Solutions to Assignment #1 for **Managerial Economics**
Fall 2017

Due date: Mon. Sept. 18.

Readings: The optimization theory in the problems below, and, as needed, your microeconomics textbook. The next assignment will include optimization treatments of examples found in Klein et al. Ch. 1.

Optimization techniques with applications

1. Unconstrained Optimization

For a microeconomist, the central assumption is that people, as decision makers, are doing as well as they can with the knowledge and resources they have. The mathematics of optimization is the key to understanding the implications of this assumption.

1.1. **The General Form.** Throughout, $\mathbf{x} = (x_1, \ldots, x_n)$ will represent the levels of n different decision variables. When studying different situations, the interpretations of the n variables will change. Examples include production levels for both private and public goods, investment levels, pollution abatement levels, the amount of time spent on projects, and many many others.

The **unconstrained** optimization problems will have the general form

$$\max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \text{ or } \max_{x_1,\ldots,x_n \geq 0} f(x_1, \ldots, x_n). \tag{1}$$

$f(\cdot)$ is called the **objective function**. It represents what the decision maker is trying to optimize. The second variant of the maximization problem represents situations in which negative levels of the decision variables do not represent anything of interest.

Solutions to the problems in (1) are, by assumption, what the decision maker chooses to do. They will be denoted $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$. They will have economic interpretations.

1.2. **Solving Unconstrained Problems.** Solving $\max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n)$ for $\mathbf{x}^*$, will often be found by solving the system of n equations in n unknowns given by

$$\begin{aligned}
\tfrac{\partial f(x_1,\ldots,x_n)}{\partial x_1} &= 0 \\
\tfrac{\partial f(x_1,\ldots,x_n)}{\partial x_2} &= 0 \\
\vdots \quad &= 0 \\
\tfrac{\partial f(x_1,\ldots,x_n)}{\partial x_n} &= 0.
\end{aligned}$$

Since these equations involve first order derivatives, they are called **First Order Conditions (FOCs)**.

Often, but not always, the FOCs just given can also be used solve the problem $\max_{x_1,\ldots,x_n \geq 0} f(x_1, \ldots, x_n)$. There will be important cases in which this is not true. For those cases, we will need to modify the FOCs, and we will cover this modification later.

1.3. **Homework problems on unconstrained maximization.** You MUST know how to do the following problems in order to get through this class.

A. Give the FOCs and the solutions to the following problems.

1. $\max_x f(x)$ when $f(x) = 9 + 3x - 4x^2$.
   Ans. FOCs — $f'(x) = 3 - 8x = 0$. **Soln** — $x^* = \frac{3}{8}$.
2. $\max_x f(x)$ when $f(x) = -[2x^2 + 4(100 - x)^2]$.
   Ans. FOCs $f'(x) = -[4x - 8(100 - x)] = 0$. **Soln** — $x = \frac{200}{201} \simeq 0.995$.
3. $\max_x f(x)$ when $f(x) = 250 + 19x - e^x$.
   Ans. FOCs $f'(x) = 19 - e^x = 0$. **Soln** — $x^* = \log(19) \simeq 2.944$.
4. $\max_{x\geq 0} f(x)$ when $f(x) = 50\sqrt{x} - 0.8 \cdot x$.
   Ans. FOCs $f'(x) = \frac{25}{\sqrt{x}} - 0.8 = 0$. **Soln** — $x^* = \left(\frac{25}{0.8}\right)^2 = 976.5625$.
5. $\max_{x\geq 0} f(x)$ when $f(x) = 500\sqrt{x} - 0.01 \cdot x^2$.
   Ans. FOCs $f'(x) = \frac{250}{\sqrt{x}} - 0.02x = 0$. **Soln** — $x^* = \left(\frac{250}{0.02}\right)^{2/3} \simeq 538.61$.
6. $\max_{x\geq 0} f(x)$ when $f(x) = 5\log(x) - 0.1 \cdot x$.
   Ans. FOCs $f'(x) = \frac{5}{x} - 0.1 = 0$. **Soln** — $x^* = 50$.
7. $\max_{x\geq 0} f(x)$ when $f(x) = 5\log(x) + 3\log(10 - x)$.
   Ans. FOCs $f'(x) = \frac{5}{x} - \frac{3}{10-x} = 0$. **Soln** — $x^* = 6.25$.
8. $\max_{x\geq 0} f(x)$ when $f(x) = 5x - 0.01x^2$.
   Ans. FOCs $f'(x) = 5 - 0.02x = 0$. **Soln** — $x^* = 250$.

B. Give the FOCs and the solutions to the following problems.

1. $\max_{x_1,x_2} f(x_1, x_2)$ when $f(x_1, x_2) = 29 + 3x_1 + 4x_2 - (3x_1^2 - 2x_1x_2 + 4x_2^2)$.
   Ans. FOCs —
$$\frac{\partial f}{\partial x_1} = 3 - 6x_1 + 2x_2 = 0$$
$$\frac{\partial f}{\partial x_2} = 4 + 2x_1 - 8x_2 = 0.$$
   Soln — $x_1^* = \frac{16}{22}$, $x_2^* = \frac{15}{22}$.
2. $\max_{x_1,x_2} f(x_1, x_2)$ when $f(x_1, x_2) = -[x_1^2 + 4x_2^2 + 2(100 - (x_1 + x_2))^2]$.
   Ans. FOCs —
$$\frac{\partial f}{\partial x_1} = -[2x_1 - 4(100 - (x_1 + x_2))] = 0$$
$$\frac{\partial f}{\partial x_2} = -[8x_2 - 4(100 - (x_1 + x_2))] = 0$$
   Soln — from $2x_1 = 4(100 - (x_1 + x_2))$ and $8x_2 = 4(100 - (x_1 + x_2))$, conclude $x_1 = 4x_2$, plug into $\partial f/\partial x_2 = 0$ to find $8x_2 - 400 + 4(4x_2 + x_2) = 0$, that is, $28x_2 = 400$ or $x_2^* = \frac{100}{7}$ and $x_1^* = \frac{400}{7}$.
3. $\max_{x_1,x_2\geq 0} f(x_1, x_2)$ when $f(x_1, x_2) = 5\log(x_1) + 30\log(x_2) - 0.4 \cdot x_1 - x_2$.
   Ans. FOCs —
$$\frac{\partial f}{\partial x_1} = \frac{5}{x_1} - 0.4 = 0$$
$$\frac{\partial f}{\partial x_2} = \frac{30}{x_2} - 1 = 0$$
   Soln — $x_1^* = 12.5$, $x_2^* = 30$.

4. $\max_{x_1,x_2\geq 0} f(x_1,x_2)$ when $f(x_1,x_2) = 15x_1 + 30x_2 - 0.4e^{x_1} - 0.1e^{x_2}$.
Ans. FOCs —

$$\frac{\partial f}{\partial x_1} = 15 - 0.4e^{x_1} = 0$$
$$\frac{\partial f}{\partial x_2} = 30 - 0.1e^{x_2} = 0$$

Soln — $x_1^* = \log(37.5) \simeq 3.6434$, $x_2^* = \log(30) \simeq 3.4012$.

C. A person has an original amount a of a good. By sacrificing x of it, they can produce $y = g(x)$ of another good. The person solves the utility maximization problem $\max_{x\geq 0} u(a-x, g(x))$. Suppose that $u(c,y) = \log(c)+y$ and that $g(x) = x$.

1. Give the FOCs for the maximization problem $\max_{x\geq 0} u(a-x, g(x))$.
Ans. FOCs — $\max_{x\geq 0} f(x,a) = \log(a-x) + x$ has $\partial f/\partial x = -\frac{1}{a-x} + 1 = 0$. Note that if $a < 1$, this cannot be solved.
2. Solve for x^*.
Ans. $-\frac{1}{a-x} + 1 = 0$ requires $x^*(a) = a - 1$, and if $a \geq 1$, this is the solution.

