Notes for a Course in Managerial Economics

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Contents

CHAPTER

Preface

This course will cover three major principles crucial to managerial economics in enough detail and in enough contexts that their importance becomes clear. At a low level of detail, these are:

- monotone comparative statics — if the marginal net reward to some activity goes up (resp. down), then the optimal level of that activity goes up (resp. down);
- first order conditions[1] — when allocating resources across different activities, the best, most efficient/profitable choices are those involving equalization of marginal net benefits; and
- opportunity cost — most centrally, in almost all of the settings we will look at, the right notion of cost for calculating net benefit is the cost of the best forgone opportunity.

Let us talk through a pair of examples in which these three principles are in play.

- Consider choosing whether or not to invest the upfront costs, C, of a project with rewards, B, that will not accrue for some time, say at time T in the future. If the best alternative use of those funds will grow, say, at a 12% rate, then this gives the opportunity cost of capital. Specifically, one could use C to earn $C \cdot (1.12)^T$ at time T. The appropriate comparison is whether or not B is larger than $C \cdot (1.12)^T$. In terms of value at present, the comparison is between C and $B/(1.12)^T$. The **net present value** of the project is $B/(1.12)^T - C$. That this is the correct criterion for evaluating projects emerges from an analysis of the opportunity cost of capital. There are further questions related to changes in the marginal net reward to the project: What happens as the opportunity cost of capital goes up? Down? What happens as T goes up? Down? Or what happens as B and T become more uncertain? The last question takes us toward the idea of **risk premia**.
- Consider choosing between projects 1 and 2. Project 1 returns a benefit B_1 that depends on the investment made, C_1, which we write as $B_1(C_1)$, and project 2, which returns $B_2(C_2)$. Suppose further that these rewards will accrue at times T_1 and T_2. With a total of C to allocate across the two projects, an optimal allocation, C_1^* and C_2^*, will have the property that the marginal net benefits are equalized — if suppose that an extra dollar given to (say) project 1 makes $[B_1(C_1^*+1) - B_1(C_1^*)] > 0$, and that this is larger than the loss to profits from project 2, $[B_2(C_2^*) - B_2(C_2^* - 1)]$, then shifting the dollar provides a net gain. This is the logic of equalizing

[1]So called because they are conditions on the first derivatives of net benefit functions.

marginal net benefits across projects. Here the opportunity cost of capital spent on project 2 is the benefit it would give if spent on project 1. Again, a further question includes, "What happens as the B_1, B_2, T_1 and T_2 become more uncertain?" and again, risk premia appear.

This course will cover applications of these ideas in numerous contexts. Along the way, we will use a variety of tools from your previous classes, as well as some that we will develop in this class. These tools include derivatives, partial derivatives, means and variances of random variables, the means and variances of functions of random variables, the probability density functions and cumulative distribution functions for random variables, the so-called "hazard rate" of a distribution, and the ideas of "supermodularity" and "mechanism design." A short list of the applications we will cover includes the following.

- The introduction of new products — the main difference betweeen present-day capitalism and the capitalism that existed before roughly 1870, is the speed at which new products are introduced to the market. The timing of product launch decisions is influenced by the opportunity cost of capital, and by the possibility of being 'scooped' by the competition. Here the tools are: derivatives for characterizing the optimal time; the examination of how the opportunity cost of capital and the risk of being scooped affects the optimal timing uses the idea of supermodularity; an examination of the importance of the timing decision when the measurement of benefits is not precise uses the mean and variance of the associated profits.
- Transfer pricing across divisions in a firm — this is a way of equalizing the marginal net benefits of the resources allocated to the divisions, and we will, mostly see this as a result of the calculus of optimality, crudely, an implication of setting a derivative equal to 0. This is a direct use of the principle of the efficiency gains in markets, on that arises by equalizing marginal net benefits across different uses of resources. We will look at this also as part of the general equilibrium insights about the coordination of efforts by prices.
- Economies of scale and of scope — the boundaries of a firm are, in good part, determined by economies of scale and economies of scope, the cost advantages/disadvantages of being large and of producing many different products. The simplest of inventory systems have the property that the cost function follows a power law with a power less than 1. This means that running an inventory system to meet, say, twice as large a flow costs less than twice as much (the savings becomes larger when we add uncertainty to the picture). This gives a cost advantage to larger firms, it is called an economy of scale, and economies of scale are part of why successful firms are large. Economies of scope accur when there are "synergies" in production or marketing. The simplest kind of example arises when the production one good yields, as a side effect, something beneficial in the production of a second good. The classical example is bee-keepers provide both honey and pollination services, a side effect of using your bees to provide pollination services is that your cost of inputs for honey goes down.

- There is a strong complimentarity between the delegation of authority and incentives, which is the main theme in Milgrom and Robert's textbook for a managerial economics course [**12**]. We will get at this through supermodularity, and this is where mechanism design for compensation schemes shows up, something that we will get at using random variables and hidden information. The observation that managers often consult with employees on both what should be measured for future bonuses and what levels should trigger bonuses, this is clear indication that one is not looking at anything like a zero-sum situation, and that's a very useful kind of insight/general habit of thought.
- We will take a systematic look at the value of information and at the value/danger of leverage. Both of these topics lean heavily on comparing the means of different functions of random variables.

Outline.

Chapter I. Review and comparative statics.
- A. First and second derivatives for optima.
- B. Demand functions, cost functions, elasticities.
- C. Economic models of behavior.
- D. Supermodularity and calculus analyses of the models.

Chapter II. Discounting and the opportunity cost of capital.
- A. Summation and integration results.
- B. Up front costs, backloaded benefits.
- C. Continuous discounting.
- D. Risk premia.

Chapter III. Decisions under uncertainty and the value of information.
- A. $E\,u(X)$ theory.
  1. Risk aversion and concavity.
  2. Demand for insurance.
  3. Portfolio choice.
- B. Leverage, ownership structures, and bankruptcy laws.
- C. Forecasts, their value, and selection biases.

Chapter IV. Game theory and firm behavior.
- A. The advantage of being small.
- B. The power of commitment.
- C. Cournot, Bertrand, and Stackelberg competition.
- D. Repeated games.

Chapter V. Mechanism design.
- A. Moral hazard.
- B. Adverse selection (and selection biases).
- C. Designing employment contracts.

Topics that I would like to cover but am unlikely to have time for: arbitrage, portfolio theory, and financial markets; bargaining solutions and the effect of better/worse outside options;

Possible sources: Besanko et al. for talky rather than analytic coverage of examples of complementarity, substitutability, scale, scope, strategic interactions; Milgrom and Roberts [**12**]; Mazzucato's *Entrepreneurial State* [**11**]; Pisano and Shih's *Producing Prosperity* [**14**].

Possible papers: Conlisk's Costly Predation, to do it fully and easily requires some matrix analysis and the idea of Lyapunov stability; Milgrom's unraveling, to cover it fully requires some iterative deletion of strategies coverage; Brandenburger and Nalebuff on Coopetition and Raiffa's much antecedent idea of cooperative antagonists; Dybvig and Ross [**7**] survey article on arbitrage, state prices, and portfolio theory.

CHAPTER

Notation

N.1 For sets A, B: "$A \subset B$" is "A is a subset of B" or "anything that's in A is also in B;" "$A \supset B$" is "A is a superset of B" or "A contains the set B," or "$B \subset A$," or "anything that's in B is also in A."

N.2 For sets A, B: "$A \cap B$" is the intersection of A and B, that is, the set of all things that are in both A and B; "$A \cup B$" is the union of A and B, that is, the set of all things that are either in A, or in B, or in both A and B.

N.3 It may seem weird, but it is very useful to have a "set" that contains nothing, it is denoted "$\emptyset$" and called "the empty set." For example, if A is the set of even integers and B is the set of odd integers, than $A \cap B = \emptyset$.

N.4 $\mathbb{N}$ is the set of integers, $\{1, 2, 3, \ldots\}$ where "$\{\ \}$" indicates that the stuff between the braces is a set, and the "..." means carry on in the indicated fashion.

N.5 $\mathbb{R}$ is the set of real (royal) numbers, that is, the numbers you used in your calculus class, the numbers on the axes of graphs you've drawn.

N.6 The large arrow, "$\Rightarrow$" will often appear between statements, say $\mathbb{A}$ and $\mathbb{B}$. We read "$\mathbb{A} \Rightarrow \mathbb{B}$" as "$\mathbb{A}$ implies $\mathbb{B}$," equivalently for our purposes, as "whenever $\mathbb{A}$ is true, $\mathbb{B}$ is true." This last makes this relationship between statements sound something like $A \subset B$ for A and B corresponding to the times when $\mathbb{A}$ and $\mathbb{B}$ are true. This is not an accident.

N.7 The three letters, "iff" stand for the four words, "if and only if." Throughout, there are statements of the form "$\mathbb{A}$ iff $\mathbb{B}$," these mean that statement $\mathbb{A}$ is true when and only when $\mathbb{B}$ is true. For a really trivial example, think of "we left an hour later" iff "we left 60 minutes later."

N.8 We superscript variables with an asterix to denote their optimal value. For example, for the problem $\max_{x>0}(x - x^2)$ (which you should read as find the value of $x > 0$ that makes $(x - x^2)$ as large as possible), we have $x^* = 1/2$. Often, the solutions will depend on some aspect of the problem, supposing that $p > 0$, consider the the problem $\max_{x>0}(px - x^2)$, the solution is $x^*(p) = p/2$. Taking $p = 1$ recovers the problem $\max_{x>0}(x - x^2)$ and $x^*(1) = 1/2$.

N.9 The upside-down capital A symbol, "$\forall$," is read as "for All," is in "$\forall x \neq 0,\ x^2 > 0$," which makes the true statement that "for all non-zero numbers, the square of that number is strictly positive."

N.10 The backwards capital F symbol, "$\exists$," is read as "there exists," as in the "$\exists x \neq 0, x^2 < -3$, which makes the false statment that "there exists a non-zero number whose square is strictly smaller than -3."

CHAPTER I

Decision Theory

A. Introduction

Economists assume that people are rational, understanding rationality as internal consistency of choices, if my choices reveal that I like option x better than option y, consistency requires that I not choose y when x is available in some other choice situation. This kind of consistency is equivalent to **maximizing a utility function** and equivalent to people doing the best they can for themselves according to their own interests. This assumption of maximizing, goal-oriented behavior has a *descriptive* and a *prescriptive* side to it: if we have a good description of the interests and concerns of the people in a situation, we have a chance of figuring out what they will do, and what they will do if the situation changes; an understanding of what rationality looks like allows us to figure out what we need to know in order to make a good decision in a complicated situation, and what we might want to change to make things work out better.

Often, especially but not only in the study of firms, we take the utility to be accurately measured by profits or money. This is a separate assumption according well with intuition and evidence in many cases, but it is not always valid, and we will be crucially interested in the many situations and ways in which it fails to be valid. Trying to figure out what rational people are doing passes through the latin question, "*Cui bono*?" or "Who benefits?" With the assumption that the people understand their benefits in monetary terms, this can be reduced to the advice from a famous movie, "Follow the money."

Both varieties of this kind of analysis provides insight into a wide variety of situations.

A.1. Some Examples. In all of these examples, one can see motivations by looking at the payoffs, and finding these motivations gives us a descriptive understanding of what is going on that is sufficiently causal in its nature that we are willing to undertake a prescriptive analysis. There will be a distinction between insights that are inherently **decision theoretic**, that is, inherently about the interests of a single decision maker making choices that affect only her/his interests, and **strategic** or **game theoretic**, that is, about how decisions are chosen in situations where one person's choices affect other's incentives, hence influence their decisions.

A.1.1. *The green revolution.* When fertilizers and hybrid crops first gave the promise of large yield increases in poor parts of the world, many farmers resisted. One could understand this as ignorance and/or superstition, that is, as irrational responses. Alternately, one could start from the assumption that they had their own good reasons for what the were doing and then ask what those reasons are. This "assume rationality" approach was useful, the "assume irrationality" approach made the analysts feel superior, but got them no further. The new agricultural practices involved single-crop cultivation, poor farmers and/or members of their family die if they have planted a single crop and it fails.

Descriptively, we can understand from this that their adherence to multiple crop strategies were a form of diversification in the face of risk, a thoroughly rational decision. Prescriptively, we can guess that some form of insurance or other form of risk pooling is needed in order to get people to try riskier options with higher expected returns.

A.1.2. *Bed frames.* In the former Soviet Union, bed frame factory managers were given bonuses as a function of the number of tons of bedframes they produced. Soon bedframes required 5 strong men to move them.

Descriptively, if you understand what managers are maximizing, you can understand their behavior. Prescriptively, one would like the managers to be faced with consequences in line with the values of consumers, of those who buy and use the bedframes. If the market is working, it provides this kind of feedback to firm managers.

A.1.3. *Bond trading firms.* By the mid 1980's, the Salomon Brothers company had developed the idea of packaging mortgages into securities and was actively marketing them. This was part of the huge increase in the size, volatility, and profitability in the bond market. Until 1990, the pay system at Salomon Brothers consisted of a base salary plus a bonus, and most (commonly 2/3) of the yearly money for employees came in the bonus. The bonus system priced almost all transactions, charged for overhead and risk, and credited the profits to the individuals and departments involved, essentially a piece rate system.

Descriptively, one can understand that the upshot of so self-centered a reward system was self-centered behavior, a lack of cooperation between departments that went so far as not even passing on useful information to non-competing departments within the firm. Prescriptively, after May 1990, to solve these kinds of problems, the value of the bonus was tied to the overall market value of the entire firm 5 years in the future. This was arranged by having substantial parts of one's pay in the form of restricted stock in the company, the restriction being that the stock could not be sold for 5 years.

A.1.4. *Big pharma.* Most R&D (research and development) done by the large pharmaceutical firms involves finding minor tweaks on molecules, tweaks that are large enough to be patentable, hence granting the firm another many years of monopoly, but small enough that previous research on efficacy and dangers need not be re-done. One frequent example involves reformulations of the molecule that involve needing a new dosage, e.g. because of different rates of metabolic uptake. Another constraint that the large pharmaceutical companies must pay attention to is that their research targets must be found quickly, they must be small enough that the internal rate of return on the research project be financially rational. The research involved in finding new therapeutic molecules is, for the most part, far too expensive, long term, and chancy for any private firm or venture capitalist to undertake. It is governments that undertake to subsidize most of that kind of research, often in the form of the more politically salable form of tax "breaks," for example in the Orphan Drugs Act.[1] The large pharamaceutical firms have attracted huge amounts of capital and been paid by governments for the riskier research projects, but, have, for their history so far, not been otherwise profitable.

Descriptively, if you understand what kind of inventions are most profitable, and profits are revenues minus costs, you will understand where firms direct their research. Prescriptively, one can see what aspects of patent law and government funding of research need to be changed in order to direct research into less wasteful channels.

A.1.5. *Letting people profit from their own labors.* Through the mid to late 1800's, married women in different states in the US began to first acquire the right to sign legally binding contracts. Until that time, only their husbands had the right to sign contracts for their wives. Descriptively, as the right was acquired and they could profit from their own efforts, women began, among other activities, to patent more inventions [**8**]. Prescriptively, allowing people to control the fruits of their own labors leads to more innovation.

In 1349, after years of poor crops and famines, the Black Death killed of almost 50% of the European population. With the ratio of land to labor so drastically increased, yeomen were able to purchase land for themselves and enclose it, ending the feudal open field system. With control of the output of their own labor, after about two centuries of decreasing output per laborer, output per laborer began to grow, and continued to grow strongly, for the next century and a half. Descriptively, allowing people to benefit from their own labor encourages work and creativity (in the U.S. Declaration of Independence, the pursuit of life, liberty and happiness are "inalienable rights"). Prescriptively, delegation of authority, you make decisions for your own land, and incentives, you benefit from what comes from your land, are mutually reinforcing across a huge range of situations.

Between 1809 and 1821, Lachlan Macquarie broke the monopoly that the New South Wales Corps had on "rum," often used as the medium of exchange, allowed emancipists with valuable skills government employment, even as magistrates. To help make the colony food self-sufficient, he offered other emancipists 30 acre land grants. Descriptively, providing people with a motivation to serve their sentence and come out to a better, free life, reduced prison problems and the labors of the former convicts started Australia on its way its way to being a colony, a nation.[2] Prescriptively, we see again how productive it can be to align authority, over one's own life, and incentives, the right to benefit from your own labor.

Slavery is the flip side of these examples, but we see a flip side in the much less extreme form of discrimination. The old rule of "four factors of two," a woman needs to do twice as much twice as well in half the time as a man to be counted as half so good, is less close to true than it used to be. Someone who is discriminated against will often find smaller rewards to any efforts they put in, sometimes this is mitigated by their furious striving to be even better, a rather unfair burden. Two examples with widely applicable prescriptive lessons, one economic and one academic: after adjusting for differences in the size of the firm and the industry in

[1]The Orphan Drug Act has been tremendously effective, Lazonick and Tulum [**10**] calculate that the share of orphan drugs as a percentage of total product revenues for the six leading biopharmaceutical firms is 60% of the product revenues.

[2]As he wrote, "I found New South Wales a gaol and left it a colony. I found a population of idle prisoners, paupers and paid officials, and left a large free community in the prosperity of flocks and the labour of convicts."

which they are, firms with female CEOs tend to be slightly more profitable; Lucien LeCam built one of the world's best statistics department at U.C. Berkeley in good part by recognizing and hiring talent in people few others would look at because of racial and gender reasons.

A.1.6. *Limited liability and the right of corporations to be sued.* The Industrial Revolution is usually dated as starting in the mid-1700's in England and running for 60 or 70 years. The Second Industrial Revolution, also known as the Technological one, is often dated as starting in the mid-1800's and running through to WWI. Until someplace around the beginning of the middle of the First Industrial revolution, firms were typically owned by one person, or by a small number of people. If the firm failed to live up to its obligations, the owner(s) were sued. The industrial revolution led to firms needing more capital than a single owner or small number of owners could provide. However, if a rich person invested in a firm and the firm went bankrupt, the debtors had the most incentive to go after the person who could best make good their losses.

