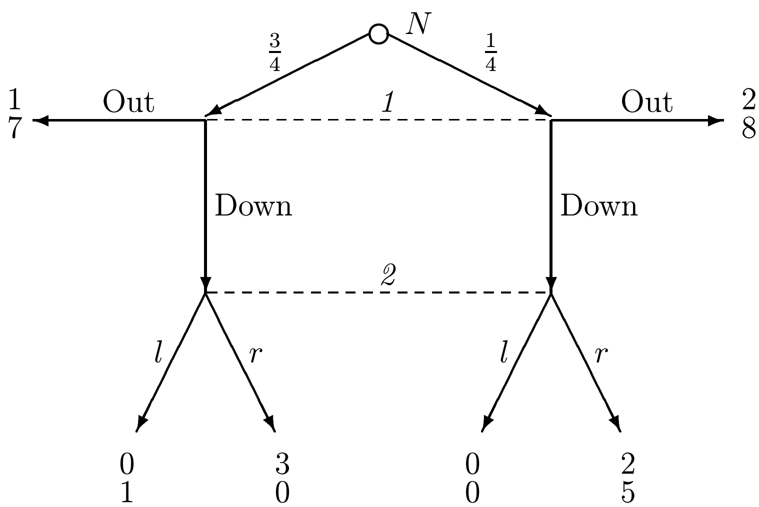
7.10. **Sequential Equilibria as Special PBE'a.** PBE'a require that at all h, $\sigma^*(h) = s^*(h)$. Sequential equilibria require that beliefs at h, $\beta(\cdot\big|h)$, have a special property called consistency. The starting point is the observation that if $\sigma \gg 0$, then for all information sets h, $P(h\big|\sigma) > 0$. Let $\beta(\cdot\big|h, \sigma)$ be the beliefs determined by Bayes's Law when $\sigma \gg 0$.
Consistency: A belief system β is **consistent for the strategy vector** σ if there exists a sequence $\sigma^n \to 0$, $\sigma^n \gg 0$, such that for all h, $\beta(\cdot\big|h) = \lim_{n\to\infty} \beta(\cdot\big|h, \sigma^n)$.

A PBE with consistent beliefs is called a **sequential equilibrium**.

[25]Note that as $n \uparrow \infty$ in the Cournot games, the outcome converges to the competetive on.

Often, asking for a PBE gives a sufficiently small set of equilibria. If this doesn't work, often asking for a sequential equilibrium gives a sufficiently small set of equilibria. But sometimes this doesn't work either. Then it is time to pull out the big guns.

Homework 7.10.1. *This question refers to the following extensive form game in which Nature, N, moves first with the probabilities indicated below.*



1. *Give the normal form for this game.*
2. *This game has two types of Nash equilibria. Find them and their associated utilities.*
3. *This game shows that a Perfect Bayesian Equilibrium (PBE, best response to strategies and beliefs at all information sets, beliefs given by Bayes' law if applicable) can play a weakly dominated strategy. Explain.*
4. *This game shows that a PBE need not be sequential. Explain.*

7.11. **Iterated Deletion of Equilibrium Dominated Strategies.** This is where things become **really** interesting for game theory. Our first example is a story about entry deterrence told as a story about Beer and Quiche. The essential idea is to identify sets of E of equilibria that all give the same distribution over terminal nodes (i.e. the same distribution over outcomes), and then to only accept sets E that are internally consistent. The internal consistency requirement looks at domination relative to a set — a strategy t_i is **dominated by a strategy π relative to a set** E if for all π' in E, $u_i(\pi'\backslash\pi_i) > u_i(\pi'\backslash t_i)$. A set E of sequential equilbria is internally consistent if E is contains a subset of the sequential equilibria of the smaller game that we get when anything dominated relative to E is removed.

This sounds complicated, in practice, it is fairly easy. If we have time, we'll look at a labor market signaling game.

7.11.1. *Beer and quiche.* The following is a version of entry deterrence, and is due to Cho and Kreps [1987]. It is called Beer-Quiche.