D. Person 1 has an original amount a_1 of a good while person 2 has an original amount a_2 of a good. By sacrificing x_1 and x_2 of it, the two of them can produce $y = g(x_1 + x_2)$ of another good. Suppose that $u_1(c_1, y) = \log(c) + y$ and that $u_2(c_2, y) = \log(c) + y$. This is a very simple representation of the idea of a public good — whatever the level of y that is produced, both people enjoy it. As above, suppose that $g(x) = x$.

1. Give the FOCs for the maximization problem
$$\max_{x_1,x_2\geq 0} [u_1(a_1 - x_1, x_1 + x_2) + u_2(a_2 - x_2, x_1 + x_2)].$$
Ans. Want the FOCs for
$$\max_{x_1,x_2\geq 0} [\log(a_1 - x_1) + (x_1 + x_2)] + [\log(a_2 - x_2) + (x_1 + x_2)].$$
These are $-\frac{1}{a_1-x_1} + 2 = 0$ and $-\frac{1}{a_2-x_2} + 2 = 0$.
2. Solve for $\mathbf{x}^* = (x_1^*, x_2^*)$.
Ans. $\mathbf{x}^* = (a_1 - \frac{1}{2}, a_2 - \frac{1}{2})$.

2. Parametrized Optimization Problems

The objective functions of interest are often of interest because they are **parametrized**, $f = f(\mathbf{x};\theta)$. This changes the problems to

(2) $$V(\theta) = \max_{x_1,\ldots,x_n} f(x_1,\ldots,x_n;\theta) \text{ or } V(\theta) = \max_{x_1,\ldots,x_n\geq 0} f(x_1,\ldots,x_n;\theta).$$

2.1. Three Aspects.
There are three aspects to this.

- First, θ is **not** something that the decision maker can choose, it is something outside of their control. When studying different situations, the interpretations of θ will change. Examples include resources, prices of outputs, prices of inputs, pollution reduction targets, measures of benefits and costs included in calculations. There are many others.

- Second, optimizing behavior now depends on the value of the parameter, and we represent this by $\mathbf{x}^*(\theta) = (x_1^*(\theta), \ldots, x_n^*(\theta))$. We care about how $\mathbf{x}^*(\cdot)$ depends on the parameter θ.
- Third, we now include the parametrized **value** of the decision problem, $V(\theta)$.

To find $V(\theta)$ explicitly, we will solve the problem and "plug the solution back in," that is,

$$V(\theta) = f(x_1^*(\theta), \ldots, x_n^*(\theta); \theta). \tag{3}$$

2.2. **Examples from Microeconomics.**

- x is the production level for a firm, p is the market price for the good, $c(x)$ is the cost of producing x, and the problem is
$$V(p) = \max_x \ (px - c(x)),$$
that is, $f(x;p) = px - c(x)$. Here: the price p is the parameter; $x^*(p)$ is the supply that the firm produces when the price is p; $V(p) = px^*(p) - c(x^*(p))$ is the profit function.
- x is the production level for a firm, p is the market price for the good, $c(x, w)$ is the cost of producing x when the price of inputs is w, and the problem is
$$V(p, w) = \max_{x \geq 0} \ (px - c(x, w)),$$
that is, $f(x; p, w) = px - c(x, w)$. Here: the parameter is the vector of prices, (p, w); $x^*(p, w)$ is the supply that the firm produces when the price of the output is p and the price of the inputs is w; $V(p, w)$ is the profit function. In intermediate micro, $x^*(\cdot, \cdot)$ is the supply function expressed as a price of both inputs and outputs. We are now explicitly including the dependence of profits on the price of the output and the price of the inputs.
- x_1, x_2 are the production levels for two goods, p_1, p_2 are the market prices for the two goods, $c(x_1, x_2)$ is the cost of production, and the problem is
$$V(p_1, p_2) = \max_{x_1, x_2} \ [(p_1x_1 + p_2x_2) - c(x_1, x_2)].$$
that is, $f(x_1, x_2; p_1, p_2) = (p_1x_1 + p_2x_2) - c(x_1, x_2)$. The solution vector, $\mathbf{x}^*(p_1, p_2) = (x_1^*(p_1, p_2), x_2^*(p_1, p_2))$ is the joint supply function for the function, and $V(p_1, p_2) = (p_1x_1^* + p_2x_2^*) - c(x_1^*, x_2^*)$ is the profit function.
- $x_1, \ldots, x_n$ are the n inputs into the production of good y, the prices of inputs are $w_1, \ldots, w_n$, $y = g(x_1, \ldots, x_n)$ expresses output using the production function $g(\cdot)$, output is sold at a price p. The parameter is now the price of the output as well as the vector of the prices of the inputs,
$$V(p, w_1, \ldots, w_n) = \max_{x_1, \ldots, x_n} pg(x_1, \ldots, x_n) - (w_1x_1 + \cdots + w_nx_n),$$
that is, $f(x_1, \ldots, x_n; p, w_1, \ldots, w_n) = pg(x_1, \ldots, x_n) - (w_1x_1 + \cdots + w_nx_n)$. The solution vector $\mathbf{x}^*(p, w_1, \ldots, w_n)$ is the vector of goods the firm produces when

the output price is p and the input prices are $w_1, \dots, w_n$. Plugging this vector of solutions back into the objective function gives the profit function.

- When the firm in the previous example is large enough that their decisions affect the price and the demand function is $p(q)$, then we have the price as a function of the decision variables, $p = p(g(x_1, \dots, x_n))$. In this case, the problem is

$$V(w_1, \dots, w_n) = \max_{x_1,\dots,x_n} p(g(x_1, \dots, x_n)) \cdot g(x_1, \dots, x_n) - (w_1x_1 + \cdots + w_nx_n).$$

  Here $f(x_1, \dots, x_n; w_1, \dots, w_n) = p(g(x_1, \dots, x_n)) \cdot g(x_1, \dots, x_n) - (w_1x_1 + \cdots + w_nx_n)$ and the solution vector is the supply vector as function of the input prices. In the previous example, the price p was a parameter, it was not under the control of the decision maker. Here the decision maker *does* control p, hence it is not a parameter.

2.3. Homework problems on parametrized maximization.

E. Give $x^*(p)$ and $V(p) = \max_{x\geq 0} (px - c(x))$ when $c(x) = \frac{1}{2}x^2$.
Ans. FOCs are $p - c'(x) = p - x = 0$, which yields $x^*(p) = p$, from which we have $V(p) = px^*(p) - c(x^*(p)) = p^2 - \frac{1}{2}p^2 = \frac{1}{2}p^2$.

F. Give $x^*(p, w)$ and $V(p, w) = \max_{x\geq 0} (px - c(x, w))$ when $c(x, w) = w(e^x - 1)$.
Ans. FOCs are $p - c'(x) = p - we^x = 0$, which yields $x^*(p, w) = \log(p/w)$. For this to make any sense, we must have $p/w > 1$ to avoid $x^* < 0$. Under this condition, $V(p, w) = px^*(p, w) - c(x^*(p, w), w) = p\log(p/w) - w(\log(p/w) - 1)$.