Descriptively, this explains the reticence of people with wealth to invest and led to investments being pooled by unincorporated associations with so many members, thousands sometimes, that suing them was nearly useless. However, if an organization cannot be held to blame when it misbehaves, its incentives to behave well are weakened. Prescriptively, it explains that laws needed to be changed in order to have capital move to where it might be most productive.

The Joint Stock Act of 1844 (in Britain) allowed for the formation of joint-stock companies that were, for the first time, *not* government granted monopolies, the Limited Liability Act of 1855 meant that one could lose up to and including one's entire investment in a firm, but no more. When combined with notions of corporate personhood, especially the right to be sued (present already in Roman and in ancient Indian law), it meant that one could trust a corporation insofar as misbehaving brought negative consequences. This is an inherently strategic insight rather than a decision theoretic one: letting others have the right to hurt me can be good for me because that is what makes them willing to do business with me.[3]

A.1.7. *The Hudson Bay Company.* From the 1500's on, the European governments "outsourced" the exploration, settling, and exploitation of their international empires by granting monopolies. For example, the Dutch, the French, and the British India companies were supported in a variety of ways, naval power for example, and, in theory at least, they were protected from domestic competition. Chartered by King Charles II of England, the Hudson Bay Company had a royal grant of monopoly to all trade in lands draining into the Hudson Bay, a geographical expanse larger than much of Europe. The company traded only from a few trading posts on the shores of the Hudson Bay, salaries of employees were independent of how well the trading posts did, they were punished for any violations of rules and procedures decided upon in Britain by people who had never been to the area. As a result, the tribes near the bay quickly become the intermediaries between the upriver sources and the Hudson Bay Co.

Descriptively, this explains why the nearby tribes so fiercely defended their own monopolies, the double monopoly problem explains the inefficiencies in the trade patterns that arose, and explains why the employees had no incentive either to see these problems or to try to fix it. It also explains how the North West Company,

[3]This is a life lesson as well as a lesson about conduct in business situations.

sourcing from the areas beyond the reach of the Hudson Bay Co. and thereby operating at an absolutely massive technological disadvantage, could, by aligning their employees incentives with their own profitability, drive the Hudson Bay Co. nearly to bankruptcy (the value of the company dropped by approximately 3/4 over the period of two decades). Prescriptively, in 1809, the Hudson Bay Co. built trading posts inland, allocated half of profits to officers in the field, and increased the independence of action and incentives of other employees, reversing the decline and shifting the advantage so decisively to the Hudson Bay Co. that it soon acquired its pesky competitor.

A.1.8. *Stock buy backs.* Since the 1980's and 90's, many of the large corporations in the US have amassed large cash reserves and systematically used them to manipulate their stock prices in fashions designed to maximize the payoffs of managers with contractual rewards dependent on stock prices. This is not a small issue, in 1970, about 1% of executive compensation came in the form of stocks or stock options, by 2000, about 50% of executive compensation came in this form. This part of executive compensation comes in many forms: stock options, the option to buy a at a date of the executive's choosing at the stock market price on that day, a class of rewards that led to widespread back-dating scandals (at the end of the year or quarter, the executives would look back at the stock prices and specify which date in the past they wanted to be named as the date at which they exercised their option, giving a huge incentive to create artificial dips in the prices); restricted stock awards, a set of gifts or heavily subsidized shares in a firm with restrictions on when/if they can be sold often in the form of hitting growth, profitability, or stock price goals; phantom stock plans, entitling the executive to receive the share price appreciation, never depreciation, and dividends that the executive would have received if he/she had owned the stocks; and the related stock appreciation rights, the right to collect the amount of share price appreciation on some specified amount of stock.

One of the design questions for such contracts is how to set the various goals that will trigger the executive's payoffs, and it is here that "analysts" enter the picture. In the best of all possible worlds,[4] analysts would perform an important function, gathering statistical and other information about the performance of firms in a fashion useful to investors. Analysts' accuracy rates are shockingly high, no statistician can be that good. Consultations with executives or people close to the executives have been a good explanation for this accuracy, especially when the executives can use their cash reserves to drive the prices up by stock re-purchases.

There are potentially innocent explanations for and potential benefits from stock re-purchases. For example, if the executives know that the stock price undervalues the firm, they can make money for the firm by purchasing undervalued assets and holding them until the price again reflects some true underlying value. Such behavior would not systematically result in purchases being concentrated at times when the stock price is high [**9**, See esp. pp. 908-9].

Descriptively, this set of managerial incentives helps explain why corporations are no longer as good a source of middle-class employment and careers in the United States as they used to be, managerial compensation is more tied to managing the firm's stock prices than to managing the firm. Prescriptively, it suggests that management contracts need to be re-thought and re-written, perhaps in terms of

[4] I would refer to Voltaire rather than to Leibniz on this matter.

customers created and the possibilities of creating future value through creating future customers.[5] One hope is that the satisfaction of getting away from the inherent dishonesties of the present system may make the alternatives attractive.

A.2. Overview of the Topics in this Chapter. This chapter covers the **maximization of utility functions**, first as a representation of internally consistent behavior, and later as a somewhat more stringent understanding of rationality in the face of uncertainty.

With utility maximization in place, the chapter spends a good deal of time on **comparative statics**. In all of the examples given above and all of the situations discussed in this book, once one understands what incentives are at work, one can begin to see how behavior might change if the incentives change. The general study of these kinds of changes is called "comparative statics." The 'statics' part is the assumption that the situation is well enough understood by the people making the decisions that they are maximizing, that they have figured out what the best decision actually is. The 'comparative' part is the comparison between what their maximizing choices are before and after the changes.

Uncertainty and decisions are often dynamic, one does not learn something important for a present decision until later. The next section covers **decision trees**, a method of modeling complicated decisions by breaking them down into constituent parts in a fashion that makes their solution easier. The last section in this chapter applies decision trees to a sub-class of **dynamic games**, investigating issues of commitment such as the advantage of having the right to be sued.

The study of multiple player decision problems, ones in which one person's decisions affect the well-being (utility) of others, usually in a fashion that changes their optimal choices, bears the somewhat unfortunate name **game theory**. Because person i's optimal choice may depend on what person j chooses, we cannot simply study optimal choices, the very idea is not sensible for an individual considered in isolation. Instead, we study **equilibria**. These are a list of choices, one for each person in the situation, that are optimal for each one *given* what everyone else is choosing.

We will spend much of the semester either implicitly or explicitly talking about the insights into firms' optimal behavior in game theoretic terms. A major sub-class of the 'games' that we will talk are called 'games of perfect information.' This involve people knowing what others have previously chosen when they come to take their own decision, and knowing that anyone choosing after them will know what they have chosen.

A.3. Some Themes Underlying the Topics. To end this introductory section, I would like to recall a wonderful quote from Winston S. Churchill, just after he had been turned out of office by the voters, "Democracy is the worst form of government, except for all the others." Throughout this set of notes, there are a couple of underlying themes related to this quote.

◦ Markets regularly misfunction, but generally, less than other forms of economic organization misfunction. Many bad market outcomes are directly due to the behavior of the people in the market, as Adam Smith's words remind us, "People of the same trade seldom meet together, even for merriment and diversion, but the

[5]Such changes may be difficult to effect, as Upton Sinclair noted, "It is difficult to get a man to understand something, when his salary depends upon his not understanding it."

conversation ends in a conspiracy against the public, or in some contrivance to raise prices."

- More than a century ago, Max Weber wrote that "Bureaucratic administration means fundamentally domination through knowledge." Bureaucracies have taken over the world, they have problems, but they are better, for almost anything complicated, than every other form of organization we have found.

A.4. Degrees of Difficulty. Some sections, subsections, or subsubsections are marked with a "**Adv**" or a "**XAdv**." These indicate that the material is advanced or extra advanced, requiring at least fair exposure to multi-variate calculus to be fully understood. Even without the requisite background, it may be worthwhile skimming the material to see what is being discussed and how it is being discussed.

B. Utility Maximizing Representations of Consistent Behavior

B.1. Summary. We consider the following class of choice situations: there is a set of options X; a budget set, B, is a non-empty set of feasible options, that is, $B \subset X$. The basic behavioral rule in economics has a simple form: from any budget set B that the decision maker is faced with, they choose that option that is best for their interests, interests being broadly conceived. The foundational result in this field runs as follows: provided choices across different budget sets B are internally consistent, this behavior is the same as the behavior that results from maximizing a utility function.

A little bit more carefully now. Suppose that we observe a person's choice behavior, that is, suppose that we know what they chose when face with each B, that is, suppose we know the choices, denoted $C^*(B)$. From these choices, for each pair of options, $x, y \in X$, we define x being **revealed preferred** to y, denoted $x(RP)y$, if there is a budget set, B, with two properties: first, it contains both x and y; second, that x is in $C^*(B)$ while y is not in $C^*(B)$. The internal consistency condition on choices is that if $x(RP)y$, then it is not the case that $y(RP)x$. Put another way,

if in some choice situation where you had the option to pick x or y, you picked x but you did not pick y,
then there is no choice situation/budget set containing both x and y in which you pick y and not x.

Introductory economics talks about utility maximizing choices. A utility function gives you the number, $u(x)$, the 'utility' or 'happiness' associated with x, for each option $x \in X$. The utility maximizing choice set for this utility fuction is $C^*_u(B) = \{x \in B : \text{ for all } y \text{ in B } u(x) \geq u(y)\}$. This corresponds to picking the utility maximizing choice(s) from whatever set the decision maker is faced with.

The result is, with a couple of details here supressed, that a choice rule $C^*(\cdot)$ satisfies the internal consistency condition if and only if there is a utility function, u, with the property that for every budget set, $C^*(B) = C^*_u(B)$.

Again, internally consistent behavior is the same as utility maximizing behavior, this is the most basic form of what we call **rational choice theory**. Now, there are a number of additional assumptions that will come into play and these assumptions can be objectionable or even wrong. Mostly these assumptions concern what it is that drives utility up or down, what are the things that motivate the decision maker.

To make the arguments precise without a great deal of mathematics, we are going to only cover these results for finite sets of options.[6] For those looking to go to advanced study in economics, the most relevant fields of mathematics for this kind of material are called "real analysis" and "functional analysis," this last to be distinguished from the sub-field of the same name in sociology.

B.2. Preference Relations$^{\mathbf{Adv}}$. Preference relations on a set of options are at the core of economic theory. A decision maker's preferences are encoded in a preference relation, $\succsim$, and "$a \succsim b$" is interpreted as "a is at least as good as b for the decision maker." It is critically important to keep clear that preference relations are assumed to be a property of the individual, perhaps even a defining property. I am sure that your $\succsim$ is different than mine. When the decision maker being studied is a firm, we will often presume that $\succsim$ reflects a preference for profits, that is, that $a \succsim b$ is the same as "a is more profitable than b."

The two results here, Theorems I.1 and I.4, are the foundational results in the theory of *rational choice*: utility maximization is equivalent to preference maximization for complete and transitive preferences; and preference maximizing behavior, equivalently, utility maximizing behavior, is equivalent to a choice rule satisfying the *weak axiom of revealed preference*, a very minimal assumption on the internal consistency of behavior across different choice problems. Theorems I.2 and I.3 show that rational choice theory is not a mathematically empty theory and gives the first, and most basic comparative result for optimal choice sets.

B.2.1. *Rational Preference Relations.* Let X be a finite set of options. We want to define the properties a relation $\succsim$ on X should have in order to represent preferences that are rational. $X \times X$ denotes the class of all ordered pairs, (x, y), with $x \in X$ and $y \in X$. The idea is that the left-to-right order in which we write (x, y) means that x is first and y is second. Thus, $(x, y) \neq (y, x)$ unless $x = y$. A relation is defined as a subset of $X \times X$, and we write $x \succsim y$ for the more cumbersome $(x, y) \in \succsim$.

Definition I.1. *A relation $\succsim$ on X is* ***complete*** *if for all $x, y \in X$, $x \succsim y$ or $y \succsim x$ (or perhaps both), it is* ***transitive*** *if for all $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies that $x \succsim z$, and it is* ***rational*** *if it is both complete and transitive.*

Relations need not be complete, and they need not be transitive.

Example I.1. *One of the crucial order properties of the set of numbers, $\mathbb{R}$, is the property that $\leq$ and $\geq$ are both complete and transitive. The relation $\neq$ is not complete and it is not transitive: letting x and y both be the number 7, we have neither $x \neq y$ or $y \neq x$ because $7 \neq 7$ and $7 \neq 7$; $4 \neq 5$ and $5 \neq 4$ but it is not the case that $4 \neq 4$. The relation $>$ is not complete, but it is transitive: it is not the case that $7 > 7$; but for any three numbers, x, y, z, if $x > y$ and $y > z$, then $x > z$.*

They need be complete or transitive, however, to sensibly describe behavior using the idea of preference maximization, they must be both.

In thinking about preference relations, completeness is the requirement that any pair of choices can be compared for the purposes of making a choice. Given how much effort it is to make life decisions (jobs, marriage, kids), completeness is a strong requirement (think of William Styron's novel *Sophie's Choice*). If a

[6]From Leopold Kronecker, "*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*," roughly, "God made the integers, all else is the work of man."

preference relation is not complete, then faced with a choice between x and y, no decision based on picking the most prefered option can be made and some other kind of explanation of behavior must be found for what people are choosing to do.

There is another aspect to failures of completeness. When a relation is not complete, there are choices that cannot be compared, the previous problem, and there may be two or more optimal but non-comparable choices in the set.

Example I.2. *Consider the relation $\supset$ on the set of all subsets of $A = \{1, \ldots, 10\}$ except the full set, except A itself. Suppose we are looking for the largest or a largest subset in this ordering. Each of the subsets with 9 elements is a largest element and they cannot be compared with each other.*

Transitivity is another rationality requirement. If violated, vicious cycles could arise among three or more options — any choice would have another that strictly beats it. To say "strictly beats" we need the following.

Definition I.2. *Given a relation $\succsim$, we write "$x \succ y$" for "$x \succsim y$ and it is not the case that $y \succsim x$," and we write "$x \sim y$" for "$x \succsim y$ and $y \succsim x$."*

When talking about preference relations, "$x \succ y$" or "$y \prec x$" is read as "x is strictly preferred to y" and "$x \sim y$" is read as "x is indifferent to y."

Example I.3. *Suppose you're at a restaurant and you have the choice between four meals, Pork, Beef, Chicken, or Fish, each of which costs the same. Suppose that your preferences, $\succsim$, and strict preferences, $\succ$, are given by*

pork	$\otimes$			$\otimes$
beef			$\otimes$	$\otimes$
fish		$\otimes$	$\otimes$	$\otimes$
chic	$\otimes$	$\otimes$	$\otimes$	
	chic	*fish*	*beef*	*pork*

$\succsim$

pork	$\otimes$			
beef				$\otimes$
fish			$\otimes$	$\otimes$
chic		$\otimes$	$\otimes$	
	chic	*fish*	*beef*	*pork*

$\succ$

The basic behavioral assumption in economics is that you choose that option that you like best. Here $p \succ b \succ f \succ c \succ p$. Suppose you try to find your favorite meal. Start by thinking about (say) c, discover you like f better so you switch your decision to f, but you like b better, so you switch again, but you like p better so you switch again, but you like c better so you switch again, coming back to where you started. You become confused and starve to death before you make up your mind.

B.2.2. *Utility Functions.* There are other ways to represent preference relations, one can think of them as measuring the happiness associated with different options.

Definition I.3. *A utility function $u : X \to \mathbb{R}$ **represents** $\succsim$ if $x \succ y$ holds when and only when $u(x) > u(y)$ and $x \sim y$ holds when and only when $u(x) = u(y)$.*

Since u is a function, it assigns a numerical value to every point in X. Since we can compare any pair of numbers using $\geq$, any preference represented by a utility function is complete. Since $\geq$ is transitive, any preference represented by a utility function is transitive.

Theorem I.1. *$\succsim$ is rational iff there exists a utility function $u : X \to \mathbb{R}$ that represents $\succsim$.*

Since X is finite, we can replace $\mathbb{R}$ by $\mathbb{N}$ in this result.

PROOF. Suppose that $\succsim$ is rational. We must show that there exists a utility function $u : X \to \mathbb{N}$ that represents $\succsim$. Let $W(x) = \{y \in X : x \succsim y\}$, this is the set of options that x beats or ties. A candidate utility function is $u(x) = \#W(x)$, that is, $u(x)$ is the number of options that x beats or ties. It's pretty clear that higher values of $u(x)$ correspond to better options, but being "pretty clear" is not, in general, a complete argument.

By transitivity, $[x \succsim y] \Rightarrow [W(y) \subset W(x)]$. By completeness, either $W(x) \subset W(y)$ or $W(y) \subset W(x)$, and $W(x) = W(y)$ if $x \sim y$. Also, $[x \succ y]$ implies that $W(y)$ is a proper subset of $W(x)$. Combining, if $x \succ y$, then $u(x) > u(y)$, and if $x \sim y$, then $W(x) = W(y)$ so that $u(x) = u(y)$.

Now suppose that $u : X \to \mathbb{R}$ represents $\succsim$. We must show that $\succsim$ is complete and transitive. For $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ (or both). By the definition of representing, $x \succsim y$ or $y \succsim x$. Suppose now that $x, y, z \in X$, $x \succsim y$, and $y \succsim z$. We must show that $x \succsim z$. We know that $u(x) \geq u(y)$ and $u(y) \geq u(z)$. This imply that $u(x) \geq u(z)$, so that $x \succsim z$. $\square$

B.3. Choice Rules$^{\mathbf{Adv}}$. A choice rule is a function C, taking budgets to what people choose from their budget. For choice functions, we assume that people choose something and that that something is actually available to them: $C(B) \neq \emptyset$ if $B \neq \emptyset$ and $C(B) \subset B$. The interpretation is that $C(B)$ is the set of options that might be chosen from the menu B of options. The best known class of choice rules are of the form $C^*(B) = C^*(B, \succsim) = \{x \in B : \forall y \in B,\ x \succsim y\}$. In light of Theorem I.1, $C^*(B) = \{x \in B : \forall y \in B,\ u(x) \geq u(y)\}$, that is, $C^*(B)$ is the set of utility maximizing elements of B.