Example 7.11.1. *There is a fellow who, on 9 out of every 10 days on average, rolls out of bed like Popeye on spinach. When he does this we call him "strong." When strong, this* *fellow likes nothing better than Beer for breakfast. On the other days he rolls out of bed like a graduate student recovering from a comprehensive exam. When he does this we call him "weak." When weak, this fellow likes nothing better than Quiche for breakfast.*

In the town where this schizoid personality lives, there is also a swaggering bully. Swaggering bullies are cowards who, as a result of deep childhood traumas, need to impress their peers by beating on others. Being cowards, they would rather pick on someone when they are down, that is, weak, than to pick on somebody strong. They get to make their decision about whether or not to pick on the first (schizoid) fellow after having observed what he had for breakfast. Formally, the game tree is (ignoring the $h_{i,z}$ *for the instant):*

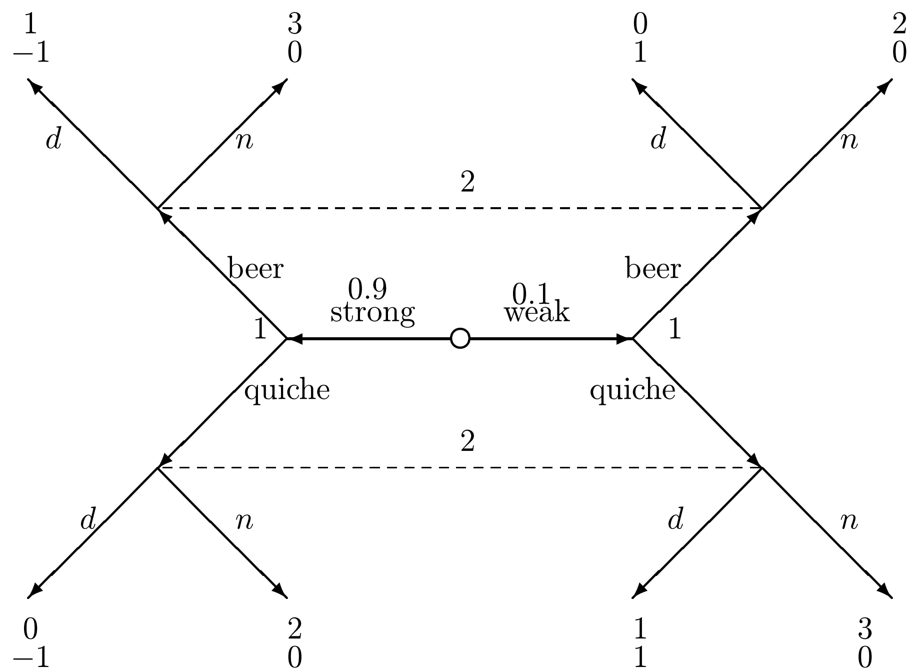


Figure 5

Show that there are two sequential equilibria, they are both pooling. We are going to kill the one in which player 1 has quiche for breakfast. There are many arguments against this equilibrium.

1. It's too weird.
2. It's not sensible. Cho and Kreps [1987], and Cho [1987] argue in terms of the different types of the agents. Because the weak type's payoff in this equilibrium, 3, is better than either of the payoffs to having beer for breakfast, 0 or 2, while the same is not true for the strong type (2 *vs.* 1 or 3), the swaggering bully should believe that it is infinitely more likely that it is the strong type whenever he sees beer for breakfast. But in this case, the equilibrium falls apart, because the bully's best response to these beliefs is to never challenge a beer drinker, and if beer drinkers are never challenged, the schizoid should always have beer when he feels strong.

There is a speechifying version of the previous argument.

There is also an iterated deletion of equilibrium dominated strategies argument. Give it. Fix the set $E \subset Eq(\Gamma)$ of quiche equilibria. We will agree that E is "sensible" or "internally consistent" by asking if it passes an E-test, that is, if the strategie in E are still equilibria after we have deleted any weakly dominated strategies and/or any strategies that are weakly dominated with respect to E. For this we need the definition,

A strategy $\sigma_i \in \Delta_i$ **weakly dominates** $t_i \in S_i$ **relative to** $T \subset \Delta$ if

$$(\forall \sigma' \in T)[u_i(\sigma' \backslash \sigma_i) \geq u_i(\sigma' \backslash t_i)] \text{ and } (\exists \sigma^\circ \in T)[u_i(\sigma^\circ \backslash \sigma_i) > u_i(\sigma^\circ \backslash t_i)].$$

If $T = \Delta$, this is the previous definition of weak dominance. Let $D_i(T)$ denote the set of $t_i \in S_i$ that are weakly dominated relative to T. Smaller T's make the first condition easier to satisfy and make the second condition more difficult to satisfy, so there is no general relation between the size of T and the size of $D_i(T)$.