G. Give $\mathbf{x}^*(p_1, p_2)$ and $V(p_1, p_2) = \max_{x_1,x_2}(p_1x_1 + p_2x_2) - c(x_1, x_2)$ when $c(x_1, x_2) = x_1^2x_2^3$.
Ans. With $f(x_1, x_2; p_1, p_2) = (p_1x_1 + p_2x_2) - x_1^2x_2^3$, the FOCs are

$$\frac{\partial f}{\partial x_1} = p_1 - 2x_1x_2^3 = 0$$
$$\frac{\partial f}{\partial x_2} = p_2 - 3x_2^2x_2^2 = 0.$$

From these, we see that $\frac{p_1}{p_2} = \frac{2x_1x_2^3}{3x_2^2x_2^2} = \frac{2}{3}\frac{x_2}{x_1}$. Rearranging, $x_1 = \frac{2}{3}x_2$ so that $p_1 = \frac{4}{3}x_2^4$ or $x_2^* = (\frac{3p_1}{4})^{1/4}$ and $x_1^* = \frac{2}{3}(\frac{3p_1}{4})^{1/4}$. As usual, $V(p_1, p_2) = f(x_1^*, x_2^*; p_1, p_2)$, which is, in this case slightly messy.

H. Give $\mathbf{x}^*(w_1, \dots, w_n)$ and $V(p, w_1, \dots, w_n) = \max_{x_1,\dots,x_n} pg(x_1, \dots, x_n) - (w_1x_1 + \cdots + w_nx_n)$ when $g(x_1, \dots, x_n) = \Pi_{i=1}^n x_i^{\alpha_i}$ where each $\alpha_i > 0$ and $\sum_i \alpha_i < 1$.
Ans. This problem requires a bit more algebraic manipulation than the others. The solution below turns the complicated multivariable maximization problem into a one variable problem and then solves that. There are other ways to proceed.

The FOCs yield $x_i^{\alpha_i - 1} \cdot \Pi_{j\neq i}x_j^{\alpha_j} = w_i$. For each $i \neq j$, this gives $\frac{w_ix_i}{\alpha_i} = \frac{w_jx_j}{\alpha_j}$. Letting $\beta_i = \frac{w_i}{\alpha_i}$ we have $\beta_ix_i = \beta_jx_j$. Letting $i = 1$, we have $x_j = \gamma_jx_1$ where $\gamma_j = \frac{\beta_1}{\beta_j}$. Returning to the original problem with this turns it into the 1-variable problem,

$$\max_{x_1>0}\ p(x_1^{\alpha_1}\Pi_{j\geq 2}(\gamma_j)^{\alpha_j}x_1^{\alpha_j}) - (w_1x_1 + w_2\gamma_2x_1 + \cdots + w_n\gamma_nx_1).$$

Let $\kappa = \Pi_{j\geq 2}(\gamma_j)^{\alpha_j}$, $A = \sum_j \alpha_j$ and $\rho = (w_1 + w_2\gamma_2 + \cdots + w_n\gamma_n)$. With these, the objective function is $p\kappa x_1^A - \rho x_1$. The FOCs for this are $p\kappa x_1^{A-1} - \rho = 0$, solving yields

$$x_1^* = \left(\frac{\rho}{p\kappa}\right)^{1/(A-1)} \text{ and}$$
$$x_j^* = \gamma_j x_1^* \text{ for } j \geq 2.$$

Writing out $V(\cdot)$ is a messy exercise in substitution.

I. Give $x^*(a, \beta)$ and $V(a, \beta) = \max_{x\geq 0} [\beta \log(a - x) + x]$. Assume $a > 0$ and $\beta > 0$. **Ans**. The FOCs are $-\frac{\beta}{a-x} + 1 = 0$ which yields $x^*(a, \beta) = a - \beta$ and $V(a, \beta) = \beta \log(\beta) + (a - \beta)$. Again, we need to be a bit careful, we must have $a \geq \beta$ for this to make sense.

J. Give $\mathbf{x}^*((a_1, a_2), (\beta_1, \beta_2))$ and the value function

$$V((a_1, a_2), (\beta_1, \beta_2)) = \max_{x_1, x_2 \geq 0} [(\beta_1 \log(a_1 - x_1) + x_1) + (\beta_2 \log(a_2 - x_2) + x_2].$$

Ans. The FOCs are a variant of those in the previous problem and $x_i^* = a_i - \frac{\beta_i}{2}$ for $i = 1, 2$, and you substitute those into the objective function to find $V(\cdot)$.

3. Comparative Statics with Derivatives

We now turn to determining the dependence of behavior, $\mathbf{x}^*(\theta)$, on θ. In particular, we will often be interested in knowing whether or not the following kind of monotone results hold:

$$\text{if } \theta^\circ > \theta, \text{ then } \mathbf{x}^*(\theta^\circ) > \mathbf{x}^*(\theta); \text{ or}$$
$$\text{if } \theta^\circ > \theta, \text{ then } \mathbf{x}^*(\theta^\circ) < \mathbf{x}^*(\theta).$$

The key part of the answer is, "If the net marginal benefit of an activity increases as θ increases, then $\mathbf{x}^*(\theta^\circ) > \mathbf{x}^*(\theta)$, if the net marginal benefit decreases, then $\mathbf{x}^*(\theta^\circ) < \mathbf{x}^*(\theta)$." There are subtleties, especially when the parameter and the decision take vector form, but this will be the essential intuition in many many contexts.

3.1. **An Example: Geometry and Calculus.** There are complicated routes to this kind of result, and simple routes. The complicated route is to explicitly calculate $\mathbf{x}^*(\theta)$, then explicitly calculate $\partial\mathbf{x}^*/\partial\theta$, and then check if it is positive or negative. This is overkill. When it can be done, you will not only know how far above or below 0 the derivative $\partial\mathbf{x}^*/\partial\theta$ is, you will also be able to tell what the derivative depends on.

We can often answer the simpler, less detailed, monotone questions — i.e. does the optimum go up or down as the parameter goes up — without needing to do all of the hard work. Consider, as a starting point, the one-input/one-output profit maximization problem,

$$V(p) = \max_{x\geq 0} [px - c(x)]. \tag{4}$$

In the last set of problems, you solved a version of this problem (with $c(x) = \frac{1}{2}x^2$) by finding and then solving the FOCs. We revisit these with notation that keeps the parameter, p, more firmly in view. Let $f(x;p) = px - c(x)$, the FOCs are $\partial f(x,p)/\partial x = 0$, that is,

$$p - c'(x) = 0.$$

This is a "net marginal benefit equals 0" equation: the marginal benefit of a small increase in x is p; the marginal cost is $c'(x)$. Since $c'(\cdot)$ is (usually) an increasing function, you can solve this problem by graphing the decreasing function $p - c'(x)$ — it crosses 0 from above at the point $x^*(p)$. To answer the monotone questions, we are interested in what happens to this intersection if p increases to $p^\circ > p$.

- The geometry — if you shift a decreasing function upwards, e.g. shift from the curve $p - c'(x)$ to the everywhere higher curve $p^\circ - c'(x)$, the place where it crosses 0 must move to the right. We therefore know that $x^*(p^\circ) > x^*(p)$. In this example, the economics interpretation of the result is that the supply curve of a competitive firm is increasing in the price of the output.
- The calculus — suppose that $x^*(p)$ is the function that satisfies the FOCs for all p, that is, $p - c'(x^*(p)) \equiv 0$. Suppose also that $x^*(\cdot)$ has a derivative. Taking the derivative of the FOCs along the curve $x^*(p)$ **with respect to the parameter** p involves finding

$$\tfrac{d}{dp}(p - c'(x^*(p)),$$

  and this yields (using the chair rule from calculus)

(5) $$1 - c''(x^*(p))\tfrac{dx^*(p)}{dp} = 0, \text{ or } \tfrac{dx^*(p)}{dp} = \tfrac{1}{c''(x^*(p))}.$$

  The assumption that $c'(\cdot)$ is increasing is the assumption that $c''(x) > 0$. The detailed information is $\frac{dx^*(p)}{dp} = \frac{1}{c''(x^*(p))}$, the monotone information $\frac{dx^*(p)}{dp} > 0$.