The basic existence result tells us that the preference maximizing choice rule yields a non-empty set of choices.

THEOREM I.2. *If B is a non-empty, finite subset of X and $\succsim$ is a rational preference relation on X, then $C^*(B) \neq \emptyset$.*

PROOF. The set of numbers, $\{u(x) : x \in B\}$, is finite. Any finite collection of numbers has a largest element (a mathematician would prove this by induction). $\square$

For $R, S \subset X$, we write $R \succsim S$ if $x \succsim y$ for all $x \in R$ and $y \in S$, and $R \succ S$ if $x \succ y$ for all $x \in R$ and $y \in S$. The basic comparison result for choice theory is that larger sets of options are at least weakly better.

THEOREM I.3. *If $A \subset B$ are non-empty, finite subsets of X, $\succsim$ is a rational preference relation on X, then*
(*a*) $[x, y \in C^*(A)] \Rightarrow [x \sim y]$, *optima are indifferent,*
(*b*) $C^*(B) \succsim C^*(A)$, *larger sets are at least weakly better, and*
(*c*) $[C^*(B) \cap C^*(A) = \emptyset] \Rightarrow [C^*(B) \succ C^*(A)]$, *a larger set is strictly better if it has a disjoint set of optima.*

PROOF. If $u(x)$ and $u(y)$ are both the largest number in $\{u(z) : z \in A\}$, the $u(x) = u(y)$. If $x \in C^*(B)$, then $u(x)$ is the largest number in $\{u(z) : z \in B\}$, and since A is a subset of B, this means that it is at least as large as any number in $\{u(z) : z \in A\}$. For the last one, note that nothing in A can give as high utility as anything in $C^*(B)$. $\square$

B.4. The Weak Axiom of Revealed Preference$^{\mathbf{XAdv}}$. The main result here, only to be covered by those interested in learning to make formal arguments, has a simple outline: choosing x when y is available means that x has been revealed to be better than y; for behavior to be consistent, revealing that x is better than y cannot go hand in hand with revealing the reverse; preference maximizing behavior leads to choices satisfying this consistency requirement; and, any choice behavior with this consistency is the result of preference maximizing behavior.

Now, we start on the more advanced treatment.

We now approach the choice problem starting with a choice rule rather than starting with a preference relation. The question is whether there is anything new or different when we proceed in this direction. The short answer is "No, provided the choice rule satisfies a minimal consistency requirement."

A choice rule C defines a relation, $\succsim^*$, "revealed preferred," defined by $x \succsim^* y$ if $(\exists B \in \mathcal{P}(X))[[x, y \in B] \wedge [x \in C(B)]]$. Note that $\neg[x \succsim^* y]$ is $(\forall B \in \mathcal{P}(X))[\neg[x, y \in B] \vee \neg[x \in C(B)]]$, equivalently, $(\forall B \in \mathcal{P}(X))[[x \in C(B)] \Rightarrow [y \notin B]]$. In words, x is revealed preferred to y if there is a choice situation, B, in which both x and y are available, and x belongs to the choice set.

From the relation $\succsim^*$ we define "revealed strictly preferred," $\succ^*$, as in Definition I.2 (p. 21). It is both a useful exercise in manipulating logic and a useful way to understand a piece of choice theory to explicitly write out two versions what $x \succ^* y$ means:

$$\text{(B.1)} \quad (\exists B_x \in \mathcal{P}(X))[[x, y \in B_x] \wedge [x \in C(B_x)]] \wedge \\ (\forall B \in \mathcal{P}(X))[[y \in C(B)] \Rightarrow [x \notin B]], \text{ equivalently,} \\ (\exists B_x \in \mathcal{P}(X))[[x, y \in B_x] \wedge [x \in C(B_x)] \wedge [y \notin C(B_x)]] \wedge \\ (\forall B \in \mathcal{P}(X))[[y \in C(B)] \Rightarrow [x \notin B]].$$

In words, the last of these says that there is a choice situation where x and y are both available, x is chosen but y is not, and if y is ever chosen, then we know that x was not available.

A set $B \in \mathcal{P}(X)$ reveals a strict preference of y over x, written $y \succ_B x$, if $x, y \in B$, and $y \in C(B)$ but $x \notin C(B)$.

Definition I.4. *A choice rule satisfies the* ***weak axiom of revealed preference*** *if* $[x \succsim^* y] \Rightarrow \neg(\exists B)[y \succ_B x]$.

This is the minimal consistency requirement. Satisfying this requirement means that choosing x when y is available in one situation is not consistent with choosing y but not x in some other situation where they are both available.

Theorem I.4. *If C is a choice rule satisfying the weak axiom, then $\succsim^*$ is rational, and for all $B \subset X$, $C(B) = C^*(B, \succsim^*)$. If $\succsim$ is rational, then $B \mapsto C^*(B, \succsim)$ satisfies the weak axiom, and $\succsim = \succsim^*$.*

Proof. Suppose that C is a choice rule satisfying the weak axiom.

We must first show that $\succsim^*$ is complete and transitive.

Completeness: For all $x, y \in X$, $\{x, y\} \in \mathcal{P}(X)$ is a non-empty set. Therefore $C(\{x, y\}) \neq \varnothing$, so that $x \succsim^* y$ or $y \succsim^* x$.

Transitivity: Suppose that $x \succsim^* y$ and $y \succsim^* z$. We must show that $x \succsim^* z$. For this, it is sufficient to show that $x \in C(\{x, y, z\})$. Because $C(\{x, y, z\})$ is a non-empty subset of $\{x, y, z\}$, we know that there are three cases: $x \in C(\{x, y, z\})$;

$y \in C(\{x,y,z\})$; and $z \in C(\{x,y,z\})$. We must show that each of these cases leads to the conclusion that $x \in C(\{x,y,z\})$.

Case 1: This one is clear.

Case 2: $y \in C(\{x,y,z\})$, the weak axiom, and $x \succsim^* y$ imply that $x \in C(\{x,y,z\})$.

Case 3: $z \in C(\{x,y,z\})$, the weak axiom, and $y \succsim^* z$ imply that $y \in C(\{x,y,z\})$. As we just saw in Case 2, this implies that $x \in C(\{x,y,z\})$.

We now show that for all $B \in \mathcal{P}(X)$, $C(B) = C^*(B, \succsim^*)$. Pick an arbitrary $B \in \mathcal{P}(X)$. It is sufficient to show that $C(B) \subset C^*(B, \succsim^*)$ and $C^*(B, \succsim^*) \subset C(B)$.

Pick an arbitrary $x \in C(B)$. By the definition of $\succsim^*$, for all $y \in B$, $x \succsim^* y$. By the definition of $C^*(\cdot,\cdot)$, this implies that $x \in C^*(B, \succsim^*)$.

Now pick an arbitrary $x \in C^*(B, \succsim^*)$. By the definition of $C^*(\cdot,\cdot)$, this implies that $x \succsim^* y$ for all $y \in B$. By the definition of $\succsim^*$, for each $y \in B$, there is a set B_y such that $x, y \in B_y$ and $x \in C(B_y)$. Because C satisfies the weak axiom, for all $y \in B$, there is no set B_y with the property that $y \succ_{B_y} x$. Since $C(B) \neq \emptyset$, if $x \notin C(B)$, then we would have $y \succ_B x$ for some $y \in B$, a contradiction. Problem B.6 asks you to complete the proof. $\square$

So, What Have we Done?

It is important to note the reach and the limitation of Theorem I.4.

Reach: We did not use X being finite at any point in the proof, so it applies to infinite sets. Second, the proof would go through so long as C is defined on all two and three point sets. This means that we can replace $\mathcal{P}(X)$ with a family of sets $\mathcal{B}$ throughout, provided $\mathcal{B}$ contains all 2 and 3 point sets.

Limitation: In many of the economic situations of interest, the 2 and 3 point sets are not the ones that people are choosing from. For example, the leading case has $\mathcal{B}$ as the class of affordable bundles.

B.5. Exercises.

PROBLEM B.1. *The table below gives a relation $\succsim$. Give the corresponding strict relation $\succ$. If possible, give a utility function that represents $\succsim$, if it is not possible, explain why.*

pork	$\otimes$	$\otimes$		$\otimes$
beef			$\otimes$	$\otimes$
fish	$\otimes$	$\otimes$	$\otimes$	
chic	$\otimes$			
	chic	*fish*	*beef*	*pork*

PROBLEM B.2. *The table below gives a relation $\succsim$. Give the corresponding strict relation $\succ$. If possible, give a utility function that represents $\succsim$, if it is not possible, explain why.*

pork	$\otimes$	$\otimes$	$\otimes$	$\otimes$
beef	$\otimes$	$\otimes$	$\otimes$	$\otimes$
fish	$\otimes$	$\otimes$	$\otimes$	
chic	$\otimes$			
	chic	*fish*	*beef*	*pork*

Problem B.3. *The table below gives a relation $\succsim$. Give the corresponding strict relation $\succ$. If possible, give a utility function that represents $\succsim$, if it is not possible, explain why.*

pork		$\otimes$	$\otimes$	$\otimes$
beef	$\otimes$	$\otimes$	$\otimes$	$\otimes$
fish	$\otimes$	$\otimes$	$\otimes$	
chic	$\otimes$			$\otimes$
	chic	*fish*	*beef*	*pork*

Problem B.4. *The table below gives a relation $\succsim$. Give the corresponding strict relation $\succ$. If possible, give a utility function that represents $\succsim$, if it is not possible, explain why.*

pork	$\otimes$	$\otimes$	$\otimes$	$\otimes$
beef	$\otimes$	$\otimes$	$\otimes$	$\otimes$
fish	$\otimes$	$\otimes$	$\otimes$	$\otimes$
chic	$\otimes$	$\otimes$	$\otimes$	$\otimes$
	chic	*fish*	*beef*	*pork*

Problem B.5. *The table below gives a relation $\succsim$. Give the corresponding strict relation $\succ$. If possible, give a utility function that represents $\succsim$, if it is not possible, explain why.*

pork	$\otimes$	$\otimes$	$\otimes$	
beef	$\otimes$	$\otimes$		$\otimes$
fish	$\otimes$		$\otimes$	$\otimes$
chic		$\otimes$	$\otimes$	$\otimes$
	chic	*fish*	*beef*	*pork*

Problem B.6. *What is left to be proved in Theorem* I.4*? Provide the missing step(s).*

C. Opportunity Cost: Getting the Utility Function Right

C.1. Summary. For decision theory to be of use prescriptively, we must get the the utility function correct, think of the bedframes example, the Hudson Bay Co. example, or the historical property rights of married women example. One of the crucial insights from economics is that opportunity cost, a sometimes subtle, sometimes glaringly obvious concept, is the right one to measure the costs of of different choices. The basic idea is simple, the opportunity cost of using a resource is the forgone benefit of using it for something else. For this to be useful for decisions, that "something else" needs to be the best alternative use.

C.2. Re-Assigning an Employee Temporarily. The I.T. person at a firm keeps everything running smoothly, easily worth the \$10,000/month she is paid. A subdivision in upgrading their computer system, it will take a month, hiring an outside consultant will cost more than \$10,000. The decision is whether or not to move her over for a month? Faulty thinking on this problem involves the argument that the firm will pay her the \$10,000 in either case, hence it is a saving of the outside consultant fees. What will be the cost to the firm if she is taken away from her regular activities? If she is "easily worth" \$10,000 per month, it may be cheaper, perhaps even much cheaper, to hire the outside consultant.

C.3. Allocating Scarce Productive Resources. This will be the first time these notes touch on the theme of the ability of prices to coordinate the productive actions and decisions of many people simultaneously. This is a prime example of the complimentarity between incentives and delegation, and what we are after the right mix of them. The wrong mix provides mis-coordination, and is inefficient in the sense that economists use the word.

C.3.1. *Problem Description.* Suppose that an organization has 4 subdivisions, each subdivision has 3 possible projects, projects $k = 1, \ldots, 12$, project k, if run at proportion α, $0 \leq \alpha \leq 1$, gives benefit αB_k and costs αC_k of a scarce resource. The company has a total of $1,200$ of the scarce resource. We are going to work through how to solve the company's problem of picking the right projects to fund, first by asking what would happen if each division is allocated 300 of the $1,200$ in resources, that is, if each of the four is allocated $\frac{1}{4}$ of the total. To do this, we need the data on the projects' benefits and costs, which is

Division	Project	B_k	C_k	B_k/C_k
I	1	600	100	6
	2	1,400	200	7
	3	1,000	200	5
II	4	500	50	10
	5	750	250	3
	6	1,000	200	5
III	7	900	100	9
	8	3,500	500	7
	9	1,600	400	4
IV	10	800	100	8
	11	1,000	250	4
	12	1,200	400	3

C.3.2. *Partial Solutions.* Suppose first that Division I is given 300 of the resource, it could find all of project $k = 1$ (getting a benefit of 600 and costing 100), half of project $k = 2$ (getting a benefit of $700 = \frac{1}{2} \cdot 1,400$ and costing $100 = \frac{1}{2} \cdot 200$), and half of project $k = 3$ (getting a benefit of $500 = \frac{1}{2} \cdot 1,000$ and costing $100 = \frac{1}{2} \cdot 200$). The total benefit would be $1,800$, but is not optimal for Division I. What is?

Solve the problem

$$\text{(C.1)} \qquad \max_{\alpha_1,\alpha_2,\alpha_3} (\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3) \text{ subject to } 0 \leq \alpha_k \leq 1,\ k = 1, 2, 3,$$
$$\text{and } \ (\alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3) \leq 300.$$

Here is the key observation: one finds the solution by picking off the highest B/C ratios, fully funding those projects, using the leftovers on the project having the next highest B/C ratio.

Solve the corresponding problems for the other divisions, each given $300 = \frac{1}{4} \cdot 1,200$ of the resource to use. The solution gives $2,000 + 1,650 + 2,300 + 1,600 = 7,550$. This funds projects $1, 2$ from Division I, projects $4, 6$, and one fifth of project

5 from Division II, project 7 and two fifths of project 8 from Division III, and project 10 and four fifths of project 11 from Division IV.

C.3.3. *Solving the Entire Problem.* Now solve the problem for the entire four division company: this funds projects $4, 7, 10, 8, 2, 1$, and three quarters of either project 3 or 6, and gives a total benefit of $8,250$, roughly a 9% improvement.

There is a 9% waste in not coordinating the subdivisions. To put it another way, the opportunity cost of giving each division a proportional share of the assets is 9% of possible profits.

There is more information in the solution: if one took a unit of the resource from the divisions, it would be at an opportunity cost of 5, one would only do this if the other use(s) gave more than this cost; if there was more of the scarce resource to allocate to the divisions, the benefit would be 5 per unit, one would do this if the other use(s) of the resource had a lower benefit.

It's fine and easy to give the advice, "Find the optimal coordination." It seems somewhat harder to answer the question, "How?" In the small problem we just gave, one could have probably come to the solution without reading these notes with no more than pencil and paper. In large corporations with multiple divisions producing multiple products for an inter-related set of markets, spread sheets would help, but one would still require information from all of the divisions, and getting that information, in a truthfull and timely fashion might do nothing but waste time. In principle, the solution is easy, it involves getting the prices right, and that is how we are going to talk about it. Advice: keep this principle in mind throughout the course, economists really believe in it.

A simple rule for the center to announce to implement the best, coordinated solution, **without the central organizing office needing to know about the benefits and costs of the different projects**, is

> Fund any project with $B/C > 5$, talk to us about projects with $B/C = 5$, and forget projects with $B/C < 5$.

An alternative formulation of $B/C > 5$ is $B > 5 \cdot C$. Therefore, an alternative formulation of the rule is

> Value the resource at a price $p = 5$ and pick projects to maximize profits, talk to us about projects that break even.

One number, the price p to be paid for the resource, plus the simple and decentralizable rule, "maximize profits," achieves coordination on the scale of the firm.

When the scarce resource is produced by another division within the firm, the price to the divisions using that scarce resource is called a **transfer price**. There are still problems to be solved about figuring out what that transfer price should be, but the answer one should always be looking for is that the price should reflect the opportunity cost of the transferred resource. If there is an outside market for the scarce resource, the appropriate price is pretty clear. If not, there is still work to do. Problem C.1 gives a supply curve crossing demand curve solution to finding the appropriate price for the data given in Table C.3.1.

There is a crucial further observation: whatever the transfer price is, it should be the same for all of the divisions using the resource. If it is not, then transferring it from the low value to the high value use is a pure improvement.

C.3.4. *Diseconomies of Scale.* A **diseconomy of scale** happens when increasing the output requires a more than proportional input of the resources. In this example, this is a result of the optimality of the allocation. We can see this by varying the amount of the scarce resource available, tracing out its decreasing marginal product, and then noting that the inverse of the marginal product is the marginal cost.

Consider now the problem as the amount of the scarce resource varies. We solved the problem at $R = 1,200$, but the same principle applies to the solution at different levels of R. For $0 \leq R < 50$, the marginal benefit, or marginal product, of another unit of the resource is 10, this from investment in Project 4 in Division II; for $50 \leq R < 150$, the marginal benefit/product of another unit of the resource is 9, this from investment in Division III's Project 7; for $150 \leq R < 250$, the marginal benefit/product is 8; and so on, with decreasing marginal benefits from the scarce resource. This decreasing marginal product pattern happens because it is optimal to pick the better projects first, they are the ones that should have priority claims to the resource.

The inverse of the marginal product is the marginal cost: for benefits levels from 0 to 500, the marginal resource cost of a unit of benefit is 1/10; for benefit levels between 500 and 1,400, the marginal resource cost of a unit of benefit is 1/9; and so on, with increasing marginal costs. This is an example of something often called a **diseconomy of scale**, the marginal costs of getting more benefits out of the system keep increasing.