This kind of self-referential test is called an **equilibrium dominance test**. Verbally, this makes (some kind of) sense because, if everyone knows that only equilibria in a set E are possible, then everyone knows that no-one will play any strategy that is either weakly dominated or that is weakly dominated *relative to E itself.* That is, E should survive an E-test.

Detour: There is a problem with this idea, one that can be solved by restricting attention to a class $\mathcal{E}$ of subsets of $Eq(\Gamma)$. The class $\mathcal{E}$ is the class of closed and connected subsets of $Eq(\Gamma)$. If you've had a reasonable amount of real analysis or topology, you will know what the terms "closed" and "connected" mean. If not, this is gibberish. **End of Detour**

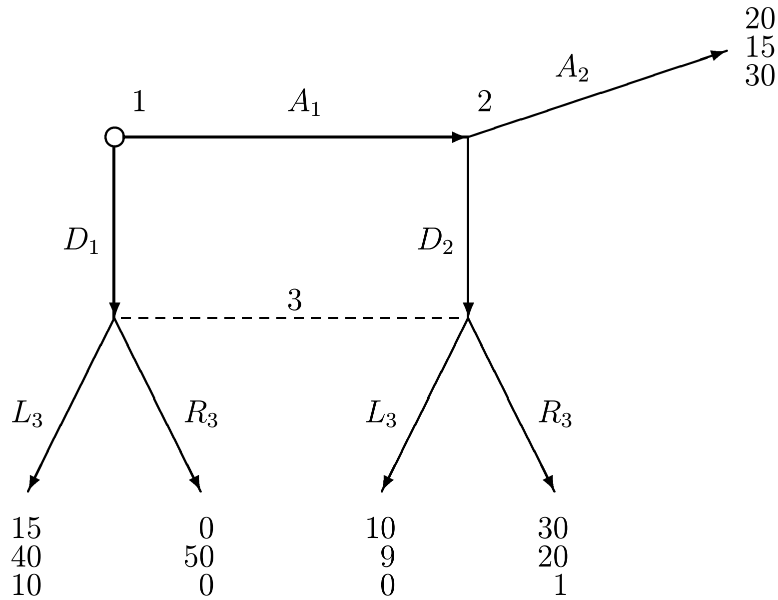
Formally, fix a set $E \subset Eq(\Gamma)$, set $S_i^1 = S_i$, $E^1 = E$, given S_i^n for each $i \in I$, set $\Delta^n = \times_{i \in I} \Delta(S_i^n)$, and iteratively define S_i^{n+1} by

$$S_i^{n+1} = S_i^n \setminus \{D_i(\Delta^n) \cup D_i(E^n)\}.$$

$E \in \mathcal{E}$ **passes the iterated equilibrium dominance test** if at each stage in the iterative process, there exists a non-empty $E^{n+1} \in \mathcal{E}$, $E^{n+1} \subset E$, such that for all $\sigma \in E^{n+1}$ and for all $i \in I$, $\sigma_i(\{D_i(\Delta^n) \cup D_i(E^n)\}) = 0$.

Okay, now show that the quiche equilibrium does not survive this kind of self-referential test. There is a theorem that there is always some non-empty (as well as closed and connected) set of equilibria that does survive this test, so the beer equilibria do survive.

7.11.2. *A horse game.* These games are called horse games because the game tree looks like a stick figure horse, not because they were inspired by stories about the Wild West.



There are three sets of equilibria for this game, Listing 1's and 2's probabilities of playing D_1 and D_2 first, and listing 3's probability of playing L_3 first, the equilibrium set can be partitioned into $Eq(\Gamma) = E_A \cup E_B \cup E_C$,

$$E_A = \{((0, 1), (0, 1), (\gamma, 1 - \gamma)) : \gamma \geq 5/11\}$$

where the condition on γ comes from $15 \geq 9\gamma + 20(1 - \gamma)$,

$$E_B = \{((1, 0), (\beta, 1 - \beta), (1, 0)) : \beta \geq \frac{1}{2}\}$$

where the condition on β comes from $15 \geq 10\beta + 20(1 - \beta)$, and

$$E_C = \{((0, 1), (1, 0), (0, 1))\}.$$

Note that $\mathbb{O}(\cdot)$ is constant on the sets E_A, E_B, and E_C. In particular, this means that for any $\sigma, \sigma' \in E_k$, $u(\sigma) = u(\sigma')$. I assert without proof that the E_k are closed connected sets.[26]