3.2. **Adding Another Parameter.** Now let us add a second detail to the problem in (4) a bit, replacing it with

(6) $$V(p,w) = \max_{x \geq 0} \, [px - w \cdot c(x)].$$

The new question is how the optimal behavior depends on w, the price of the input into the productive process. The FOCs are $\partial f(x;p,w)/\partial x = 0$.

- Geometry — the net marginal benefit is the decreasing function $p - wc'(x)$, if w increases to $w^\circ > w$, the decreasing function shifts down to $p - w^\circ c'(x)$ and the intersection, $x^*(p,w)$, moves to the left.
- Calculus — if $p - wc'(x^*(p,w)) \equiv 0$, taking derivatives of both sides **with respect to the parameter** w yields (using the product rule from calculus)

(7) $$-c'(x^*(p,w)) - wc''(x^*(p,w))\tfrac{\partial x^*(p,w)}{\partial w}, \text{ or } \tfrac{\partial x^*(p,w)}{\partial w} = -\tfrac{c'(x^*(p,w))}{wc''(x^*(p,w))}.$$

  The detailed information is the complicated expression for $\frac{\partial x^*(p,w)}{\partial w}$, the monotone information is that $\frac{\partial x^*(p,w)}{\partial w} < 0$ because $c'(x) > 0$ and $c''(x) > 0$.

3.3. **Homework problems on comparative statics with derivatives.** It will be helpful for the course if, before you start doing the calculations, you ask yourself if net marginal benefits are increasing or decreasing in the parameter in question.

K. Give the FOCs for $V(p, w) = \max_{x \geq 0} (px - wc(x))$ when $c(x) = x^2 + (e^x - 1)$. [If you can solve the FOCs for $x^*(p, w)$ as a function of p and w in terms of known functions, then you have made a mistake.] Show that $\partial x^*(p, w)/\partial p > 0$ and $\partial x^*(p, w)/\partial w < 0$ by checking the conditions discussed just above.
Ans. Part of why I assigned this problem is that I want you to appreciate how much simpler sub- and super-modularity are.
The FOCs are $p - wc'(x) = 0$. Suppose that $x^*(p, w)$ is the function for which we have
$$p - wc'(x^*(p, w)) \equiv 0.$$
Taking derivatives on both sides w.r.t. p yields
$$1 - wc''(x^*(p, w,))\tfrac{\partial x^*(p,w)}{\partial p} = 0,$$
or $\frac{\partial x^*(p,w)}{\partial p} = 1/wc''(x^*(p, w))$. Now, $c''(x) = e^x > 0$ for all x, hence $\frac{\partial x^*(p,w)}{\partial p} > 0$.
Taking derivatives on both sides w.r.t. w and rearranging yields
$$\tfrac{\partial x^*(p,w)}{\partial w} = -\tfrac{c'(x^*(p,w,)}{wc''(x^*(p,w))},$$
and we know that $w > 0$, $c' > 0$, and $c'' > 0$, hence $\frac{\partial x^*(p,w)}{\partial w} < 0$.

L. Suppose that $u(c, y) = \beta \log(c) + y$, that $g(x) = \sqrt{x + x^{1/3}}$, and give the FOCs for $V(\beta) = \max_{x \geq 0} \beta \log(a - x) + g(x))$. Is $\partial x^*(\beta)/\partial\beta > 0$? Or < 0?
Ans. FOCs are $\frac{\beta}{a-x} = g'(x)$ which yield
$$x^* = a - \tfrac{\beta}{g'(x)}$$
where $g'(x) = \frac{1+\frac{1}{3x^{2/3}}}{\sqrt{x+x^{1/3}}}$. You could try, and probably succeed, in taking derivatives of $a - \frac{\beta}{g'(x^*(\beta))}$ w.r.t. β Instead, note that the marginal benefits of sacrifice, the x, is strictly decreasing in β, so we know already that $\frac{\partial x^*(\beta)}{\partial\beta} \leq 0$, and, looking at the FOCs, when β increases, the same x cannot solve them, hence $\frac{\partial x^*(\beta)}{\partial\beta} \neq 0$, which leaves as the only possibility that $\frac{\partial x^*(\beta)}{\partial\beta} < 0$.

M. Suppose that $u(c, y) = \beta \log(c) + y$, that $g(x) \geq 0$, that $g'(x) > 0$, and that $g''(x) < 0$. Give the FOCs for $V(\beta) = \max_{x \geq 0} \beta > \log(a - x) + g(x))$, $\beta > 0$. Is $\partial x^*(\beta)/\partial\beta > 0$? Or < 0?
Ans. This solution is the same as the previous problem, since $g''(x) < 0$, a solution to the FOCs at a higher β cannot be the same as the solution at a lower β, hence $\frac{\partial x^*(\beta)}{\partial\beta} < 0$.

N. Suppose that $u(c, y) = \beta \log(c) + y$, that $g(x) \geq 0$, that $g'(x) > 0$, and that $g''(x) < 0$. Give the FOCs for $V(\beta, \gamma) = \max_{x \geq 0} \beta \log(a - x) + \gamma g(x))$, $\beta, \gamma > 0$. Is $\partial x^*(\beta, \gamma)/\partial\gamma > 0$? Or < 0?

Ans. Here an increase in γ increases the marginal reward to sacrifice hence $\partial x^*(\beta,\gamma)/\partial\gamma \geq 0$. To check that $\partial x^*(\beta,\gamma)/\partial\gamma \neq 0$, write out the FOCs and note that an increase in γ means that the same x cannot satisfy them.

4. Monotone Comparative Statics: Super- and Sub-modularity

Above, we took the derivatives of FOCs with respect to parameters to find the monotone results. We are now going to replace the derivative-based analysis with something that is simultaneously easier and more general. This is possible because it often happens that the more general approach is simpler — it lets you focus on the essentials and ignore the complicated details.

There is an 'entry cost' for this kind of analysis, learning to manipulate inequalities. This looks more difficult than it is. You should find that the derivative analysis above often provides an easy guide.

4.1. **The Supermodular Setting.** We start with a set $X \subset \mathbb{R}$, a set $\Theta \subset \mathbb{R}$, and a function $f : X \times \Theta \to \mathbb{R}$. For any $\theta \in \Theta$, let $x^*(\theta)$ be the solution (or set of solutions) to the stripped-down problem

$$\max_{x \in X} f(x,\theta). \tag{8}$$

We are interested in the comparison of $x^*(\theta^\circ)$ and $x^*(\theta)$ when $\theta^\circ > \theta$.

For $x \in X \subset \mathbb{R}$, $\theta \in \Theta \subset \mathbb{R}$, a function $f : X \times \Theta \to \mathbb{R}$ **supermodular** if for all $\theta^\circ > \theta$ and all $x^\circ > x$,

$$f(x^\circ,\theta^\circ) - f(x,\theta^\circ) \geq f(x^\circ,\theta) - f(x,\theta), \tag{9}$$

and it is **strictly supermodular** if the inequalities are strict.

Another name for supermodularity is **increasing differences** — the difference $f(x^\circ,\theta) - f(x,\theta)$ is higher at higher values of θ, that is, the difference increases as θ increases.

The function is **submodular** if for all $\theta^\circ > \theta$ and all $x^\circ > x$,

$$f(x^\circ,\theta^\circ) - f(x,\theta^\circ) \leq f(x^\circ,\theta) - f(x,\theta), \tag{10}$$

and it is **strictly submodular** if the inequalities are strict.

Another name for submodularity is **decreasing differences**.