C.4. The Equalization of Marginal Benefits. A key observation: each subdivision must be told the same price. Telling subdivisions different prices means that the firm is losing money, losing money through miscoordination.

Here is another formulation of the optimality of using the same price across divisions: if one division is funding projects with a benefit cost ratio of p, another with a benefit cost ratio of $p' > p$, then switching a unit of the resource from p to p' gains the firm $p' - p$; the only way that such gains are *not* possible is when all of the p's are the same. We will see that when there are many costly inputs, each firm using the same *set* of prices will have the same coordinating effect.

C.4.1. *Simplest Calculus Version.* **Notation Alert**: We superscript variables with an asterix to denote their optimal value. For example, for the problem $\max_{x>0}(x - x^2)$, we have $x^* = \frac{1}{2}$.

You should have seen something like this in intermediate micro: equalization of marginal benefits is a condition for optimality. Here is the simplest calculus formulation of the problem that delivers equality of marginal benefits. You have R of a resource, you are going to devote x of it to project 1 and $R - x$ to project 2, receiving rewards/profits $f(x)$ plus $g(R - x)$. The questions of interest is "What must necessarily be true at an optimal pair $(x^*, (R - x^*))$?" Consider the problem

$$\max_{x \geq 0} \; [f(x) + g(R - x)]. \tag{C.2}$$

The derivative of the above expression is $[f'(x) - g'(R - x)]$. Having the derivative equal to 0 is often taught as a necessary condition for optimality. This is not quite correct, but it is correct provided that the optimum does not happen at $x^* = 0$ or $x^* = R$, the so called **boundary cases**. In any case, setting the derivative equal to 0 delivers

$$f'(x^*) = g'(R - x^*), \tag{C.3}$$

that is, the marginal benefits are equalized in the two potential uses of the resource.

C.4.2. *Other Calculus Versions*$^{\boldsymbol{Adv}}$. You have $X > 0$ of one resource to allocate the K possible uses. Using x_k of the resource for use k generates $f_k(x_k)$ of benefit. Solve the problem

$$\max_{x_1,\ldots,x_K} \sum_{k=1}^{K} f_k(x_k) \text{ subject to } x_k \geq 0,\ k = 1,\ldots,K,\ \sum_{k=1}^{K} x_k \leq X. \tag{C.4}$$

We are going to assume that each $f_k(\cdot)$ is productive, i.e. $f_k'(x_k) > 0$, and we are going to assume that each has decreasing returns to scale, that is, that the marginal product is declining, $f_k''(x_k) < 0$.

There are several methods available for solving this kind of problem.

- Use Lagrangeans, set

$$L(x_1,\ldots,x_K;\lambda) = \sum_{k=1}^{K} f_k(x_k) + \lambda(X - \sum_{k=1}^{K} x_k), \tag{C.5}$$

take the derivatives, $\partial L/\partial x_k$ and $\partial L/\partial \lambda$, and set them equal to 0. If this is possible, it yields $df_k(x_k^*)/dx_k = \lambda^*$ for $k = 1,\ldots,K$. If it is not possible, then we look for solutions with $df_k(x_k^*)/dx_k = \lambda^*$ for for all k with $x_k^* > 0$ and $\partial f_k(0) < \lambda^*$ for all k with $x_k^* = 0$. That is, at the optimum, the marginal benefit is the same for each activity that is being used, and the marginal benefit of starting an unused activity is lower.
- Solve at different prices for the resource, adjust the price until the resource use is X. In more detail, consider the problem

$$\max_{x_1,\ldots,x_K} \sum_{k=1}^{K} f_k(x_k) - p \cdot \sum_{x} x_k. \tag{C.6}$$

At the solution, $df_k(x_k^*)/dx_k = p$ if $x_k^* > 0$ and $df_k(0)/dx_k < p$ if $x_k^* = 0$. Here p is replacing the λ from above. Let $x_k^*(p)$ denote the solution as a function of p. As $p \uparrow$, $x_k^*(p) \downarrow$ because of decreasing marginal returns. Start from a low p and increase until $\sum_x x^*k(p) = X$ and you will have solved the problem.

C.4.3. *More Lessons to be Drawn.* There are many useful versions of the divisions in this type of problem. Here are some.

1. There are K possible projects for saving statistical lives on the highway linking Brisbane to Sydney and you have a fixed budget. The problem is to figure out the best allocation of the budget to the K projects.
2. There are M different states, each having K_m different possible highway projects for saving statistical lives on their highways, optimally allocating the federal dollars to the states is the question.
3. A grocery store has a fixed amount of shelf space to allocate to the display of different goods.
4. A chain of shoe stores has a fixed amount of hyper-cool Italian boots to allocate across its different locations.

C.4.4. *Words of Caution.* One can go too far, become too enthusiastic about this kind of analysis.

> We should be careful to get out of an experience only the wisdom that is in it and stop there lest we be like the cat that sits down on a hot stove lid. She will never sit down on a hot stove lid

again and that is well but also she will never sit down on a cold one anymore. (Mark Twain)

Return on Assets (ROA) is the ratio of returns per unit of the asset used. In the problem we analyzed above, it is $8,250/1,200 = 6.875$. If we wanted the highest ROA, we would look at project 4 in Division II, its ROA is 10, an investment of 50 yields 500, but it is very very far from optimal to only run that project even though it does maximize the ROA. Optimality depends on getting the *marginal* ROA correct.

C.5. On Some Accounting Measures. The following is an extensive quote from Steve Dennings *Forbes* article from Nov. 2011 covering a talk given by Clayton Christensen. In it, he argues that standards accounting measures give the wrong incentives, and that this is one of the crucial drivers of an ongoing destruction of the U.S. economy.

> How whole sectors of the economy are dying:
>
> Christensen retells the story of how Dell progressively lopped off low-value segments of its PC operation to the Taiwan-based firm ASUSTek – the motherboard, the assembly of the computer, the management of the supply chain and finally the design of the computer. In each case Dell accepted the proposal because in each case its profitability improved: its costs declined and its revenues stayed the same. At the end of the process, however, Dell was little more than a brand, while ASUSTeK can – and does – now offer a cheaper, better computer to Best Buy at lower cost.
>
> Christensen also describes the impact of foreign outsourcing on many other companies, including the steel companies, the automakers, the oil companies, the pharmaceuticals, and now even software development. These firms are steadily becoming primarily marketing agencies and brands: they are lopping off the expertise that is needed to make anything anymore. In the process, major segments of the US economy have been lost, in some cases, forever.
>
> Business school thinking is driving this:
>
> Why is this happening? According to Christensen, the phenomenon is being "driven by the pursuit of profit. Thats the causal mechanism for these things. The problem lies with the business schools which are at fault. What we've done in America is to define profitability in terms of percentages. So if you can get the percentage up, it feels like we are more profitable. It causes us to do things to manipulate the percentage. I'll give you a few examples."
>
> "There is a pernicious methodology for calculating the internal rate of return on an investment. It causes you to focus on smaller and smaller wins. Because if you ever use your money for something that doesnt pay off for years, the IRR is so crummy that people who focus on IRR focus their capital on shorter and shorter term wins. There's another one called RONA — rate

of return on net assets. It causes you to reduce the denominatorassetsas Dell did, because the fewer the assets, the higher the RONA. We measure profitability by these ratios. Why do we do it? The finance people have preached this almost like a gospel to the rest of us is that if you describe profitability by a ratio so that you can compare profitability in different industries. It 'neutralizes' the measures so that you can apply them across sectors to every firm."

The thinking is systematically taught in business and followed by Wall Street analysts. Christensen even suggests that in slavishly following such thinking, Wall Street analysts have outsourced their brains.

"They still think they are in charge, but they aren't. They have outsourced their brains without realizing it. Which is a sad thing."

The case of the semi-conductor industry:

How is this working out across the economy? In the semiconductor industry, for instance, there are almost no companies left in America that fabricate their own products besides Intel. Most of them have become "fab-less" semiconductor companies. These companies are even proud of being "fab-less" because their profit as a percent of assets is much higher than at Intel. So they outsource the fabrication of the semi-conductors to Taiwan and China.

Christensen notes that when he visits these these factories, they have nothing to do with cheap labor. Its very sophisticated manufacturing, even though it's (not yet) design technology. The plants cost around 10 billion dollars to build.

Christensen recalls an interesting talk he had with the Morris Chang the chairman and founder of one of the firms, TSMC, who said: "You Americans measure profitability by a ratio. Theres a problem with that. No banks accept deposits denominated in ratios. The way we measure profitability is in 'tons of money.' You use the return on assets ratio if cash is scarce. But if there is actually a lot of cash, then that is causing you to economize on something that is abundant."

Christensen agrees. He believes that the pursuit of profit, as calculated by the ratios like IRR and ROA, is killing innovation and our economy. It is the fundamental thinking drives that decisions that he believes are "just plain wrong."

Can IRR be defended?

A case could be made that it is wrong to blame the analytic tools, IRR and RONA, rather than the way that the tools being used.

Thus when a firm calculates the rate of return on a proposal to outsource manufacturing overseas, it typically does not include:

- ○ The cost of the knowledge that is being lost, possibly forever.
- ○ The cost of being unable to innovate in future, because critical knowledge has been lost.
- ○ The consequent cost of its current business being destroyed by competitors emerging who can make a better product at lower cost.
- ○ The missed opportunity of profits that could be made from innovations based on that knowledge that is being lost.

The calculation of the IRR based on a narrow view of costs and benefits assumes that the firms ongoing business will continue as is, ad infinitum. The narrowly-defined IRR thus misses the costs and benefits of the actions that it is now taking that will systematically destroy the future flow of benefits. The use of IRR with the full costs and benefits included would come closer to revealing the true economic disaster that is unfolding.

C.6. Exercises.

PROBLEM C.1. *Referring to the data in Table C.3.1, give, as a function of the price p charged for the resource each division's demand for the resource. Summing these demands gives the total demand curve for the resource. The supply curve is fixed at $1,200$. Find the intersection of the demand and the supply curves.*

PROBLEM C.2. *You have 12 workers and must decide which 6 of them will work on the six machines of type A and which 6 will work on the six machines of type B. If worker i works on the machine of type A they make profits of $\pi_{i,A} > 0$ for the firm, if on B, they make $\pi_{i,B} > 0$ for the firm. This problem uses the data from the following table. (Note that the last row contains the column averages, e.g. $452.8 \simeq (\sum_{i=1}^{12} \pi_{i,A})/12$.)*

i	$\pi_{i,A}$	$\pi_{i,B}$	$\pi_{i,A} - \pi_{i,B}$
1	361	273	88
2	522	731	-209
3	823	821	2
4	170	762	-592
5	364	957	-593
6	389	832	-443
7	493	296	197
8	293	779	-486
9	845	757	88
10	442	623	-181
11	239	607	-368
12	493	991	-498
Avg.	452.8	701.6	-248.8

a. *Find the assignment of workers to machines that maximizes profit and compare the maximal total profits to the expected profits that would result from a random assignment of workers to machines.*

b. *Suppose the initial assignment is workers* 1 *through* 6 *on machines on type* A *and* 7 *through* 12 *on machines of type* B*. Further suppose that the workers have ownership rights to these assignments and to the profits they produce for the firm by working at these machines. In this part of the problem, I ask you to imagine a market for these ownership rights, to imagine that there is a price* P_A *for ownership of the right to work a machine of type* A *and* P_B *for machine* B*.*
   i. *Graph, as a function of* P_A/P_B*, the amount of ownership rights for the right to work on machine A that would be supplied.*
   ii. *Graph, as a function of* P_A/P_B*, the amount of ownership rights for the right to work on machine A that would be demanded.*
   iii. *Find the price or range of prices at which demand is equal to supply and show that the market allocation at these prices is the same as the profit maximizing allocation you found above.*

c. *Suppose again that the initial assignment is workers* 1 *through* 6 *on machines on type A and* 7 *through* 12 *on machines of type* B*. Suppose that the workers own stock options whose value increases as company profit increases. Also, any pair of workers are free to swap assignments if it is mutually agreeable. Provided the workers don't have preferences over which types of jobs they are assigned to, which pair(s) workers would like to swap? Show that after all possible mutually agreeable swaps have happened, the allocation is the same as the profit maximizing allocation you found above.*

PROBLEM C.3. *This is a one-problem review of much of the material about competitive firms from introductory microeconomics. The technology for production has a fixed cost of* $F = 20$*, and the marginal cost of production at* $q \geq 0$ *is* $MC(q) = 10 + 0.1q$*. If it helps, think of the units as being tonnes.*

a. *Give the variable cost function,* $V(q)$*, the total cost function,* $C(q)$*, and the average cost function,* $AC(q)$*.*

b. *Graph the average cost function and the marginal cost function, indicating where they cross. Give the most efficient scale for the firm to operate.*

c. *If the going price for the output is* p *and the quantity produced by the firm has no effect on this price, give the firm's short-run profit maximizing supply. Indicate the range of prices for which the firm will optimally produce* 0*, for which it will produce a positive amount but lose money, and for which it will make money.*

d. *What are the differences between the long-run and the short-run profit maximizing quantities in your previous answer?*

e. *At what price will the firm find it more profitable to open a second factory that duplicates the technology of the first one?*

PROBLEM C.4. *You just won a free ticket to see a concert by the band Midnight Oil. Because it is a won ticket, it has no resale value. Cold Chisel is performing on the same night and is your next-best alternative activity. Tickets to see Cold Chisel cost 30\$. On any given day, you would be willing to pay up to 45\$ to see Cold Chisel. Assume there are no other costs of seeing either band. Based on this information, what is the opportunity cost of seeing Midnight Oil? (a) 0\$, (b) 15\$, (c) 30\$, or (d) 45\$. Explain.* [*Think of consumer surplus before you leap to an answer here.*]

PROBLEM C.5. *When one looks at statistics measuring the competence with which firms are run, after adjusting for the industry, one finds a weak effect in* *favor of firms with female CEO's, and a much stronger effect in favor of larger firms. In this problem, you are going to investigate a different advantage of being large, the decreasing average cost aspect of simple inventory systems. Decreasing average costs sometimes go by the name of economies of scale, and economies of scale are a crucial determinant of the horizontal boundary of a firm. In this problem, you will find a power law relating size to costs.*

Your firm needs Y units of, say, high grade cutting oil per year. Each time you order, you order an amount Q at an ordering cost of $F + pQ$, where F is the fixed cost of making an order (e.g. you wouldn't want just anybody to be able to write checks on the corporate account and such sytems are costly to implement), and p is the per unit cost of the cutting oil. This means that your yearly cost of ordering is $\frac{Y}{Q} \cdot (F + pQ)$ because $\frac{Y}{Q}$ is the number of orders per year of size Q that you make to fill a need of size Y.

Storing anything is expensive, and the costs include insurance, the opportunity costs of the space it takes up, the costs of keeping track of what you actually have, and so on. We suppose that these stockage costs are s per unit stored. Computerized records and practices like bar-coding have substantially reduced s over the last decades. Thus, when you order Q and draw it down at a rate of Y per year, over the course of the cycle that lasts Q/Y of a year, until you must re-order, you store, on average $Q/2$ units. This incurs a per year cost of $s \cdot \frac{Q}{2}$. Putting this together, the yearly cost of running an inventory system to keep you in cutting oil is

$$C(Y) = \min_{Q} \left[\frac{Y}{Q} \cdot (F + pQ) + s \cdot \frac{Q}{2} \right], \tag{C.7}$$

and the solution is $Q^*(Y, F, p, s)$.

a. *Without actually solving the problem in equation (C.7), find out whether Q^* depends positively or negatively on the following variables, and explain, in each case, why your answers makes sense: Y; F; p; and s.*
b. *Now explicitly find the optimal tradeoff between fixed costs and storage costs to solve for $Q^*(Y, F, p, s)$ and $C(Y)$.*
c. *Find the marginal cost of an increase in Y. Verify that the average cost, $AC(Y)$, is decreasing and explain how your result about the marginal cost implies that this must be true.*
d. *With the advent and then lowering expenses of computerized inventory and accounting systems, the costs F and s have both been decreasing. Does this increase or decrease the advantage of being large?*

PROBLEM C.6. *Synergies in production can be a driving force in the expansion of a firm via merger or acquisition. To get at this very simply, let us suppose that there are two firms, unimaginatively, 1 and 2, producing goods q_1 and q_2 and costs $c_1(q_1)$ and $c_2(q_2)$, and facing demand curves $p_1(q_1)$ and $p_2(q_2)$ (note that $p_1(\cdot)$ does not depend on q_2 and vice versa). Synergies in production can be thought of as the joint firm having a cost function satisfying $c_J(q_1, q_2) < c_1(q_1) + c_2(q_2)$.*

a. *Give three examples of technological synergies.*
b. *Show that in the presence of synergies, the joint firm, that is, the firm after merger or acquisition, will make higher profits.*
c. *Give conditions under which you could conclude that the joint firm produces more/less of both q_1 and q_2 than the two firms did separately. Explain.*

Problem C.7. *One of the main ways that firms expand their horizontal and vertical boundaries is through mergers and acquisitions. On average, when one firm acquires or merges with another, value is destroyed, that is, after the acquisition or merger, the market value of the resulting firm is lower than the sum of values of the two firms, at least as valued by the stock market at periods of 3, 6, and 12 months after the acquisition/merger. Remember, this is an average, some of these corporate manoeuvres create value, other destroy value, and good decisions can result in bad outcomes, this is just the nature of randomness.*

There are a number of explanations for this statistic, business researchers have looked, for example, at the role of overconfidence in CEO's, or the role of the independence and strength of the board of directors. Some of the explanation is that the incentives in the contracts of the people making the decision to acquire are badly written, and we will here look at a simple version of this part of the problem.