There are no weakly dominated strategies for this game:

1. $u_1(s\backslash D_1) = (15, 15, 0, 0)$ while $u_1(s\backslash A_1) = (10, 20, 30, 20)$ so no weakly dominated strategies for 1,
2. $u_2(s\backslash D_2) = (40, 9, 50, 20)$ while $u_2(s\backslash A_2) = (40, 15, 50, 15)$ so no weakly dominated strategies for 2,
3. $u_3(s\backslash L_3) = (10, 0, 10, 30)$ while $u_3(s\backslash R_3) = (0, 1, 0, 3)$ so no weakly dominated strategies for 3.

Each E_k survives iterated deletion of weakly dominated strategies. However, E_A and E_B do not survive self-referential tests, while E_C does.

[26] Intuitively, the sets are closed because they are defined by weak inequalities, and they are connected because, if you were to draw them, you could move between any pair of points in any of the E_k without lifting your pencil.

1. E_A — the strategy D_1 is dominated for 1 **relative to** E_A. Removing D_1 makes L_3 weakly dominated for 3, but every $\sigma \in E_A$ puts mass on the deleted strategy, violating the iterative condition for self-referential tests. (We could go further, removing L_3 make A_2 dominated for 2, and every $\sigma \in E_A$ puts mass on A_2.)
2. E_B — the strategy R_3 is dominated for 3 relative to E_B, removing R_3 make D_2 weakly dominated for 2, meaning that every $\sigma \in E_B$ puts mass on the deleted strategy, violating the iterative condition for self-referential tests.

The set E_C contains only one point, and it is easy to check that 1 point survives iterated deletion of strategies that are either weakly dominated or weakly dominated relative to E_C.

7.11.3. *Burning money.* This is just another version of Example 7.6.1. The first person, 1 chooses between a sure payoff of 2 or else play of "Battle of the Sexes" (Figure 6).

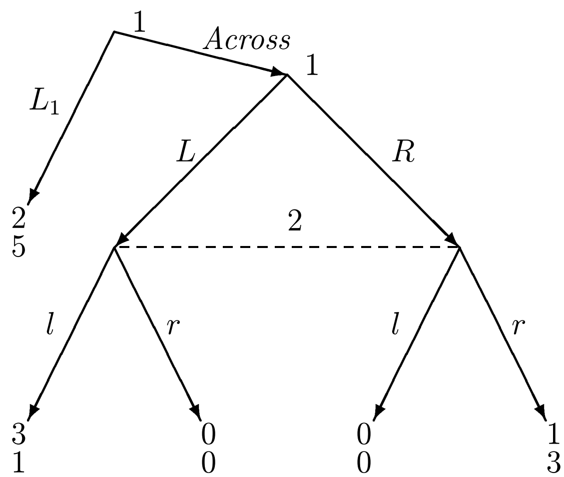
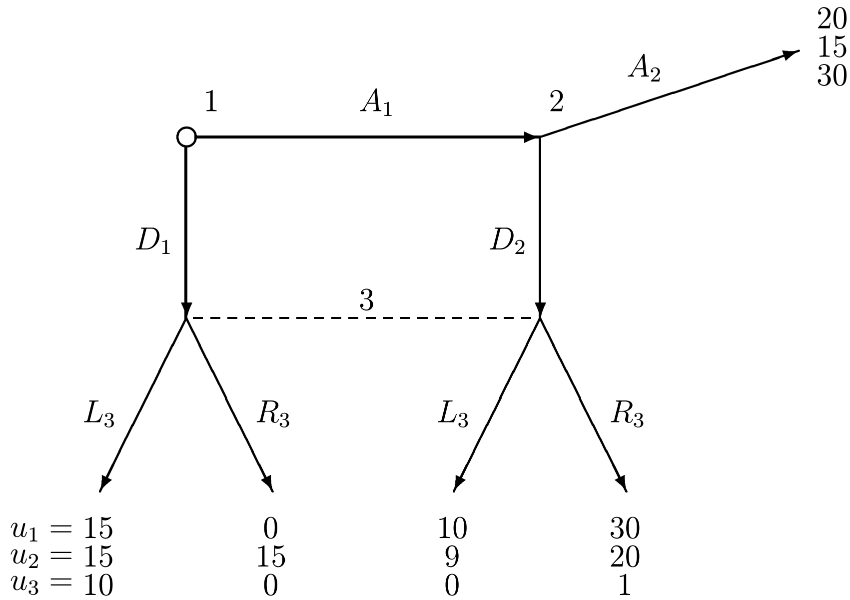