With some qualifications (having to do with the possibility of multiple optima), the essential result is the following.

Super-modularity and comparative statics. If $f(\cdot,\cdot)$ is supermodular and $\theta^\circ > \theta$, then $x^*(\theta^\circ) \geq x^*(\theta)$.

Proof. For now we limit ourselves to the case that there is only one optimal x at any given θ. In other words, we are giving the argument for the super-modularity and comparative statics result under the additional assumption that $x^*(\theta)$ contains at most one element for each θ.

Suppose that $f(\cdot,\cdot)$ is supermodular that $\theta^\circ > \theta$, that x^* is optimal at the lower value, θ, and that x is some point less that x^*. Because x^* is optimal at θ, we know that $f(x^*,\theta) > f(x,\theta)$, that is,

$$f(x^*,\theta) - f(x,\theta) > 0.$$

Because $f(\cdot,\cdot)$ is supermodular, equation (9) holds, i.e. $f(x^*,\theta^\circ) - f(x,\theta^\circ) \geq f(x^*,\theta) - f(x,\theta)$. Therefore

$$f(x^*,\theta^\circ) - f(x,\theta^\circ) > 0.$$

This means that any $x < x^*$ cannot be optimal at θ°. From this, we can conclude that if there is an optimum at the higher value of the parameter, θ°, then that optimum must be greater than or equal to x^*. □

Sub-modularity and comparative statics. If $f(\cdot,\cdot)$ is submodular and $\theta^\circ > \theta$, then $x^*(\theta^\circ) \leq x^*(\theta)$.

The argument is almost the same.

4.2. **Some Examples.** We begin with a general observation that will make checking super/submodularity easier.

4.2.1. *On Cancellations.* Return to the single-input/single output competitive firm, the one that solves

$$\max_{x\in X} f(x,(p,w)) = px - wc(x)$$

where $X \subset \mathbb{R}_+$ is the set of possible production levels. We will show that $f(\cdot,\cdot)$ is supermodular in x and p and submodular in x and w.

Supermodularity. Pick $p^\circ > p$ and $x^\circ > x$. To show supermodularity we must show that

$$\begin{aligned} f(x^\circ,(p^\circ,w)) - f(x,(p^\circ,w)) &\geq f(x^\circ,(p,w)) - f(x,(p,w)), \text{ that is} \\ [p^\circ x^\circ - wc(x^\circ)] - [p^\circ x - wc(x)] &\geq [px^\circ - wc(x^\circ)] - [px - wc(x)]. \end{aligned}$$

The $wc(x^\circ)$ and the $wc(x)$ terms appear on both sides and cancel.

This kind of cancellation happens all the time. Pay attention to it.

After the cancellation, all that we need to check is

$$\begin{aligned} [p^\circ x^\circ - p^\circ x] &\geq [px^\circ - px], \text{ that is} \\ p^\circ[x^\circ - x] &\geq p[x^\circ - x]. \end{aligned}$$

We know that $[x^\circ - x] > 0$ and we know that $p^\circ > p$, so $f(\cdot,\cdot)$ is strictly supermodular in x and p.

Submodularity. Pick $w^\circ > w$ and $x^\circ > x$. To show submodularity we must show that

$$\begin{aligned} f(x^\circ,(p,w^\circ)) - f(x,(p,w^\circ)) &\leq f(x^\circ,(p,w)) - f(x,(p,w)), \text{ that is} \\ [px^\circ - w^\circ c(x^\circ)] - [px - w^\circ c(x)] &\leq [px^\circ - wc(x^\circ)] - [px - wc(x)]. \end{aligned}$$

The px° and the px terms appear on both sides and cancel. *The cancellation happened again!* All that we need to check that

$$[w^\circ c(x) - w^\circ c(x^\circ)] \leq [wc(x) - wc(x^\circ)] \text{ that is}$$
$$w^\circ[c(x) - c(x^\circ)] \leq w[c(x) - c(x^\circ)].$$

Because $x < x^\circ$, $c(x) \leq c(x^\circ)$. If they are equal then we have the requisite inequality holding as an equality, if they are unequal, then the requisite inequality holds strictly.

More generally, suppose that $f(x, \theta) = g(x, \theta) + h(x) + m(\theta)$ where $h(\cdot)$ and $m(\cdot)$ are arbitrary functions. To check the inequalities for checking supermodularity of $f(\cdot, \cdot)$, we can ignore the $h(x)$ and the $m(\theta)$ terms — they will cancel.

The general lesson: you only need to pay attention to terms that include the action, x, and the parameter θ. To see why, pick $x^\circ > x$ and $\theta^\circ > \theta$. We have

$$f(x^\circ, \theta^\circ) - f(x, \theta^\circ) = [g(x^\circ, \theta^\circ) - g(x, \theta^\circ)] +$$
$$[h(x^\circ) - h(x)] + [m(\theta^\circ) - m(\theta^\circ)].$$

We also have

$$f(x^\circ, \theta) - f(x, \theta) = [g(x^\circ, \theta) - g(x, \theta)] +$$
$$[h(x^\circ) - h(x)] + [m(\theta) - m(\theta)].$$

To check that $[f(x^\circ, \theta^\circ) - f(x, \theta^\circ)] \geq [f(x^\circ, \theta) - f(x, \theta)]$, note that the $[h(x^\circ) - h(x)] + [m(\theta^\circ) - m(\theta^\circ)]$ and the $[h(x^\circ) - h(x)] + [m(\theta) - m(\theta)]$ terms cancel.

Returning to the example above, to show that $f = px - wc(x)$ is supermodular in p and x, we only need check that $g = px$ is supermodular. The inequality

$$[p^\circ x^\circ - p^\circ x] > [px^\circ - px]$$

is immediate because $x^\circ > x$ and $p^\circ > p$. To check submodularity in w and x, we only need check that $g = -wc(x)$ is submodular. If you prefer, it is equivalent to check that $wc(x)$ is supermodular in w and x because multiplying the inequalities in (9), those that define supermodularity, by -1 changes their direction, giving the inequalities that define submodularity, (10). Anyhow, For $w^\circ > w$ and $x^\circ > x$, this involves checking

$$[w^\circ c(x^\circ) - w^\circ c(x)] \geq [wc(x^\circ) - wc(x)],$$

that is, $w^\circ[c(x^\circ) - c(x)] \geq w[c(x^\circ) - c(x)]$ which is immediate.

4.2.2. *On Monopoly and Monopsony.* When markets break down, it almost aways has an inimical effect on society. When there is only one person/organization on the supply side of the market, we have a **monopoly**, when there is only one person/organization on the demand side of the market, we have a **monopsony**. Both forms of market breakdown have happend at different points in history, and the consequences have varied from merely bad to outright evil. Let us give the bloodless analysis first, then cite some examples.

Monopoly. The demand curve is $p(q)$, the cost curve is $C(q)$, the monopolist solves the problem

$$\max_{q \geq 0} \; [qp(q) - C(q)]. \tag{11}$$

When q' is sold at the price $p' = p(q')$, consumer surplus is

$$S(q) := \int_0^q [p(q) - p']\, dq.$$

Consumer surplus is an increasing function, for $q^\circ > q$, $S(q^\circ) > S(q)$.

Society's problem is

$$\max_{q \geq 0}\ [(qp(q) + S(q)) - C(q)]. \tag{12}$$

The problems (11) and (12) can be put together by setting $f(q, \theta) = [(qp(q) + \theta S(q)) - C(q)]$ and setting $\Theta = \{0, 1\}$.

- The problem (11) is $\max_{q \geq 0} f(q, 0)$, while
- the problem (12) is $\max_{q \geq 0} f(q, 1)$.