Let the decision $x = 0$ represent making no acquisition, and let $x = 1$ represent the decision to make the acquisition, and consider the maximization problem

$$\max_{x=0,1} \; [F \cdot x + \gamma \max\{(M(1) - M(0)), 0\}] \tag{C.8}$$

where $F > 0$ are the fees collected for arranging the acquisition, $M(1)$ is the market value of the post-acquisition firm, $M(0)$ is the value of the pre-acquisition firm, and $\gamma > 0$ reflects the part of the reward that the decision maker receives from increasing the market value of the firm (e.g. from stock options or the like).

a. *Give the solution to the problem in (C.8).*
b. *Look up the term "clawbacks" in the business press and explain how the shareholders might want to change the contract in (C.8) to get the decision $x^* = 1$ when and only when $(M(1) - M(0)) > 0$.*
c. *How does this analysis relate to the market for corporate control as discussed in Besanko et al.?*

Problem C.8. *When one looks at historical statistics about R&D rates, one finds that it is concentrated in the larger firms. Such figures do not include a recent phenomenom, the growth in the number of firms that specialize in doing contract R&D, often for the government, but increasingly in the recent past, for the large pharmaceutical firms who have been "outsourcing their brains." In this problem, you are going to investigate a simple case of how being large can give a decreasing risk-adjusted average cost of doing R&D. Behind the results you will find here is the notion of portfolio diversification.*

We are going to suppose that research projects cost C, that C is "large," and that research projects succeed with probability p, that p is "small," and that if the project does not succeed, then it fails and returns 0. Thus, the distribution of returns for a project are $(R - C)$ with probability p and $0 - C$ with probability $1 - p$. Since R, if it happens, will be off in the future and the costs, C, must be borne up front, we are supposing that R measures the net present value of the eventual success if it happens.

The expected or average return on a research project is $p(R-C) + (1-p)(-C)$ which is equal to $pR - C$, expected returns minus expected costs. We assume that $pR > C$, that is, that expected returns are larger than expected costs. We are also going to assume that success on different projects are independent of each other. Specifically, if you take on two projects, then the probability that both succeed is p^2, *the probability that both fail is $(1-p)^2$, and the probability that exactly one of them succeeds is $[1-p^2-(1-p)^2]$, that is, $2p(1-p)$.*

A heavily used measure of the risk of a random return is its standard deviation, which is the square root of the average squared distance of the random return from its average. We let $\mu = pR - C$ be the average or expected return of a single project, the standard deviation is then $p\sqrt{(R-C)-\mu} + (1-p)\sqrt{(-C)-\mu}$, which is denoted σ. Of particular interest is the ratio $\frac{\sigma}{\mu}$, a unitless measure giving the risk/reward ratio for the project. Of interest is the comparison of the risk/reward ratio when you have one project and when you have two. Its inverse, $\frac{\mu}{\sigma}$ is a risk adjusted measure of the average return.

a. *If $R = 10^7$ and $C = 100,000$, find the set of p for which the expected value, μ, is positive. For these p, give the associated σ and $\frac{\mu}{\sigma}$. Graph your answers in an informative fashion.*
b. *Now suppose that your research budget is expanded, and you can afford to undertake two projects. Verify that the expected value is now $2 \cdot \mu$. Verify that the new σ for the R&D division is $\sqrt{2}$ times the answer you previously found. What has happened to the risk adjusted measure of the average return?*
c. *Repeat the previous two problems with $R = 10^8$ and $C = 200,000$.*
d. *In the inventory problem above, there was a power law giving the advantage of being larger. Give the general form of the power law relating the research budget to the risk adjusted rate of return.*

D. Monotone Comparative Statics I

D.1. Summary. If something in the environment changes, the best, i.e. profit-maximal or utility maximal, choice of action changes. Comparative statics is the study of the dependence of the best choice on aspects of the environment. A powerful underlying principle for this kind of analysis is the idea of increasing differences. We will apply this idea to a number of changes: input prices; exchange rates; competitors leaving or entering the market; the appearance of new ways of organizing production flows; the appearance of new ways of getting and acting on market information.

D.2. Increasing Differences. We take X and T to be subsets of $\mathbb{R}$ with the the usual less-than-or-equal-to order. Note that nothing rules out the sets X and T being discrete, e.g. we will often have T being the two point set $\{0, 1\}$. We are interested in the behavior of the maximizing choice, $x^*(t)$, for the problems

$$\max_{x \in X} f(x,t),\ t \in T. \tag{D.1}$$

The idea here is that t is not under control of the decision maker, they can pick x, and when faced with different values of t, they may make different decisions. In particular, we want to know if $x^*(t') > x^*(t)$ when $t' > t$. We are after what is called a **monotone comparative statics result** — if $f(\cdot,\cdot)$ satisfies the following condition, then we can safely conclude that the optimizing set $x^*(t')$ is larger than the optimizing set $x^*(t)$ when $t' > t$. This is easiest when there is a unique optimum for each value of $t \in T$.

Definition I.5. *For linearly ordered X and T, a function $f : X \times T \to \mathbb{R}$ has **increasing differences** or is **supermodular** if for all $x' > x$ and all $t' > t$,*

$$f(x',t') - f(x,t') \geq f(x',t) - f(x,t), \tag{D.2}$$

equivalently

$$f(x',t') - f(x',t) \geq f(x,t') - f(x,t). \tag{D.3}$$

It is ***strictly supermodular*** *or has* ***strictly increasing differences*** *if the inequalities are strict. For* ***submodularity/decreasing differences****, and* ***strictly submodular/decreasing differences*** *functions, reverse the inequalities.*[7]

At t, the benefit of increasing from x to x' is $\Delta(f,t) = f(x',t) - f(x,t)$, at t', it is $\Delta(f,t') = f(x',t') - f(x,t')$. This assumption asks that Δ, the difference in f from moving from the low x to the higher x', be higher for higher values of t.

Crudely, the main result is the following:

> If the function $f(\cdot,\cdot)$ has increasing differences, then $x^*(\cdot)$ is increasing.

Be careful here, increasing differences is a sufficiently strong condition to tell us that $x^*(\cdot)$ is increasing, but it is not necessary for the function $f(\cdot,\cdot)$ to have increasing difference in order for $x^*(\cdot)$ to be increasing.

Three sufficient conditions in the case that $f(\cdot,\cdot)$ is differentiable case are: $\forall x$, $f_x(x,\cdot)$ is nondecreasing; $\forall t$, $f_t(\cdot,t)$ is nondecreasing; and $\forall x,t$, $f_{xt}(x,t) \geq 0$.

D.2.1. *A Purely Mathematical Example of the Main Result.* Consider the function $f(x,t) = 7 - (x-t)^2$. We will now check that $f(\cdot,\cdot)$ has strictly increasing differences. Pick any $t' > t$ and any $x' > x$, we must check that $f(x',t') - f(x',t) > f(x,t') - f(x,t)$. Re-writing f as $7 - x^2 + 2tx - t^2$, this involves checking

$$\begin{aligned}&[7-(x')^2+2t'x'-(t')^2] - [7-(x)^2+2t'x-(t')^2] > \\ &\qquad [7-(x')^2+2tx'-(t)^2] - [7-(x)^2+2tx-(t)^2].\end{aligned} \tag{D.4}$$

This comparison of the sum of 8 terms against the sum of 8 other terms may not look as easy as it will end up being. What makes it easy is that all the terms that do not involve both x and t simultaneously cancel each other out: the two positive 7's and the two negative 7's are the same on each side of the inequality, so we can take them out, leaving us to check

$$\begin{aligned}&[-(x')^2+2t'x'-(t')^2] - [-(x)^2+2t'x-(t')^2] > \\ &\qquad [-(x')^2+2tx'-(t)^2] - [-(x)^2+2tx-(t)^2].\end{aligned} \tag{D.5}$$

In just the same way, the $-(x')^2$ terms and the $+(x)^2$ terms cancel each other out, leaving us to check

$$\begin{aligned}&[2t'x'-(t')^2] - [2t'x-(t')^2] > \\ &\qquad [2tx'-(t)^2] - [2tx-(t)^2].\end{aligned} \tag{D.6}$$

Now the negative and the positive $(t')^2$ terms add to 0 as do the negative and the positive $(t)^2$ terms, leaving us to check

$$2t'x' - 2t'x > 2tx' - 2tx, \text{ or } 2t'(x'-x) > 2t(x'-x). \tag{D.7}$$

Well, $(x'-x) > 0$ because $x' > 0$, so this is true because $t' > t$.

[7] Talking about supermodular functions can impress your friends and make your enemies fear you, but the more straightforward "f has increasing differences" phrasing has always helped me remember what is going on.

> A very important lesson in checking for increasing differences: you only need to check the terms involving both x and t; everything else will cancel out.

So, now we know that $x^*(\cdot)$ is an increasing function of t. In this case, it is so simple to solve explicitly for $x^*(t)$ and check it directly, that even I am willing to do it. For each value of t, the function $f(\cdot, \cdot)$ is a quadratic in x that opens downwards. The standard notation for quadratic functions is $q(x) = ax^2 + bx + c$, here, $a = -1$, $b = 2t$, and $c = 7 - t^2$. The **quadratic formula** for the roots of the quadratic $q(\cdot)$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{D.8}$$

From high school algebra, you should have learned that halfway between the two roots is the critical point of the quadratic, the bottom point if the quadratic opens upwards, the top if it opens downwards. Since $x^*(t)$ is the top point of the quadratic, we know that $x^*(t) = \frac{-2t}{-2} = t$. This is certainly an increasing function of t.

As the easy alternative to using the quadratic formula, one could take the derivative of $f(\cdot, t)$ with respect to x and set it equal to 0, that is, set $-2(x - t) = 0$ to find $x^*(t) = 0$.

D.2.2. *The Main Result.* It took us economists quite a while to figure out just how useful the following result truly is.

THEOREM I.5 (Topkis). *If X and T are linearly ordered, $f : X \times T \to \mathbb{R}$ is supermodular and $x^*(t)$ is the largest solution to $\max_{x \in X} f(x, t)$ for all t, then $[t' > t] \Rightarrow [x^*(t') \geq x^*(t)]$. Further, if there are unique, unequal maximizers at t' and t, then $x^*(t') > x^*(t)$.*

The following set of arguments has a very intuitive geometry to it, one that is made even easier if we assume that there is only one optimizer at t and only one at t'. This is the case covered in lecture. Below is the more complete argument, an argument that takes care to look at the comparison between the largest optimizer at t and at t'. If there is only one optimizer, then it must be the largest one.

PROOF. The idea of the proof is that having $x^*(t') < x^*(t)$ can only arise if f has strictly decreasing differences.

Suppose that $t' > t$ but that $x' := x^*(t') < x := x^*(t)$. Because $x^*(t)$ and $x^*(t')$ are maximizers, $f(x', t') \geq f(x, t')$ and $f(x, t) \geq f(x', t)$. Since x' is the largest of the maximizers at t' and $x > x'$, i.e. x is larger than the largest maximizer at t', we know a bit more, that $f(x', t') > f(x, t')$. Adding the inequalities, we get $f(x', t') + f(x, t) > f(x, t') + f(x', t)$, or

$$f(x, t) - f(x', t) > f(x, t') - f(x', t'),$$

i.e. strictly decreasing differences in x and t. □

D.2.3. *Quasi-Supermodularity.* From consumer demand theory, we have the following observation about the irrelevance of monotonic transformations: if $u : \mathbb{R}^k_+ \to \mathbb{R}$ is a utility function, and $x^*(p, w)$ solves the problem

$$\max_{x \in \mathbb{R}^k_+} u(x) \text{ subject to } px \leq w, \tag{D.9}$$

then for any strictly increasing function $\psi : \mathbb{R} \to \mathbb{R}$, $x^*(p, w)$ also solves the problem

$$\max_{x \in \mathbb{R}^k_+} \psi(u(x)) \text{ subject to } px \leq w. \tag{D.10}$$

In the same way, if $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then the sets $x^*(t)$ are exactly the same for the following two problems,

$$\text{(D.11)} \qquad \max_{x \in X} f(x,t) \text{ and } \max_{x \in X} \psi(f(x,t)).$$

A function $f : X \times T \to \mathbb{R}$, $X, T \subset \mathbb{R}$, having increasing differences is a sufficient condition for the highest and the lowest elements of $x^*(t)$ to be non-decreasing in t. It is not a necessary condition and there is room for improvement. A yet weaker sufficient condition is the following.

Definition I.6. *For linearly ordered X and T, a function $f : X \times T \to \mathbb{R}$ has is* ***quasi-supermodular (q-supermodular)*** *if for all $x' > x$ and all $t' > t$,*

$$\text{(D.12)} \qquad [f(x',t) - f(x,t) \geq 0] \Rightarrow [f(x',t') - f(x,t') \geq 0], \text{ and}$$

$$\text{(D.13)} \qquad [f(x',t) - f(x,t) > 0] \Rightarrow [f(x',t') - f(x,t') > 0].$$

It is ***strictly quasi-supermodular*** *if we can always conclude that the inequality on the right is strict.*

There are three results to remember: supermodularity implies quasi-supermodularity, Exercise D.3; quasi-supermodularity survives monotonic transformations, Exercise D.4; supermodularity need not survive monotonic transformations, Exercise D.5.

D.2.4. *First Examples.* Going back to the polluting monopolist of Example I.5 (p. 44), the supermodularity of f reduces to the supermodularity of $-c$. Thus assuming $-c$ (and hence f) is supermodular, we can use Theorem I.5 to conclude that $x^*(t)$ is increasing. None of the second derivative conditions except $c_{xt} < 0$ are necessary, and this can be replaced by the looser condition that $-c$ is supermodular.

Clever choices of T's and f's can make some analyses criminally easy.

Example I.4. *Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$ so that $CS(p) = \int_p^\infty x(r)\,dr$ is the consumer surplus when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.*

The monopolist's profit when they produce x is $\pi(x) = x \cdot p(x) - c(x)$ where $c(x)$ is the cost of producing x. The maximization problem for the monopolist and for society are

$$\text{(D.14)} \qquad \max_{x \geq 0} \ \pi(x) + 0 \cdot CS(p(x)), \text{ and.}$$

$$\text{(D.15)} \qquad \max_{x \geq 0} \ \pi(x) + 1 \cdot CS(p(x)).$$

Set $f(x,t) = \pi(x) + tCS(p(x))$ where $X = \mathbb{R}_+$ and $T = \{0,1\}$. Because $CS(p(x))$ is nondecreasing, $f(x,t)$ is supermodular (and you should check this). Therefore $x^*(1) \geq x^*(0)$, the monopolist always (weakly) restricts output relative to the social optimum.

Here is the externalities intuition: increases in x increase the welfare of people the monopolist does not care about, an effect external to the monopolist; the market gives the monopolist insufficient incentives to do the right thing. To fully appreciate how much simpler the supermodular analysis is, we need to see how complicated the differentiable analysis would be. We'll not do that here, check [**5**] if you are interested in the details.

D.3. From the Theory of the Competitive Firm. What we are after here is the basic structure of an argument, the more elaborate examples and discussions below will be more understandable if you can map back to what you see in the simple examples we start with.

D.3.1. *The marginal revenue equals marginal cost approach.* A firm produces y, sold at a price p, using an input x, bought at a price w, and has the technology to change x into y given by $y = f(x) = \sqrt{x}$. Profits are (as always), revenues minus costs, $u(x; p, w) = [pf(x) - wx]$, and the profit maximization problem is

$$\max_{x \geq 0} \, [pf(x) - wx]. \tag{D.16}$$

The marginal revenue product of the last unit of input is $pf'(x)$, the marginal cost is equal to w, at the optimum, we expect these to be equal, $pf'(x) = w$, equivalently, $f'(x^*) = \frac{w}{p}$. Now, $f'(x^*)$ is the marginal product of the last unit used at the optimum, that is, it is the rate of technological transformation of input, x, into output, $y = f(x)$. On the right-hand side of $f'(x^*) = \frac{w}{p}$ is the ratio $\frac{w}{p}$, this is equal to market rate of substitution of x into y — selling one unit of x gets you w\$, with w\$, you can buy $\frac{w}{p}$ units of y if the price of y is p.

In this particular case with $f(x) = \sqrt{x}$, so $f'(x) = \frac{1}{2\sqrt{x}}$, solving $f'(x) = \frac{w}{p}$ in this case yields the firm's demand curve for inputs, $x^*(p, w) = (\frac{p}{2w})^2$. Since $f(x^*(p, w))$ is what the firm produces when it uses the amount $x^*(p, w)$ of input, the firm's supply curve is $\sqrt{x^*(p, w)} = \frac{p}{2w}$, and their profit function is $\Pi(w, p) = \frac{p^2}{4w}$. From this you can read off the following set of results:

- if the price of output goes up, then the demand for the input goes up, $\partial x^*/\partial p > 0$;
- if the price of input goes up, then the demand for the input goes down, $\partial x^*/\partial w < 0$;
- if the price of the output goes up, then the supply goes up, $\partial y^*/\partial p > 0$,
- if the price of the input goes up, then the supply goes down, $\partial y^*/\partial w < 0$; and
- profits go in the corresponding directions, $\partial \Pi/\partial p > 0$ and $\partial \Pi/\partial w < 0$.

The intuition for this problem is clearer than one sees in doing that last bit of algebra, and applies to all increasing production functions: you produce more and have a higher profit when the price of what you're selling goes up; you produce less and have a lower profit when the price of what you're using for production goes up.

D.3.2. *The increasing differences approach.* This is a special case of the idea of **increasing differences**: consider the function $u(x; p, w) = [pf(x) - wx]$; note that if p increases to p', then the utility/profits increase more from any increase from x' to x, that is for $p' > p$ and $x' > x$, we have

$$u(x'; p', w) - u(x; p', w) > u(x'; p, w) - u(x; p, w). \tag{D.17}$$

In checking that this is true, we see again an example of a very general pattern,

$$u(x'; p', w) - u(x; p', w) = [p'f(x') - wx'] - [p'f(x) - wx], \tag{D.18}$$

$$u(x'; p, w) - u(x; p, w) = [pf(x') - wx'] - [pf(x) - wx]. \tag{D.19}$$

To check that $u(x'; p', w) - u(x; p', w) > u(x'; p, w) - u(x; p, w)$ holds, note that the wx' and the wx terms cancel each other out, so we are left to check that $p'(f(x') - f(x)) > p(f(x') - f(x))$, and since $p' > p$ and $(f(x') - f(x)) > 0$, this is true. This says that higher output prices mean the rewards to any increase in the use of inputs goes up. The conclusion is that we use more inputs.