Figure 6

The only equilibrium that satisfies Iterated Deletion of Weakly Dominated strategies is $((Across, L), l)$. In terms of making speeches, this amounts to letting agent 1 say "I am going to turn the game over to you. As you know, this means that I will give up a sure payoff of 2. Think well whether or not this means that I am going to play L with its associated payoff of 3 or R with its payoff of 1, or randomize $(\frac{1}{4}, \frac{3}{4})$ over the two actions with its payoff of $\frac{3}{4}$." It is the possibility of interpreting the rather abstract formal definition of forward induction with this kind of speech that earns it the name Forward Induction.

7.11.4. *One More Time (into the breach).*

Homework 7.11.1. *Consider the 3 person game with the following extensive form representation.*



The story is: Firm 1 owns a patent. If 1 decide to use it, D_1, they enter the market presently controlled by 3. If 1 decides not to use it, A_1, they offer a subsidiary firm 2 the chance to lease it. If firm 2 choose to lease it, D_2, they enter the market presently controlled by 3. If firm 2 chooses not to lease, A_2, firm 3 is left as a monopolist in the market (with profits of 30). Note that firm 3 cannot distinguish entrants, but has different incentives if firm 1 or firm 2 is the one to enter.

1. *Give a normal (or strategic) form representation of this game.*
2. *Show that there are no weakly dominated strategies in this game.*
3. *Let γ be the probability that 3 plays L_3 if there is an entrant. Find the set of γ such that $(A_1, A_2, (\gamma, 1-\gamma))$ is a Nash equilibrium for this game.*
4. *Show that each equilibrium $(A_1, A_2, (\gamma, 1-\gamma))$ is also a sequential equilibrium.*
5. *Show that none of the equilibria $(A_1, A_2, (\gamma, 1-\gamma))$ involve 3 having "reasonable" beliefs by showing that the requisite beliefs are not consistent with the self-referential tests described above.*

7.11.5. *Rubinstein-Ståhl sequential offers.* Here is the model: player 1 moves first offers a division $x \in [0, 1]$ that player 2 may then either accept or reject. If 2 rejects, then she may make a counter-offer which 1 may accept or reject. etc. etc. etc. Payoffs to the division x at time period t are given by $(x\delta_1^{t-1}, (1-x)\delta_2^{t-1})$ where $0 < \delta_i < 1$ is i's discount factor. The really remarkable thing is

Lemma: *There is only one subgame perfect equilibrium outcome for this game: 1 offers $\frac{1-\delta_2}{1-\delta_1\delta_2}$ in the first period and 2 accepts.*

You can say a good deal more about this subgame perfect equilibrium: the only SGP **strategies** have i offering $\frac{1-\delta_j}{1-\delta_i\delta_j}$ when it is her turn to offer and accepting (rejecting any share greater than or equal to (less than) $\delta_i\frac{1-\delta_j}{1-\delta_i\delta_j}$ when it is her turn to accept or reject.

The Ståhl version of this model had a finite horizon T so that the model is quite easy to analyse – if $(m, 1-m)$ is the unique SGP offer at time t when 1 offers, then at $t-2$, the unique SGP offer is $f(m) = 1 - \delta_2(1 - \delta_1 m)$. Thus, at $t-4$, the offer is $f \circ f(m)$, at $t-6$ it is $f \circ f \circ f(m)$, etc. In the limit, we arrive at $f^\infty(m)$ which is exactly $\frac{1-\delta_2}{1-\delta_1\delta_2}$.

In general, infinite horizon games have many more equilibria than you find by looking at the limits of finite horizon equilibria, but this game is not like that. Check the textbook for details about the infinite horizon game.

Department of Economics, U. T. Austin, Austin TX 78712-1173