The function $f(q, \theta)$ is strictly supermodular in q and θ — to check, we need only look at the term $\theta S(q)$, pick $q^\circ > q$ and $1 = \theta^\circ > \theta = 0$, and check that

$$[1S(q^\circ) - 1S(q)] > [0S(q^\circ) - 0S(q)],$$

which holds because consumer surplus is an increasing function. This means that the monopolist produces less than the amount that maximizes the sum of producer and consumer surplus. By producing more, society is made better off, and in moving to the higher quantity, that which maximizes society's welfare, the winners i.e. the consumers, can compensate the losers i.e. the owner(s) of the monopoly.

Examples. Lachlan Macquarie breaking the English army's monopoly on the medium of exchange in Australia, "rum," meant that the economy could move from barter to market-mediated exchange. In the late 19'th and early 20'th century, the Northern Securities Co. had a railroad monopoly on freight from northern mid-west farms to cities in the U.S. In the U.S. broadband is far slower and costs far more than in other countries, a result of low levels of competition enforced by a number of Federal and State laws. When you are the monopoly supplier of arms used in war and you price as a monopolist, you are *war profiteering.* This was defined, and prosecuted, as treason against the U.S. in WWII. It was not prosecuted during the invasion of Iraq.

Monopsony. The labor supply curve is an increasing function $w(q)$. Revenue for the single firm buying labor in the local market is $R(q)$. The easy case is $R(q) = pf(q)$ where $f(\cdot)$ is the production function, we expect $f'(\cdot)$ to be a positive, decreasing function, and we expect $R(\cdot)$ to have the same properties. The monopsonist solves the problem

$$\max_{q \geq 0}\ [R(q) - qw(q)]. \tag{13}$$

When a quantity of labor q' is hired at wages $w' = w(q')$, the surplus of the workers is $S(q') := \int_0^{q'} [w' - w(q)]\, dq$, an increasing function. Society problem is

$$\max_{q \geq 0}\ [(R(q) + S(q)) - qw(q)]. \tag{14}$$

Set $f(q, \theta) = (R(q) + \theta S(q)) - qw(q)$, for $\theta = 0$ we have the monopsonist's problem, for $\theta = 1$, we have society's problem, check that $f(\cdot, \cdot)$ is strictly supermodular in q and θ, which means that the monopsonist decreases wages relative to the social optimum, and that it is possible to raise monopsony wages and have the winners compensate the losers.

Examples. Company towns. Suppliers, e.g. of fighter jet engines, who only have one buyer.

4.3. **The Relation to the Derivative Arguments.** Though the argument for the super-modularity and comparative statics result made no use of derivatives, and may therefore feel unfamiliar, it is related to the "derivative of the FOCs" work you did above.

- Let $x^\circ = x + dx$ for some small, positive dx. Dividing both sides of equation (9) by dx yields

$$\frac{f(x+dx,\theta^\circ)-f(x,\theta^\circ)}{dx} \geq \frac{f(x+dx,\theta)-f(x,\theta)}{dx}. \tag{15}$$

- Taking $dx \downarrow 0$ (as you did in your calculus classes) yields

$$\frac{\partial f(x,\theta^\circ)}{\partial x} \geq \frac{\partial f(x,\theta)}{\partial x}, \tag{16}$$

  that is, higher values of the parameter θ shift the marginal net benefit curve upward, hence shift where it crosses 0 to the right.
- The previous can be re-written in terms of the cross-partial derivatives of $f(\cdot,\cdot)$. Let $\theta^\circ = \theta + d\theta$ for a small positive $d\theta$ and then send $d\theta \downarrow 0$ (as in your calculus classes). This yields

$$\frac{\partial f(x,\theta+d\theta)}{\partial x} - \frac{\partial f(x,\theta)}{\partial x} \geq 0 \tag{17}$$

$$\frac{\frac{\partial f(x,\theta+d\theta)}{\partial x} - \frac{\partial f(x,\theta)}{\partial x}}{d\theta} \geq 0 \tag{18}$$

$$\frac{\partial^2 f(x,\theta)}{\partial x \partial \theta} \geq 0. \tag{19}$$

When the function $f(\cdot,\cdot)$ is differentiable, the following result often makes it easy to check for supermodularity.

Super-modularity for differentiable functions. If $f(\cdot,\cdot)$ is twice continuously differentiable and for all (x,θ), $\frac{\partial^2 f(x,\theta)}{\partial x \partial \theta} \geq 0$, then $f(\cdot,\cdot)$ is supermodular.

4.4. **Homework Problems on Supermodular Comparative Statics.**

O. Give the arguments for the following statement. If $f(\cdot,\cdot)$ is submodular and $\theta^\circ > \theta$, then $x^*(\theta^\circ) \leq x^*(\theta)$. Assume that there is at most one optimizing solution at any θ.

Ans. Suppose that $\theta^\circ > \theta$ and that $x^*(\theta^\circ) > x$. Because there is only one optimizer at θ°, we know that $f(x^*(\theta^\circ),\theta^\circ) - f(x,\theta^\circ) > 0$. By sub-modularity

$$f(x^*(\theta^\circ),\theta^\circ) - f(x,\theta^\circ) \leq f(x^*(\theta^\circ),\theta) - f(x,\theta)$$

so that $f(x^*(\theta^\circ),\theta) - f(x,\theta) > 0$. But this means that x cannot be optimal at θ. Since this is true for **any** $x < x^*(\theta^\circ)$, we know that $x^*(\theta) \geq x^*(\theta^\circ)$, or, to put it another way,

$$x^*(\theta^\circ) \leq x^*(\theta).$$

P. A biotech firm spends $x \geq 0$ researching a cure for a rare condition (for example, one covered by the Orphan Drug Act), its expected benefits are $B_1(x)$, the social benefits not capturable by the firm are $B_2(x)$, and both are increasing functions.

1. Show that the optimal x is larger than the one the firm would choose.
   Ans. For $\theta = 0$ and $\theta = 1$, consider the problem

   $$\max_{x \geq 0} \; B_1(x) + \theta B_2(x).$$

   When $\theta = 0$, we have the firms problem, when $\theta = 1$, we have society's problem, and the function $f(x, \theta) = B_1(x) + \theta B_2(x)$ is supermodular in x and θ, hence $x^*(1) \geq x^*(0)$.
2. Show that allowing the firm to capture more of the social benefits (e.g. by giving longer patents or subsidizing the research), governments can increase the x that the firm chooses.
   Ans. For $0 \leq \alpha \leq 1$, let $f(x, \alpha) = B_1(x) + \alpha B_2(x)$. $f(\cdot, \cdot)$ is supermodular in x and α, hence $x^*(\alpha)$ increases as α increases.

Q. One part of the business model of a consulting company is to hire bright young men and women who have finished their undergraduate degrees and to work them long hours for pay that is low relative to the profits they generate for the company. The youngsters are willing to put up with this because the consulting company provides them with a great deal of training and experience, all acquired over the course of the, say, three to five years that it takes for them to burn out, to start to look for a job allowing a better balance of the personal and professional. The value of the training that the consulting company provides is at least partly recouped by the youngsters in the form of higher compensation at their new jobs. Show that the consulting company is probably providing an inefficiently low degree of training.
Ans. Let $B_1(x)$ be the benefits the firm receives, $C(x)$ their cost, and $B_2(x)$ the extra benefits the employees and their future employers receive when they go looking for a job. The function $f(x, \theta) = B_1(x) + \theta B_2(x) - C(x)$ is supermodular in x and θ if $B_2(\cdot)$ is increasing.