The case of **decreasing differences** is just the flip side of this: $u(x; p, w) = [pf(x) - wx]$; note that if w increases to w', then the utility/profits decrease more from any increase from x' to x, that is for $w' > w$ and $x' > x$, we have

$$u(x'; p, w') - u(x; p, w') < u(x'; p, w) - u(x; p, w). \tag{D.20}$$

In checking that this is true, we will again see an example of a very general pattern,

$$u(x'; p, w') - u(x; p, w') = [pf(x') - w'x'] - [pf(x) - w'x], \tag{D.21}$$

$$u(x'; p, w) - u(x; p, w) = [pf(x') - wx'] - [pf(x) - wx]. \tag{D.22}$$

To check that $u(x'; p', w) - u(x; p', w) < u(x'; p, w) - u(x; p, w)$ holds, note that the $pf(x')$ and the $p'f(x)$ cancel each other out, so we are left to check that $w'x - w'x' < wx - wx'$, that is, $w'(x - x') < w(x - x')$. Since $(x - x') < 0$ and $w' > w$, this is true. This says that higher input prices mean the rewards to any increase in the use of inputs goes down. The conclusion is that we use fewer of the inputs.

D.3.3. *The Calculus Version*$^{\boldsymbol{Adv}}$. Let us now consider the function $u(x; p, w) = [pf(x) - wx]$, and interpret the following observations about the cross partial derivatives of u.

- $\frac{\partial u^2}{\partial x \partial w} < 0$ tells us that the marginal utility (profit in this case) of using the input x goes down as w goes up. If you do something up until the marginal utility goes to 0 and then the marginal utility goes down, you ought to do *less* of it, that is you expect that $x^* \downarrow$ as $w \uparrow$.
- $\frac{\partial u^2}{\partial x \partial p} > 0$ tells us that the marginal utility (profit in this case) of using the input x goes up as p goes up. If you do something up until the marginal utility goes to 0 and then the marginal utility goes up, you ought to do *more* of it, that is you expect that $x^* \uparrow$ as $p \uparrow$.

That's a simple version of the essential insight, if the marginal rewards to doing something go up then do it more, if the marginal rewards go down then do it less.

There is one more bit of mathematical arguing to do here, one that we're doing partly because you'll be happy that our results tell you that you don't have to do it very often. Let us suppose that $x^*(p, w)$ is the one and only solution to the problem

$$\max_{x \geq 0} \, [pf(x) - wx], \tag{D.23}$$

and that this is equivalent to $x^*(p, w)$ solving the equation given by marginal revenues equal marginal costs,

$$pf'(x^*(p, w)) - w \equiv 0. \tag{D.24}$$

Taking the derivatives on both sides with respect to w in equation (D.24) yields, after remembering how the chain rule works,

$$pf''(x^*(p, w))\frac{\partial x^*(p,w)}{\partial w} - 1 \equiv 0, \text{ yielding } \frac{\partial x^*(p,w)}{\partial w} = \frac{1}{pf''(x^*(p,w))}. \tag{D.25}$$

Now $f''(x^*) < 0$, both in this particular case where $f''(x) = -1/4x^{3/2}$, and in general, where $f''(x^*) < 0$ is the condition from your calculus class for a local maximum for the problem in equation (D.23).

D.4. From the Theory of the Monopolistic Firm. One of the differences between a monopolist and a small firm competing against many other small firms is that when the monopolist changes their quantity, it changes the price.

D.4.1. *The marginal revenue equals marginal cost approach.* Let $p(q)$ be the demand curve for a monopolist and suppose that $q^*(w)$ is the solution to the problem

$$\max_{q \geq 0} \,[qp(q) - wc(q)]. \tag{D.26}$$

With $Rev(q) = qp(q)$ being the revenue, the solution is, as usual, to be found where marginal revenue is equal to marginal cost, or

$$Rev'(q) = p(q) + qp'(q) = wc'(q). \tag{D.27}$$

When w, the cost of inputs, goes up, intuition strongly suggests that the monopolist will produce less, but that can a bit harder to see in equation (D.27) than it ought to be.

Detour: One of the main reasons that monopolies exist is that marginal cost curves can slope downwards, and it seems possible, mathematically at least, that the marginal cost curve, $wc'(q)$, could cut the marginal revenue curve from above. You should draw what happens if we move the marginal cost curve up when it cuts the decreasing marginal revenue curve from below.

Calculus version of the detour: A little bit of work with the equations will tell you that the second derivative of the profit function, $\pi(q) = qp(q) - wc(q)$, being negative at q^* requires that the marginal revenue curve cut the marginal cost function from above, not the other way around. This restores order to this problem, and maybe provides some insight, but that insight is far removed from our intuition.

D.4.2. *The increasing/decreasing differences approach.* Consider the utility function $u(q, w) = [qp(q) - wc(q)]$ and ask again about the decreasing differences aspect. It is easy to check, and *I very strongly suggest that you do it*, that for $w' > w$ and $q' > q$,

$$u(q', w') - u(q, w') < u(q', w) - u(q, w). \tag{D.28}$$

We have returned to the observation that if the marginal reward of doing something goes down, you should do it less.

D.4.3. *Other aspects of monopolists.*

Shift the demand function

Lest you think that everything has become easy, let us consider what happens to a monopolist's supply after the demand curve shifts inwards or outwards by some factor θ. If the demand curve of the monopolist shifts from $p(q)$ to $\theta \cdot p(q)$ where $\theta > 0$, consider the problems

$$\max_{q} \;\pi(q, \theta) = [q\theta p(q) - wc(q)]\,. \tag{D.29}$$

If we knew that $\pi(\cdot, \cdot)$ had increasing differences in q and θ, we would know that outward expansions of the demand curve would increase suppy, but this does not hold here. Not to worry, increasing differences is sufficiently strong to guaratee that the optimal are an increasing function of θ, but you can have increasing optima without having increasing differences.

Rewriting the part involving θ, this is $\theta \cdot Rev(q)$. Now, we certainly expect that if $\theta > 1$, the monopolist will respond by increasing supply, but one cannot directly use an increasing differences approach on the function $\pi(q, \theta)$, because $Rev(\cdot)$ will typically have a region of increase and a region of decrease. What to do?

A first insight is that at the optimum, it must be the case that revenues are increasing — if decreasing then a decrease in the quantity would increase revenues and save on costs. Here is one device to allow us to use increasing differences again: (1) assume that costs are an increasing function; (2) show that in the presence of the first assumption, that there is no loss in replacing $Rev(\cdot)$ with the a new function, $F(\cdot)$ that has $F(q) = Rev(q)$ for $q \in [0, \overline{q}]$ where $\overline{q}$ is the point at which $Rev(\cdot)$ is maximized; and (3) analyze the problems

$$\max_q \; [\theta F(q) - wc(q)] \,. \tag{D.30}$$

Raising prices and restricting supply (redux)

Let us repeat the previous analysis of monopolists using prices instead of quantities for the analysis. In general, monopolists restrict quantities and raise prices in a fashion detrimental to society. This happens not only in the above ground economy, but also in the underground economy, think of the local monopolies of the drug cartels, or of the liquor-running and gun-running gangsters in the past.

From introductory microeconomics, if $p(q)$ is the demand curve and a price p° leads to an amount q° being sold, then the consumer surplus is the area above p and under the graph of the function $p(q)$, $CS(p^\circ) = \int_0^{q^\circ} [p(q) - p^\circ] \, dq$. Crucially, $CS(\cdot)$ must be a decreasing function of p. The revenues associated with p° are $Rev(p^\circ) = p^\circ q^\circ = p^\circ q(p^\circ)$ where $q(\cdot)$ is the inverse demand function. Using the same inverse demand function, the costs for the monopolist are $C(q^\circ)$, expressed in terms of price as $C(p^\circ) = C(q(p^\circ))$. Consider the problem

$$\max_p \; [Rev(p) + t \cdot CS(p) - C(p^\circ)] \tag{D.31}$$

for $t' = 1$ and $t = 0$. Since $CS(\cdot)$ is a decreasing function, $f(p, t) = [Rev(p) + t \cdot CS(p) - C(p^\circ)]$ has decreasing differences, hence society as a whole, the problem with $t' = 1$, would benefit from a lower than the monopolist would chose.

D.5. The More Formal Calculus Based Approach$^{\mathbf{Adv}}$. Assume that X and T are interval subsets of $\mathbb{R}$, that f is twice continuously differentiable, and that we are interested in the behavior of the $x^*(t)$ that solves the problem

$$\max_{x \in X} f(x, t). \tag{D.32}$$

Let f_x, f_t, f_{xx} and f_{xt} denote the corresponding partial derivatives of f. To have $f_x(x, t) = 0$ characterize $x^*(t)$, we must have $f_{xx} < 0$ (this is a standard result about concavity in microeconomics). From the implicit function theorem, we know that $f_{xx} \neq 0$ is what is needed for there to exist a function $x^*(t)$ such that

$$f_x(x^*(t), t) \equiv 0. \tag{D.33}$$

To find dx^*/dt, take the derivative on both sides with respect to t, and find

$$f_{xx} \tfrac{dx^*}{dt} + f_{xt} = 0, \tag{D.34}$$

so that $dx^*/dt = -f_{xt}/f_{xx}$. Since $f_{xx} < 0$, this means that dx^*/dt and f_{xt} have the same sign.

This ought to be intuitive: if $f_{xt} > 0$, then increases in t increase f_x; increases in f_x are increases in the marginal reward of x; and as the marginal reward to x goes up, we expect that the optimal level of x goes up. In a parallel fashion: if $f_{xt} < 0$, then increases in t decrease f_x; decreases in f_x are decreases in the marginal reward of x; and as the marginal reward to x goes down, we expect that the optimal level of x goes down.

There is a geometric intuition too: for each t, one could imagine walking on the hill given by the function $f(\cdot, t)$; it's flat at the top of the hill, $f_x(x^*(t), t) = 0$; if $f_{xt} > 0$, then moving in the direction given a small increase in t gets you to a place where the hill is now sloping upwards; if you're trying to get the top of a hill and it's sloping upwards, you have to go in that direction.

PROBLEM D.1. *Let $X = T = \mathbb{R}_+$, $f(x,t) = x - \frac{1}{2}(x-t)^2$. Find $x^*(t)$ and verify directly that $dx^*/dt > 0$. Also find f_x, f_{xx}, and f_{xt}, and verify, using the sign test just given, that $dx^*/dt > 0$. Draw f and verify from your picture that $f_{xt} > 0$ and that it is this fact that make $dx^*/dt > 0$.*

EXAMPLE I.5. *The amount of a pollutant that can be emitted is regulated to be no more than $t \geq 0$. The cost function for a monopolist producing x is $c(x,t)$ with $c_t < 0$ and $c_{xt} < 0$. These derivative conditions means that increases is the allowed emission level lower costs and lower marginal costs, so that the firm will always choose t. For a given t, the monopolist's maximization problem is therefore*

$$\max_{x \geq 0} f(x,t) = xp(x) - c(x,t) \tag{D.35}$$

where $p(x)$ is the (inverse) demand function. Since $f_{xt} = -c_{xt}$, we know that increases in t lead the monopolist to produce more, ***provided*** *$f_{xx} < 0$.*

The catch in the previous analysis is that $f_{xx} = xp_{xx} + p_x - c_{xx}$, so that we need to know $p_{xx} < 0$, concavity of inverse demand, and $c_{xx} > 0$, convexity of the cost function, before we can reliably conclude that $f_{xx} < 0$. The global concavity of $f(\cdot, t)$ seems to have little to do with the intuition that it is the lowering of marginal costs that makes x^* depend positively on t. However, global concavity of $f(\cdot, t)$ is **not** what we need for the implicit function theorem, only the concavity of $f(\cdot, t)$ in the region of $x^*(t)$. With differentiability, this local concavity is an **implication** of $x^*(t)$ being a strict local maximum for $f(\cdot, t)$. What a **supermodularity** analysis does is to make it clear that the local maximum property is all that is being assumed, and to allow us to work with optima of functions that are non-differentiable. Supermodularity is, in this simple content, known as **increasing differences**.

D.6. Laws of Unintended Consequences. Suppose that there is a policy, to be set at a level $x \geq 0$. For example, this could be a reward per ton of bed frames produced, the dollars spent researching minor but patentable tweaks to a therapeutic molecule, the degree to which married women can sign legally binding contracts, the ease of suing a firm for breach of contract, the degree to which an executive's compensation depends on next quarter's reported profits, the frequency of vehicle inspection, a tax level, a maximal amount of pollution that any given vehicle can emit per mile. Sometimes only parts of the benefits, $B(x)$, or parts of the costs, $C(x)$, are included. We want to see what happens to the optimal x when they are all included. We suppose for this analysis that both the social benefits and social costs are increasing in x.

1. Carefully compare the properties of the *sets* of optimal x's for the problems

$$\max_{x\geq 0}\ [B(x) - C(x)] \text{ and } \max_{x\geq 0}\ [(B(x) + B_2(x)) - C(x)]$$

where $B_2(\cdot)$ is another increasing benefit function.

2. Carefully compare the properties of the *sets* of optimal x's for the problems

$$\max_{x\geq 0}\ [B(x) - C(x)] \text{ and } \max_{x\geq 0}\ [B(x) - (C(x) + C_2(x))]$$

where $C_2(\cdot)$ is another increasing cost function.

3. Carefully compare the properties of the *sets* of optimal x's for the problems

$$\max_{x\geq 0}\ [B(x) - C(x)] \text{ and } \max_{x\geq 0}\ [(B(x) + B_2(x)) - (C(x) + C_2(x))]$$

when
a. the net benefits $B_2(\cdot) - C_2(\cdot)$, are increasing, and
b. the net benefits $B_2(\cdot) - C_2(\cdot)$, are decreasing.

It is worthwhile going through the examples listed above, seeing which fit with which of these analyses. It is also worth looking around for similar examples.

D.7. Goods with Complementarities in Production$^{\mathbf{Adv}}$. It is worth noting the word is "compl<u>e</u>ment" not "compl<u>i</u>ment," the root of the word "compl<u>e</u>ment" is the latin word meaning "complete." A compl<u>e</u>ment completes something, a compl<u>i</u>ment is what you give someone when they have done a good job.

Suppose that your company makes two products, unimaginatively labeled 1 and 2, and that there are complementarities in production, that is, the cost function, $c(y_1, y_2)$ satisfies $\partial^2 c(y_1, y_2)/\partial y_1 \partial y_2 < 0$. It might seem odd that producing more of one thing can lower the marginal cost of producing something else, but consider the following examples.

1. Having more passenger flights per day lowers the cost of producing more airmail services.
2. When you are separating crude oil, producing more residue (bitumen/tar used for paving roads) reduces the marginal cost of producing all of the lighter more volatile molecules (e.g. petrol/gasoline, naptha, kerosene).
3. Having produced a volume, y_1, of advertising for one region or language, producing a quantity of advertising, y_2 for another region or language has a much lower marginal cost.

Consider the problem

$$\max_{y_1, y_2 \geq 0}\ \pi(y_1, y_2; p_1, p_2) = p_1 y_1 + p_2 y_2 - c(y_1, y_2). \tag{D.36}$$

We saw above that an increase in p_1 increases the optimal $y_1^*(p_1, p_2)$, after all, profits are supermodular in y_1 and p_1. Profits are also supermodular in y_1 and y_2, so when the optimal y_1 goes up, so does the optimal y_2.

Descriptively, these provide an explanation for the ranges of products firms produce. Prescriptively, these push firm to search for complementary goods that they can produce at low cost. As with many pieces of advice from economists, there is the one hand, and the other hand: on the one hand, a firm staying near its core competencies, not becoming spread too thin, this can be good advice; on the other hand, a firm passing up opportunities to expand may be like the shark that stopped swimming, and died because no more water flowed over its gills. What is at issue here is whether or not the costs really are submodular and the profits supermodular. Prescriptively, the advice is "Figure it out!"

D.8. Labor Saving Technology. Technology and skill in labor are complements, increases in one increase the productivity of the other. We are, once again, living in an age full of warnings about humans becoming obsolete, that the new technologies will throw everyone out of work causing huge problems of social unrest. Starting at least as early as the 15'th century with *sabots* thrown into the wood gears of the new looms for fear that they would destroy jobs, we have worried about the unemployment and dislocation caused by new technologies. Now, for at least the third time in my life, we are to fear the coming of the robots.

Another way to look at this is that dislocation and change are both necessary to the functioning of capitalism since the Industrial Revolution, and that they can provide both a goad to action (uncomfortable perhaps), and new opportunities.

Suppose that we change from a technology with a marginal product of labor given by $MP_0(L)$ to a higher one, $MP_1(L) > MP_0(L)$. The company that switches needs less skilled labor and it offers inducements for some of its workers to look for other jobs (firing is an extreme version of this, the one that strikes fear into workers). This frees up skilled labor, lowering the cost to other firms of acquiring this human capital and meaning that there is, potentially at least, more left to pay the workers who stay behind. Skill is a complement to new technologies, giving an incentive for the other firms to increase the productivity of their own technologies.

It need not work out in so rosy a fashion, for example, it has worked out well in the German machine tool industry, less well in most parts of the U.S. machine tool industry. If management chooses between different new technologies on the basis that skilled labor has more power to bargain and one would rather bargain with people in a worse position, one can end up with technology choices giving management a bigger share of a pie that is smaller than it need be.

More formally, suppose that management is picking between technologies A and B, which earn π_A and π_B for the firm with $\pi_A < \pi_B$. What society would like in this case is the choice of B. Suppose that management's reward will be their share times the profits, that is, management is choosing between

$$s_A \pi_A \text{ and } s_B \pi_B \tag{D.37}$$

where $0 < s_A, s_B < 1$. As long as $s_B > s_A \cdot \frac{\pi_A}{\pi_B}$, we will get the correct decision, but if management incentives are sufficiently wrong, then we will get a bad decision.

A bad decision can be rational for management, but this is another case where the wider good of the firm is not correctly built into the incentives, just as in the analysis of monopolies above. When we cover bargaining later on, we will come to understand this as a version of the "hold up problem." Policies that help with the dislocation or re-education costs borne by workers have distributional overtones that have become less and less popular in the U.S. during my lifetime.

D.9. Discrete Complementarities. Computer Assisted Design (CAD) equipment is now wide-spread in the rich industrialized countries of the world. Many CAD installations produce control sequences for programmable manufacturing equipment. This reduces the cost of buying and using such equipment. Since this equipment is more flexible and having a CAD installation makes re-design an easier and cheaper process, the marginal cost of y, the number of product lines supported goes down.

Here is a simple way to think about this. One receives revenues, $R(y)$, that are increasing in y, the cost of producing y is $c(y; x_1, x_2)$ where x_1 and x_2 are either 0 or 1: $x_1 = 0$ means no CAD installation, $x_1 = 1$ means CAD installation; $x_2 = 0$ means no programmable equipment installed, $x_1 = 1$ means programmable equipment installed. Consider the problem

$$\max_{y \geq 0, x_1=0,1, x_2=0,1} R(y) - c(y, x_1, x_2) \tag{D.38}$$

and the four constituent sub-problems,

$$\max_{y \geq 0} R(y) - c(y; 0, 0), \tag{D.39}$$

$$\max_{y \geq 0} R(y) - c(y; 1, 0), \tag{D.40}$$

$$\max_{y \geq 0} R(y) - c(y; 0, 1), \tag{D.41}$$

$$\max_{y \geq 0} R(y) - c(y; 1, 1). \tag{D.42}$$

If costs are submodular in the x components/profits are supermodular in the x components, that is, if

$$\begin{aligned} &c(y; 1, 1) - c(y; 0, 1) < c(y; 1, 0) - c(y; 0, 0), \text{ equivalently} \\ &c(y; 1, 1) - c(y; 1, 0) < c(y; 0, 1) - c(y; 0, 0), \end{aligned} \tag{D.43}$$

then we would expect to see either $x^* = (0, 0)$ or $x^* = (1, 1)$.

If profits are supermodular in y and the x components, that is if

$$\begin{aligned} &c'(y; 1, 1) - c'(y; 0, 1) < c'(y; 1, 0) - c'(y; 0, 0), \text{ equivalently} \\ &c'(y; 1, 1) - c'(y; 1, 0) < c'(y; 0, 1) - c'(y; 0, 0), \end{aligned} \tag{D.44}$$

then we would expect to see higher y associated with $x^* = (1, 1)$. This can in turn lead to lower inventory costs since the production runs are shorter, and if keeping track of inventories is easier with a new computer system, ...

D.10. Exercises.

PROBLEM D.2. *Some comparative statics.*

a. *A biotech firm spends $x \geq 0$ researching a cure for a rare condition (for example, one covered by the Orphan Drug Act), its expected benefits are $B_1(x)$, the social benefits not capturable by the firm are $B_2(x)$, and both are increasing functions.*
   i. *Show that the optimal x is larger than the one the firm would choose.*
   ii. *Show that allowing the firm to capture more of the social benefits (e.g. by giving longer patents or subsidizing the research), governments can increase the x that the firm chooses.*

b. *An oil company owns the right to pump as high a flow of oil from their well located over one part of an underground sea of oil. As a function of the flow they choose, f_i, they make profits this year of $\Pi_i(f_i)$. The higher the flow chosen now, the higher the costs, $C_i(f_i)$, of pumping oil in the future (if you pump too hard, the small openings in the underground rock through which the oil flows begin to collapse). Higher flow also increases the future costs of the other oil companies pumping from the same underground sea. Show that the flow chosen by i is inefficiently high. (Oil fields often operate under what are called unitization agreements in order to solve these kinds of problems.)*

c. *One part of the business model of a consulting company is to hire bright young men and women who have finished their undergraduate degrees and to work them long hours for pay that is low relative to the profits they generate for the company. The youngsters are willing to put up with this because the consulting company provides them with a great deal of training and experience, all acquired over the course of the, say, three to five years that it takes for them to burn out, to start to look for a job allowing a better balance of the personal and professional. The value of the training that the consulting company provides is at least partly recouped by the youngsters in the form of higher compensation at their new jobs. Show that the consulting company is probably providing an inefficiently low degree of training.*

PROBLEM D.3. *If $f : X \times T \to \mathbb{R}$ is supermodular, then it is quasi-supermodular.*

PROBLEM D.4. *If $(x, t) \mapsto f(x, t)$ is q-supermodular and $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $(x, t) \mapsto \psi(f(x, t))$ is q-supermodular. If $(x, t) \mapsto f(x, t)$ is strictly q-supermodular and $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $(x, t) \mapsto \psi(f(x, t))$ is strictly q-supermodular. Suppose now that for each $t \in T$, $\varphi(\cdot, t)$ is strictly increasing, and show that the q-supermodularity of $(x, t) \mapsto f(x, t)$ implies the q-supermodularity of $(x, t) \mapsto \varphi(f(x, t), t)$.*

PROBLEM D.5. *For $x, t \in [11, 100]$, let $f(x, t) = xt$. Since $\partial^2 f/\partial x\partial t = 1$, this function has strictly increasing differences, and since $\partial f(x, t)/\partial x > 0$ for all x, t, $x^*(t) \equiv \{100\}$. Let $g(x, t) = \log(f(x, t)) = \log(x) + \log(y)$ and note that $\partial^2 g/\partial x\partial t = 0$, strictly increasing differences have disappeared, but $\partial g(x, t)/\partial t > 0$ for all x, t. Let $h(x, t) = \log(g(x, t))$, and $\partial^2 h/\partial x\partial t < 0$, strictly increasing differences have become decreasing differences, but $\partial h(x, t)/\partial x > 0$ for all x, t. The problems $\max_{x\in[11,100]} h(x, t)$ provide an example of strictly decreasing differences with a non-decreasing $x^*(\cdot)$.*

E. The Opportunity Cost of Capital

E.1. Summary. This is also called the time value of money. If you tie up money/capital, e.g. by putting it into new equipment, then that is money that you are not investing at whatever other rate of return you could be getting. In other words, the opportunity cost of money/capital is what it could be earning you if you used it someplace else, in the best possible someplace else.

The essential idea is that 1$ of capital now returns $(1 + r)$$ in one period if r is the per period **rate of return (ror)**. To put it another way, receiving 1$ one period in the future is only worth $\frac{1}{(1+r)}$$ right now. We now start looking at implications of this using the concept of the **net present value** of a flow of money. This is the key concept for valuing projects that will have costs and benefits spread over time, and projects that do not meet that description are rare indeed.

We will see both the discrete and the continuous version of the formulas, but mostly use the continuous version.

E.2. Discrete Discounting. We will call the time periods $t = 0, 1, 2, \ldots$ as 'years,' but one could work with weeks, months, days, decades, or whatever period makes sense for the situation at hand. These are discrete periods in time of equal length, and the discrete time discounting that we use involves sums of the form

$$\sum_{t=0}^{\infty} \rho^t B_t \text{ or } \sum_{t=0}^{\infty} \left(\frac{1}{(1+r)}\right)^t B_t \tag{E.1}$$

where $\rho := \frac{1}{(1+r)}$ is called the **discount factor**, the factor by which we discount the value of benefits or costs to be accrued in the future. The sum in equation (E.1) is called the **net present value (npv)** of the sequence of benefits, $B_0, B_1, B_2, \ldots$. The B_t can be positive or negative, depending e.g. on whether the benefits of the project is larger or smaller than the costs in period t.

The logic comes from noting that investing a quantity x at $t = 0$ returns $(1+r)x$ at $t = 1$, $(1+r)^2x$ at $t = 2$, $(1+r)^3x$ at $t = 3$, and so on and on, with the general answer being $(1+r)^t x$ at time t. Here $r > 0$, and we think of $100 \cdot r$ as the 'interest rate.' This means that receiving an amount y at some future $t = 1, 2, \ldots$ is only worth $x = y/(1+r)^t$ at $t = 0$. Setting $\rho = 1/(1+r)$ gives one of the rationales for studying sums of the form $\sum_{t=0}^{\infty} \rho^t B_t$.

Note that as well as depending on the sequence $B_0, B_1, B_2, \ldots$, the npv also depends on the discount factor: the closer ρ is to 1, that is, the smaller is r, the rate of return on capital, the more weight is given to the B_t's in the far future; the closer ρ is to 0, that is, the larger is r, the rate of return on capital, the more weight is given to the B_t in the near future.

E.2.1. *Geometric Sums.* With luck, you have seen geometric sums before, but even if you haven't, their basics are quite simple. The first observation is that $\sum_{t=0}^{\infty} \rho^t = \frac{1}{1-\rho}$. To see why, note that $\sum_{t=0}^{T} \rho^t = (1 + \rho + \rho^2 + \cdots + \rho^T)$, which implies that

$$\begin{aligned} (1-\rho) \cdot \left(\textstyle\sum_{t=0}^{T} \rho^t\right) = \\ (1 + \rho + \rho^2 + \cdots + \rho^T) + \\ (-\rho - \rho^2 - \cdots - \rho^T - \rho^{T+1}) = (1 - \rho^{T+1}). \end{aligned} \tag{E.2}$$

If $\rho < 1$, that is, if the rate of return is positive, then when T is large, ρ^{T+1} is approximately equal to 0. Putting this together,

$$(1-\rho)\left(\textstyle\sum_{t=0}^{\infty} \rho^t\right) = 1, \text{ rearranging, } \textstyle\sum_{t=0}^{\infty} \rho^t = \frac{1}{1-\rho}. \tag{E.3}$$

The formula in equation (E.2) has more implications and uses. For example, note that

$$\begin{aligned} \sum_{t=4}^{9} \rho^t &= \rho^4 + \rho^5 + \rho^6 + \rho^7 + \rho^8 + \rho^9 \\ &= \rho^4(1 + \rho^1 + \rho^2 + \rho^3 + \rho^4 + \rho^5) \\ &= \rho^4 \cdot \frac{(1-\rho^6)}{(1-\rho)} \end{aligned} \tag{E.4}$$

because $6 = (9-4)+1$. Personally, my impulse is to re-derive such formulas as needed rather than try to memorize them, this also helps me remember what I am trying to do. In any case, during any exams, the necessary formulas, and some unnecessary ones as well, will be provided.

E.2.2. *Up Front Costs, Backloaded Benefits.* From the pieces we can evaluate the net present value of a project that requires and investment of $C > 0$ for periods $t = 0, 1, \ldots, T-1$, and then returns a profit $B > 0$ for periods $T, T+1, T+2, \ldots, T+T'$. This has net present value

$$npv(\rho) = \left(\tfrac{1}{(1-\rho)}\right)\left[-C(1-\rho^T) + B\rho^T(1-\rho^{T'+1})\right]. \tag{E.5}$$

Mathematically, the easiest of the interesting cases has $T' = \infty$, that is, the project delivers a stream of profits, $B > 0$, that lasts for the foreseeable future. In this case,

$$npv(\rho) = \left(\frac{1}{(1-\rho)}\right)\left[B\rho^T - C(1-\rho^T)\right]. \tag{E.6}$$

There are two things to notice about this equation.

First, this equation cross 0 from below exactly once, at point we'll denote $\rho^\dagger$. Let r_{IRR} satisfy $\frac{1}{(1+r)} = \rho^\dagger$. The "$IRR$" stands for **internal rate of return**, this is the rate of return at which the project breaks even. Here are the rules that come from doing this calculation.

R.1 If the opportunity cost of capital, r, is greater than r_{IRR}, then it is *not* worth investing in the project.

R.2 If the opportunity cost of capital, r, is less than r_{IRR}, then it *is* worth investing in the project.

The second thing to notice about equation (E.6) is that if ρ is close enough to 1, that is, if r, the opportunity cost of capital, is low enough, then the net present value is positive. Here is one way to think about this result: if there aren't many productive alternatives around, it becomes worthwhile to invest in projects that only pay off in the far future.

There is another way to measure how "good" a project is, the **payback period**. Suppose that $B_0, B_1, B_2, \ldots$ is the expected stream of net benefits from a project, positive or negative. The payback period is the first T at which the running sum, $\sum_{t=0}^{T} \rho^t B_t$, gets and stays positive. At that point, everything that's been put into the project has been paid back (with interest), and the project has become a steady source of future profit.

It is an empirical observation that very few firms take on projects with payback periods any longer than 3 to 5 years, and that is an old figure that is probably an overestimate of present behavior. There are very few R&D projects that pay back their expenses over so short a time period. For example, Apple spends less than %3 of its yearly profits on R&D and has a product cycle requiring that the new products arrive frequently, guaranteeing that they are, mostly, small steps rather than large innovations — the really large innovations, microchips, the internet, touch screens, these all take a more serious investment of time and resources. Apple relies instead on the ruthlessly good design of products and interfaces, but uses technologies with origins almost exclusively in government funded research.

The conditions for innovation require what is perhaps best thought of as an industrial 'commons,' an idea with roots at least as old as the Enlightenment.

E.3. Commons Problems with a Dynamic Component. The term "commons" refers, historically in England, to the common grounds for a village, the area where everyone could run their sheep. The more sheep that are run on an area, the less productive it is. However, if I keep my sheep off to let it recover, all that will happen is that your sheep will benefit. As with many simple stories, this contains a very important truth, and Elinor Ostrom's work has examined the many varied and ingenious ways that people have devised to solve or circumvent this problem. However, it is a problem and it does need a solution. Here we are interested in common resources that pay off over long periods of time.

The first systematic analysis known to me came in the late 1600's, in the *Oisivités* of Louis XIV's defense minister, Sébastien Le Prestre de Vauban, who noted the following.

- Forests were systematically over-exploited in France, they are a public access resource, a commons.
- After replanting, forests start being productive in slightly less than 100 years but don't become fully productive for 200 years.
- Further, no private enterprise could conceivably have so long a time-horizon, essentially for discounting reasons.

From these observations, Vauban concluded that the only institutions that could, and should, undertake such projects were the government and the church. His calculations involved summming the **un-discounted benefits, delayed and large, and costs, early and small**, on the assumption that society would be around for at least the next 200 years to enjoy the net benefits.

E.3.1. *The American System of Manufactures.* At the end of the American Civil war, the U.S. government decided that it needed rifles with interchangeable parts. This required a huge change in technological competencies, the development of tools to make tools. The American system was also known, in the early days, as *armory practice*, it evolved in the government funded armories, spread from the armories in Springfield and Harper's Ferry to private companies, Colt in particular.

The private companies formed near where there were people with the needed competencies, you don't set up a firm to using specialized machinery too far from people who know how it works well enough to fix it. To put it another way, there was an industrial commons created and sustained by government expenditures, and in using this commons, private companies were able to flourish. The parallel between Apple's history and Colt's, separated by almost a century and a half, is striking.

The system was generalized to many other products, if you know how to do one thing really well, then you probably know how to do similar things pretty well (Singer sewing machines for example). The system itself, the idea of creating tools to create tools, then spread through Europe within a couple of decades at the beginning of the second Industrial Revolution, and has since taken over the world. This is but one example of a very general pattern, most of the really large innovations in the technologies now so widespread in electronics, computing, optics, agriculture, medicine, came from funding organized by and through the government. Venture capitalists, start-up firms, private firms in general, cannot, because of the opportunity cost of capital, have the patience to invest in these things. Vauban's core insights, that these long term projects are good for society and that only institutions with a very long time horizon can undertake them, these still resonate today.

E.3.2. *Other commons examples.* The essential observation is that markets generally underprovide goods when the benefits spill over to other people. If the maximizer pays attention to the problem

$$\max_x \; [B_1(x) - C(x)] \tag{E.7}$$

but society would pay attention to

$$\max_x \; [(B_1(x) + B_2(x)) - C(x)] \tag{E.8}$$

we know what will happen if $B_2(\cdot)$ is an increasing function.

Arthur Andersen

The consulting firm, Arthur Andersen, used to hire bright youngsters straight from college, work them incredibly hard, pay them not very much, but provide them with really valuable experience, a form of apprenticeship. When an employee has valuable skills, they can take those skills out on the market and earn a good salary. The Andersen apprenticeships earned the company money, they would not have behaved this way without it being profitable.

The benefits to their employees after they left, the $B_2(\cdot)$ above, were not particularly part of the firm's calculation in deciding how many people to train and how much to teach them, the x. Now, that's not quite correct inasmuch as the future benefits to employees who leave with valuable experience meant that the firm could continue to pay ambitious, hard-working young college graduates less than they were actually worth.

However, it has never been seriously suggested that firms take over the business of teaching the literacy and numeracy that underpins modern society. When the $B_2(x)$ includes these kinds of trainings, its value far exceeds what any firm can aspire to. Further, because the training takes decades to complete, no firm can afford it given the opportunity cost of capital arguments, not even if they could recoup the expenses with some form of slavery.

Microchips

Venture capitalists and technology firms were very late, and reluctant entrants into the business of making microchips. The original demand for such a product came from NASA (the National Aeronautics and Space Administration), who wanted very light, very small computing power that would survive some really awful conditions. NASA not only paid for the research into the feasibility of such a creation, it provided a guaranteed market for the first chips, paying $1,000 apiece for chips that soon cost only $20 to $30 apiece. The benefits of going through this learning by doing process have given us the computer age.

Internet

Think DARPA.