R. When one looks at statistics measuring the competence with which firms are run, after adjusting for the industry, one finds a weak effect in favor of firms with female CEO's, and a much stronger effect in favor of larger firms. A good part of this is that well-run firms are the ones that succeed and grow, so when you look at firms presently in existence, the well-run ones are larger. In this problem, you are going to investigate a different advantage of being large, the decreasing average cost aspect of simple inventory systems. Decreasing average costs sometimes go by the name of economies of scale, and economies of scale are a crucial determinant of the horizontal boundary of a firm. In this problem, you will find a power law relating size to costs.

Your firm needs Y units of, say, high grade cutting oil per year. Each time you order, you order an amount Q at an ordering cost of $F + pQ$, where F is the fixed cost of making an order (e.g. you wouldn't want just anybody to be able to write checks on the corporate account and such sytems are costly to implement), and p is the per unit cost of the cutting oil. This means that your yearly cost of ordering is $\frac{Y}{Q} \cdot (F + pQ)$ because $\frac{Y}{Q}$ is the number of orders per year of size Q that you make to fill a need of size Y.

Storing anything is expensive, and the costs include insurance, the opportunity costs of the space it takes up, the costs of keeping track of what you actually have, and so on. We suppose that these stockage costs are s per unit stored. Computerized records and practices like bar-coding have substantially reduced s over the last decades. Thus, when you order Q and draw it down at a rate of Y per year, over the course of the cycle that lasts Q/Y of a year, until you must re-order, you store, on average $Q/2$ units. This incurs a per year cost of $s \cdot \frac{Q}{2}$. Putting this together, the yearly cost of running an inventory system to keep you in cutting oil is

$$C(Y) = \min_{Q} \left[\frac{Y}{Q} \cdot (F + pQ) + s \cdot \frac{Q}{2} \right], \tag{20}$$

and the solution is $Q^*(Y, F, p, s)$.

a. Without actually solving the problem in equation (20), find out whether Q^* depends positively or negatively on the following variables, and explain, in each case, why your answers makes sense: Y; F; p; and s.
   Ans. Done in class.
b. Now explicitly find the optimal tradeoff between fixed costs and storage costs to solve for $Q^*(Y, F, p, s)$ and $C(Y)$.
   Ans. Done in class.
c. Find the marginal cost of an increase in Y. Verify that the average cost, $AC(Y)$, is decreasing and explain how your result about the marginal cost implies that this must be true.
   Ans. Done in class.
d. With the advent and then lowering expenses of computerized inventory and accounting systems, the costs F and s have both been decreasing. Does this increase or decrease the advantage of being large?
   Ans. Done in class.

S. For $x, t \in [1200, 1900]$, let $f(x,t) = xt$. Since $\partial^2 f/\partial x \partial t = 1$, this function has strictly increasing differences, and since $\partial f(x,t)/\partial x > 0$ for all x, t, $x^*(t) \equiv \{1900\}$. Let $g(x,t) = \log(f(x,t)) = \log(x) + \log(y)$ and note that $\partial^2 g/\partial x \partial t = 0$, strictly increasing differences have disappeared, but $\partial g(x,t)/\partial t > 0$ for all x, t. Let $h(x,t) = \log(g(x,t))$, and $\partial^2 h/\partial x \partial t < 0$, strictly increasing differences have become decreasing differences, but $\partial h(x,t)/\partial x > 0$ for all x, t. The problem $\max_{x \in [1200,1900]} h(x,t)$ provide an example of strictly submodular function with a constant $x^*(\cdot)$.
Ans. There is nothing to hand in for this problem. You are supposed to check for yourselves that the derivatives given above are correct.

5. Quasi-supermodularity

Lest you think that everything has become easy, let us consider what happens to a monopolist's supply after the demand curve shifts inwards or outwards by some factor $\theta > 0$. If the demand curve of the monopolist shifts from $p(q)$ to $\theta \cdot p(q)$ where $\theta > 0$, consider the problems

$$V(\theta) = \max_{q} \; \pi(q, \theta) = [q\theta p(q) - wc(q)]. \tag{21}$$

If we knew that $\pi(\cdot, \cdot)$ had increasing differences in q and θ, we would know that outward expansions of the demand curve would increase suppy, but this does not hold here — $Rev(q) = p \cdot p(q)$ increases and then decreases for all reasonable demand functions. However, $\pi(q,\theta)$ is **quasi-supermodular** in q and θ, and it is this that we use in our arguments that the optima are an increasing function of θ.

For $x \in X \subset \mathbb{R}$, $\theta \in \Theta \subset \mathbb{R}$, a function $f : X \times \Theta \to \mathbb{R}$ **quasi-supermodular** if for all $\theta^\circ > \theta$ and all $x^\circ > x$,

$$\text{if} f(x^\circ,\theta) - f(x,\theta) > 0, \text{ then } f(x^\circ,\theta^\circ) - f(x,\theta^\circ) > 0, \text{ and} \tag{22}$$

$$\text{if} f(x^\circ,\theta) - f(x,\theta) \geq 0, \text{ then } f(x^\circ,\theta^\circ) - f(x,\theta^\circ) \geq 0. \tag{23}$$

You should check that every supermodular function is quasi-supermodular. Further, the argument that we gave for supermodular functions having increasing optima actually only used quasi-supermodularity.

T. Show that $\pi(q,\theta)$ given above is quasi-supermodular but not supermodular.
Ans. Pick $\theta' > \theta$ and $q' > q$. For the first part, suppose that

$$[\theta q' p(q') - wc(q')] - [\theta q p(q) - wc(q)] = \theta\,[q'p(q') - qp(q)] - w\,[c(q') - c(q)] > 0.$$

We must show that

$$[\theta' q' p(q') - wc(q')] - [\theta' q p(q) - wc(q)] = \theta'\,[q'p(q') - qp(q)] - w\,[c(q') - c(q)] > 0.$$

The last term in both inequalities is the **same** negative number $-N := -w\,[c(q') - c(q)]$, on the presumption that $c(\cdot)$ is an increasing function. Letting $M = [q'p(q') - qp(q)]$, the first terms in both inequalities are θM and $\theta' M$. For the first inequality to hold, $\theta M > N$ must hold. Since $N > 0$, this means that $M > 0$, which in turn means that $\theta' M > \theta M$. Putting this together,

$$\theta' M - N > 0,$$

as we needed to show.

The arguments for the "$\geq$" inequalities are the same.

U. Give complete arguments for the following.
1. A supermodular function is quasi-supermodular.
Ans. Done in class.
2. If $f(x,\theta)$ is quasi-supermodular and $\varphi(\cdot)$ is an increasing function, then $h(x,\theta) = \varphi(f(x,\theta))$ is quasi-supermodular. [Problem S showed you that this is **not** true for supermodular functions.]
Ans. Pick $\theta' > \theta$ and $x' > x$. Suppose that

$$h(x',\theta) - h(x,\theta) > 0, \text{ that is } h(x',\theta) > h(x,\theta).$$

We must show that $h(x',\theta') - h(x,\theta') > 0$. Because $\varphi(\cdot)$ is an increasing function, we know that

$$f(x',\theta) > f(x,\theta), \text{that is } f(x',\theta)) - f(x,\theta)) > 0.$$

By the quasi-supermodularity of $f(\cdot,\cdot)$,

$$f(x',\theta') > f(x,\theta').$$

Because $\varphi(\cdot)$ is increasing,

$$h(x',\theta') > h(x,\theta'), \text{ which yields } h(x',\theta') - h(x,\theta') > 0.$$

The arguments for "$\geq$" are the same.

3. There are quasi-supermodular functions that are not supermodular.
Ans. Look at the example above with $g = \log(f)$ and $h = \log(g)$ so that $h = \log(\log(f))$ — f is supermodular but h is submodular.
4. If $f(\cdot, \cdot)$ is quasi-supermodular and $\theta^\circ > \theta$, then $x^*(\theta^\circ) \geq x^*(\theta)$. Assume that there is at most one optimizing solution at any θ.
Ans. Done in class.

6. Constrained Maximization and Lagrangeans

For a microeconomist, the central assumption is that people, as decision makers, are doing as well as they can with the knowledge and resources they have. We now turn to incorporating the constrainst on resources. We will discuss constraints on knowledge later in the semester.

6.1. **The General Forms.** The **constrained** optimization problems will come in one of four forms:

- one constraint,

$$V(b) = \max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \text{ subject to } g(x_1, \ldots, x_n) \leq b, \tag{24}$$

- m constraints,

$$V(b_1, \ldots, b_m) = \max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \text{ subject to} \tag{25}$$

$$g_1(x_1, \ldots, x_n) \leq b_1 \tag{26}$$

$$\vdots$$

$$g_m(x_1, \ldots, x_n) \leq b_m \tag{27}$$

- one constraint plus non-negativity,

$$V(b) = \max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \text{ subject to } g(x_1, \ldots, x_n) \leq b \text{ and } x_1, \ldots, x_n \geq 0, \tag{28}$$

- m constraints plus non-negativity,

$$V(b_1, \ldots, b_m) = \max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \text{ subject to} \tag{29}$$

$$g_1(x_1, \ldots, x_n) \leq b_1 \tag{30}$$

$$\vdots$$

$$g_m(x_1, \ldots, x_n) \leq b_m \tag{31}$$

$$x_1, \ldots, x_n \geq 0. \tag{32}$$

6.2. **Solving Constrained Problems.** Solving $\max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n)$ subject $g(x_1, \ldots, x_n) \leq b$ for $\mathbf{x}^*$, will often be found in two steps. First, one writes out the Lagrangean function,

$$L(x_1, \ldots, x_n; \lambda) = f(x_1, \ldots, x_n) + \lambda(b - g(x_1, \ldots, x_n)). \tag{33}$$

Then one solves the system of $n+1$ equations in $n+1$ unknowns given by

$$\frac{\partial L(x_1,\ldots,x_n;\lambda)}{\partial x_1} = 0 \quad (34)$$

$$\frac{\partial L(x_1,\ldots,x_n;\lambda)}{\partial x_2} = 0 \quad (35)$$

$$\vdots \quad = 0 \quad (36)$$

$$\frac{\partial L(x_1,\ldots,x_n;\lambda)}{\partial x_n} = 0 \quad (37)$$

$$\frac{\partial L(x_1,\ldots,x_n;\lambda)}{\partial \lambda} = 0. \quad (38)$$

Since these equations involve first order derivatives, they are called **First Order Conditions (FOCs)**.

When there are m constraints, this becomes $n+m$ equations in $n+m$ unknowns. Incorporating the non-negativity constraints will be covered later.

We will mostly work with functions f and g for which the following is true: if $(\mathbf{x}^*, \lambda^*)$ solves the FOCs, then $g(\mathbf{x}^*) \leq b$, $f(\mathbf{x}^*) \geq f(\mathbf{x}')$ for any $\mathbf{x}'$ satisfying $g(\mathbf{x}') \leq b$, and $\lambda^* = \partial V(b)/\partial b$.

6.3. **Homework Problems on Constrained Optimization.**

V. Let $f(x_1, x_2) = 1,500 - [(x_1 - 100)^2 + (x_2 - 100)^2]$, let $g(x_1, x_2) = x_1 + x_2$ and $b = 40$.

1. Write out the Lagrangean for the problem $\max f(x_1, x_2)$ subject to $g(x_1, x_2) \leq b$.
   Ans. The Lagrangean is
   $$L(x_1, x_2, \lambda) = 1,500 - [(x_1 - 100)^2 + (x_2 - 100)^2] + \lambda(b - (x_1 + x_2).$$
2. Write out the FOCs for the Lagrangean.
   Ans. These are $\partial L/\partial x_1 = 0$, $\partial L/\partial x_2 = 0$ and $\partial L/\partial \lambda = 0$. Specifically,
   $$-[2(x_1 - 100)] - \lambda = 0$$
   $$-[2(x_2 - 100)] - \lambda = 0$$
   $$b - (x_1 + x_2) = 0.$$
3. Solve the FOCs for $\mathbf{x}^*$ and $\partial V(b)/\partial b$ at $b = 40$.
   Ans. The first two equations in the FOCs imply that $x_1 = x_2$, plugging this into the third equation in the FOCs tells us that $x_1^*(b) = b/2$, $x_2^*(b) = b/2$. At $b = 40$, this is $x_1 = x_2 = 20$. As a bonus, we know that $\lambda^*(b) = -2(b/2 - 100) = 50 - b$.

W. Let $f(x, y) = 210 \cdot y - 0.01x$ subject to $g(x, y) \leq 0$ where $g(x, y) = y - \sqrt{x}$.

1. Write out the Lagrangean for the problem $\max f(x, y)$ subject to $g(x, y) \leq 0$.
   Ans. The Lagrangean is
   $$L(x, y, \lambda) = 210 \cdot y - 0.01x + \lambda[0 - (y - \sqrt{x})].$$
2. Write out the FOCs for the Lagrangean.
   Ans. These are $\partial L/\partial x = 0$, $\partial L/\partial y = 0$ and $\partial L/\partial \lambda = 0$. Specifically,
   $$0.01 + \lambda \tfrac{1}{2\sqrt{x}} = 0$$
   $$210 - \lambda y = 0$$
   $$[0 - (y - \sqrt{x})] = 0.$$

3. Solve the FOCs for (x^*, y^*) and $\partial V(b)/\partial b$ at $b = 0$.
   Ans. These yield $x^* = (10,500)^2$, $y^* = 10,500$, and $\lambda^* = \frac{210}{10,500}$ so that $\partial V(0)/\partial b = \frac{210}{10,500}$.

X. Solve $\max_{x_1,x_2 \geq 0} [0.3 \log(x_1) + 0.4 \log(x_2)]$ subject to $11x_1 + 2x_2 \leq 104$ and give the derivative of the value function at $b = 104$.
   Ans. $L(x_1, x_2, \lambda) = [0.3 \log(x_1) + 0.4 \log(x_2)] + \lambda[b - (11x_1 + 2x_2)]$ where $b = 104$. The FOCs are

$$\frac{3}{10x_1} - 11\lambda = 0$$
$$\frac{4}{10x_2} - 2\lambda = 0$$
$$[b - (11x_1 + 2x_2)] = 0.$$

   Solving yields $x_1^*(b) = \frac{3}{77}b$, $x_2^*(b) = \frac{22}{77}b$, and $\lambda^*(b) = \frac{7}{4b}$. Therefore $\partial V(104)/\partial b = \frac{7}{4 \cdot 104} = \frac{7}{108}$.

Y. Solve $\min_{x_1,x_2,x_3 \geq 0} [7x_1^2 + 9x_2^2 + x_3^2]$ subject to $(x_1 + x_2 + x_3) \geq X$ as a function of X and give $\partial V(X)/\partial X$.
   Ans. $L(x_1, x_2, \lambda) = -[7x_1^2 + 9x_2^2 + x_3^2] + \lambda(-X + (x_1 + x_2 + x_3))$. The FOCs yield $14x_1 = 18x_2 = 2x_3$ so that $x_2 = 7/9x_1$ and $x_3 = 7x_1$. Therefore $X = x_1(1+7/9+7)$ so that $x_1^* = 9X/79$, $x_2^* = (7 \cdot X)/79$, and $x_3^* = 63X/79$. Since $\lambda^* = 14x_1^*$, $\partial V(X)/\partial X = (14 \cdot 9X)/79$.