Material sciences

A silly example is velcro, more substantive historically, think transistors, silicon chips, touch-sensitive glass screens for modern computers. Underlying all of these, and the many examples not mentioned, is the post-graduate education system, producing science Ph. D.'s in a system heavily subsidized by the government. Again, this an investment that will not payoff for so long that the internal rate of return (IRR) is not sustainable for a commercial firm, and the benefits are not recoverable unless one starts down a system starting with indentured servants and ending in slavery.

Agricultural sciences

Hybrid crops were mostly, and the original techniques for genetic modification of crops were nearly completely, funded by governments interested in the long-term welfare of society.

E.4. Continuous Discounting. Continuous discounting makes the calculation of optimal timing of decisions and other continuous timing decisions much easier. These involve expressions involving the terms e^{rt} and/or e^{-rt} where $e \simeq 2.718281828459\ldots$ is the basis of the natural logarithms. The crucial competence to develop is the ability to recognize when you need which of the various formulas below, but the development of these formulas will, of necessity, make reference to facts you are not likely to have seen without a calculus class.

E.4.1. *Wherefore art thou e^{rt}?* From your calculus class, you should have been exposed to the following,

$$\lim_{n\to\infty} \left(1 + \tfrac{x}{n}\right)^n = e^x. \tag{E.9}$$

Taking $x = rt$ delivers e^{rt} as the limit in the above expression, but we do not here need the formal definition of a limit, just the idea that e^x is a very good approximation to $\left(1 + \frac{x}{n}\right)^n$ when n is moderately large.

If the interest rate is 6% per year, it is 6/4% per fiscal quarter, 6/12% per month, 6/52% per week, and 6/365%. If you calculate

- the quarterly rates of return four times over the course of the year where you re-invest all of your quarterly earnings, you make $(1 + \frac{0.06}{4})^4$ on each dollar invested at the beginning of the year,
- the monthly rates of return twelve times over the course of the year where you re-invest all of your monthly earnings, you make $(1 + \frac{0.06}{12})^{12}$ on each dollar invested at the beginning of the year,
- the weekly rates of return 52 times over the course of the year where you re-invest all of your weekly earnings, you make $(1 + \frac{0.06}{52})^{52}$ on each dollar invested at the beginning of the year,
- the daily rates of return 365 times over the course of the year where you re-invest all of your daily earnings, you make $(1 + \frac{0.06}{365})^{365}$ on each dollar invested at the beginning of the year.

It turns out that 365 is close enough to ∞, that is, 365 is large enough that the difference between $(1 + \frac{0.06}{365})^{365}$ and $e^{0.06}$ doesn't matter very much. If the difference is still to big, figure out how many seconds there are in a year and work with that. Once we get close enough to that limit out at ∞, we call it **continuous compounding** or **continuous time growth**, and most modern calculators have the button allowing you to calculate e^{rt} or $1/e^{rt} = e^{-rt}$ for any relevant r and t.

In continuous time, investing a quantity x at $t = 0$ returns $x \cdot e^{rt}$ at time $t > 0$. This means that receiving an amount y at $t > 0$ is only worth $y/e^{rt} = ye^{-rt}$ at $t = 0$. It is this continuity of time that makes these calculations so useful for optimal timing problems. This is not a deep or subtle observation, rather it is the observation that if one is summing $\sum_{t=0}^{T} \rho^t B_t$ and optimizing over T or considering a choice of a t that optimizes $\Pi(t)\rho^t$, having the time intervals be discrete, say years, means that the mathematical formulation of the problem can only return answers in years, whereas the true best timing might be "next September," at $T = 3/4$ roughly.

E.4.2. *The appearance of the integral.* If you have a function of time t, say $h(t) > 0$ for $a \leq t < b$, then the area between the curve and the time axis and bounded between the times a and b is denoted

$$\int_a^b h(t)\,dt \text{ or } \int_a^b h(x)\,dx. \tag{E.10}$$

There are two extra subtleties, one involving negative areas and the other involving time flows that stretch out into the indefinite future.

S.1 When $h(t) < 0$, we count the area between the curve and the time axis as being negative. For example, if $h(t) = -9$ for $3 \leq t < 9$ and $h(t) = 10$ for $9 \leq t < 23$, $\int_3^{23} h(t)\,dt$ is equal to $-9 \cdot (9-3) + 10 \cdot (23-9)$, that is $\int_3^{23} h(t)\,dt = -54 + 140 = 86$.

S.2 When we have benefits, $h(t)$, starting now, at $t = 0$, and extending off into the indefinite future, we will write $\int_0^\infty h(t)\,dt$ for their value. You should worry, for just a couple of seconds, about the problem that the area under a curve that stretches off forever is going to be infinite. This is one problem that discounting solves.

Another pair of observations: the areas under curves add up; and if we multiply a function by a constant, the area under the curve is multiplied the same constant.

O.1 The area under the curve $h(t)$ between a and b plus the area under the curve $g(t)$ between b and c is

$$\int_a^b h(t)\,dt + \int_b^c g(t)\,dt. \tag{E.11}$$

O.2 The area under the curve $2 \cdot h(t)$ between a and b is

$$\int_a^b 2 \cdot h(t)\,dt = 2 \cdot \int_a^b h(t)\,dt. \tag{E.12}$$

E.4.3. *Detour$^{\boldsymbol{Adv}}$*. The expressions in equation (E.10) are called "integrals," historically, the symbol "$\int$" comes from the letter capital "S," standing for "summation." A slightly more subtle aspect of these problems is that we change summations into integrals. To understand why this happens, let us think of the B_t in $\sum_{t=0}^\infty \rho^t B_t$ as being a flow, the benefits B_t are the benefits **per year** (or per day/week/month/etc. as appropriate). This means that $\sum_{t=0}^\infty \rho^t B_t$ can be rewritten as

$$\textstyle\int_0^1 \rho^1 B_1\,dt + \int_1^2 \rho^2 B_2\,dt + \int_2^3 \rho^2 B_2\,dt + \cdots. \tag{E.13}$$

In continuous time, the flow of benefits can vary with time, that is, it need not be the step functions of (E.13), B_1 for all $0 \leq t < 1$, B_2 for all $1 \leq t < 2$, etc. Further, the discount factor in continuous time is not the step function ρ^1 for all $0 \leq t < 1$, ρ^2 for all $1 \leq t < 2$, etc. Rather, it is e^{-rt} at $t > 0$. Putting these together, the continuous time net present value of a flow $q(t)$ is

$$\textstyle\int_0^T q(t)e^{-rt}\,dt \text{ or } \int_0^\infty q(t)e^{-rt}\,dt. \tag{E.14}$$

E.4.4. *Useful formulas.* For simplicity, we will mostly study flows of net benefits that start negative, say at $-C$, for a time period from 0 to T, and then turn positive, say at B, from T on into the indefinite future. The **continuously compounded net present value (npv)** of such a stream of benefits is

$$npv(r) = \int_0^T (-C)e^{-rt}\,dt + \int_T^\infty Be^{-rt}\,dt. \tag{E.15}$$

"Mostly" does not mean "always," we will also study that case that we can forsee the end of the benefits, i.e. we will care about expressions of the form

$$npv(r) = \int_0^T (-C)e^{-rt}\,dt + \int_T^\tau Be^{-rt}\,dt \tag{E.16}$$

where $\tau > T$ is the time at which the benefits B stop accruing.

Here are the useful formulas, these contain the basics of integrating continuously discounted piece-wise constant flows, and give the comparisons with discrete time discounting (recall that $\rho = \frac{1}{(1+r)}$ in these comparisons).

F.1 $\int_0^\infty e^{-rt}\,dt = \frac{1}{r}$, which you should compare to $\sum_{t=0}^\infty \rho^t = \frac{1}{(1-\rho)}$;

F.2 $\int_0^T e^{-rt}\,dt = \frac{1}{r}(1-e^{-rT})$, which you should compare to $\sum_{t=0}^T \rho^t = \frac{1}{1-\rho}(1-\rho^{T+1})$;

F.3 $\int_a^b e^{-rt} = \frac{1}{r}(e^{-ra} - e^{-rb}) = \frac{1}{r}e^{-ra}(1-e^{-r(b-a)})$ which you should compare to $\sum_{t=\tau}^T \rho^t = \frac{1}{1-\rho}\rho^\tau(1-\rho^{(T-\tau)+1})$.

Going back to the expression in (E.15), we have $npv(r) = \frac{1}{r}(-C)(1+e^{-rT}) + \frac{1}{r}Be^{rT}$. As in the discrete case, the IRR is the r that solves the equation $npv(r) = 0$. You should verify that the IRR solves $e^{-rT} = \frac{C}{B+C}$. Taking logarithms allows you to solve for the IRR. You should go through what different values of B, C and T do to the IRR, and ask yourself why your answers should make sense.

E.5. The Product Launch Decision. One of the decisions in for a new product is when to launch it. Waiting longer means that more glitches can be ironed out, the tradeoff is the time value of money, and/or that someone else may launch a competing product before you do. One has the same kinds of tradeoffs in the decision of when to submit a new invention to be patented, or when to harvest a stand of timber. We are going to first look at this problem from an opportunity cost point of view, then add to it the complication that a competitor might preempt your innovation. This last idea is the beginning of the study of decisions in the face of uncertainty, the beginnings of looking at the expected net present value. In this case, it turns out to effect the appropriate rate of discount that you should use in evaluating the npv, and this in turn contains information about how a really large organization with a long time horizon should discount.

E.5.1. *Rate of growth equal to the time rate of discounting.* Let $\pi(t)$ be the value of moving at time $t > 0$. We expect that $\pi(\cdot)$ will grow over time, $\pi'(t) > 0$, and that optimal timing will certainly involve moving before $\pi'(\cdot)$ turns and stays negative. We also expect that $\pi'(q)$ will start pretty high, but that eventually at least, $\pi'(q)$ will start to go down, that is $\pi''(q) < 0$.

If one receives value $\pi(t)$ at time t and the opportunity cost of capital, or the time rate of discounting, is r, then the npv is $\pi(t)e^{-rt}$. Let us first go through a discrete analysis of the condition that an optimal t must solve. If we are at time t and $dt > 0$ is a small number, then we need to consider whether to move at t or to delay until $t + dt$. What we expect is that near the optimal t, at/near the top of the curve $\pi(t)e^{-rt}$, a tiny delay will make no difference, that is, we expect that at the optimal t,

$$\pi(t + dt)e^{-r(t+dt)} \simeq \pi(t)e^{-rt} \tag{E.17}$$

with the equality becoming exact as dt shrinks toward 0. Rearranging yields

$$\frac{\pi(t + dt)}{\pi(t)} = \frac{e^{-rt}}{e^{-r(t+dt)}} = e^{r\cdot dt}. \tag{E.18}$$

Now we use the observations that $\pi'(t)$ is the slope of π near t, that r is the slope of e^{rt} near 0, and that dt is very small. This yields

$$\frac{\pi(t^*) + \pi'(t^*) \cdot dt}{\pi(t^*)} = 1 + r \cdot dt. \tag{E.19}$$

Rearranging, this yields $\frac{\pi'(t^*)}{\pi(t^*)} \cdot dt = r \cdot dt$, that is, at the optimal t^*, we will have $\frac{\pi'(t^*)}{\pi(t^*)} = r$.

Now $\pi'(t)/\pi(t)$ is the rate of growth of profits, and r is the rate of growth of the value of invested capital, or the time rate of discounting. This derivation has given us the optimal rule is to launch the project when its rate of growth slows down to the rate of growth of the value of invested capital. Before that time, delay increases profits, after that point, delay decreases profits. Summarizing,

> Move when the rate of growth of profits has slowed to the time rate of discounting.

Of interest is the question of how changes in the value of r change the optimal t. The easiest way to get at this is to use the decreasing differences result *after* taking logarithms — logarithm is a strictly increasing function, so maximizing $\ln(u(x))$ is the same as maximizing $u(x)$, hence we are solving the problem

$$\max_t \ \ln(\pi(t)e^{-rt}) = \max_t \ \ln(\pi(t)) + \ln(e^{-rt}) = \max_t \ \ln(\pi(t)) - rt. \tag{E.20}$$

Considering the utility function $u(t, r) = \ln(\pi(t)) - rt$, we note that it has decreasing differences in t and r. This means that increases in r decrease t, that is, increases in the opportunity cost of capital make one act more impatiently, move sooner.

E.5.2. *The calculus based analysis* ***Adv***. The previous derivation of the optimal time for product launch was cumbersome, and those of you have had a calculus course should recognize that we basically took a derivative and set it equal to 0. Let us now proceed in a slightly more sophisticated fashion now. The problem and the necessary derivative conditions for an optimum are

$$\max_{t\geq 0} \ u(t, r) = \pi(t)e^{-rt} \text{ and } \frac{\pi'(t^*)}{\pi(t^*)} = r. \tag{E.21}$$

i. To make the derivation easier, take logarithms before maximizing, that is, solve $\max_{t\geq 0} \log(u(t, r)) = \log(\pi(t)) + \log(e^{-rt})$. One has maximized $u(t, r)$ iff one has maximized its logarithm, hence we need only study this problem. Since $\log(e^{-rt}) = -rt$, we see that $\partial \log(u(t, r))/\partial t = 0$ is exactly the derivative conditions given in equation (E.21).
ii. Notice that $\log(u(t, r))$ has decreasing differences in t and r, so that increases in r, the cost of capital, decrease the amount of delay one will optimally incur. The optimal degree of impatience goes up as the time value of money goes up.

iii. The term $\frac{\pi'(t^*)}{\pi(t^*)}$ is the rate of growth of profits at t^*, leaving us with the following policy:

Move when the rate of growth of profits has slowed to the time rate of discounting.

One can also formulate this in terms of opportunity cost: the gain in profits to delaying a small bit of time, dt, at time t is $\pi'(t)dt$; the loss is the money that you could have earned from having $\pi(t)$, that is, $\pi(t) \cdot \left[e^{r(t+dt)} - e^{rt}\right] dt$. Since $\left[e^{r(t+dt)} - e^{rt}\right] = e^{r \cdot dt}$ and for small dt this is equal to $r \cdot dt$, the gains and losses from delay at t are equal when $\pi'(t^*) = \pi(t^*)r$, once again recovering the optimal policy we just gave.

E.5.3. *Rate of growth greater than the time rate of discounting.* We have seen that not counting all of the benefits or costs to an action can mean that you've got the wrong action. One of the opportunity costs of keeping a team working on a product for release is that they have not moved on to the next project. When thinking about two projects in sequence, the problem becomes

$$\max_{t_1, t_2 \geq 0} u(t_1, t_2, r) = \pi(t_1)e^{-rt_1} + \pi(t_2)e^{-r(t_1+t_2)}. \tag{E.22}$$

The strong intuition is that one should take the team off of the project where π' has gotten low and put them onto the project where π' is high. In other words, because the opportunity cost of keeping the team on the first project is that they are not working on the second project, we expect that the optimal t_1 should be earlier in this case. Here is a good way to show that this intuition is correct.

For t_2 fixed at any value t_2°, consider the two problems

$$\max_{t_1 \geq 0} f(t_1, \theta) = \pi(t_1)e^{-rt_1} + \theta\pi(t_2^\circ)e^{-r(t_1+t_2^\circ)}, \ \theta = 0, 1. \tag{E.23}$$

When $\theta = 0$, we have the problem in equation (E.21), when $\theta = 1$, we have the problem in equation (E.22). It is easy to check (and you should do it) that $f(\cdot, \cdot)$ has decreasing differences in t_1 and θ, hence one should be more impatient about the initial product launch.

E.5.4. *Learning by Doing.* Now suppose that the longer a team works on the first project, the better they get at product development. To capture this, we suppose that every instant of time that they are on the first project, there is a flow benefit of B. Now the problem is

$$\max_{t_1 \geq 0} u(t_1, r) = \pi(t_1)e^{-rt_1} + \int_0^{t_1} Be^{-rx}\, dx. \tag{E.24}$$

The strong intuition is that if you are getting benefit from having the team continuing to work on the first product, then you should keep them at it longer. Here is a good way to show that this intuition is correct. Consider the two problems

$$\max_{t_1 \geq 0} f(t_1, \theta) = \pi(t_1)e^{-rt_1} + \theta \int_0^{t_1} Be^{-rx}\, dx, \ \theta = 0, 1. \tag{E.25}$$

When $\theta = 0$, we have the problem in equation (E.21), when $\theta = 1$, we have the problem in equation (E.24). It is easy to check (and you should do it) that $f(\cdot, \cdot)$ has increasing differences in t_1 and θ, hence one should be more patient about the initial product launch.

E.5.5. *Risk adjusted timing.* Suppose now that a competitor may get their product launched, or get their patent application in, before you do. To make this easy as a first pass, suppose that the probability that they have launched by t is given by the cumulative distribution function $F(t) = 1 - e^{-\lambda t}$. Let us further suppose that if they pre-empt you, you get nothing. If you launch at t, then the probability that they have have *not* launched is $1 - F(t)$, that is, $e^{-\lambda t}$. Now the problem is

$$\max_{t_1 \geq 0} \; u(t_1, r, \lambda) = [\pi(t_1)e^{-rt_1}]e^{-\lambda t_1}. \tag{E.26}$$

This is nearly the problem in equation (E.21), and the solution involves $\pi'(t^*)/\pi(t) = (r + \lambda)$, rational worry about being pre-empted makes you act as if the time value of money was higher.

Taking logarithms, we can also see this by comparing the two problems,

$$\max_{t_1 \geq 0} \; f(t_1, \theta) = \log(\pi(t_1)e^{-rt_1}) + \theta \log(e^{-\lambda t}), \;\; \theta = 0, 1. \tag{E.27}$$

This is submodular/has decreasing differences in t_1 and θ, the optimal t_1 is smaller for the problem with $\theta = 1$, that is, for the problem in equation (E.26).

E.5.6. *Pre-mortems and other prescriptive lessons.* What these various alternative scenarios have done is to give a sense of the kinds of factors that should enter into an optimal timing decision. One of the better ways to get people thinking about what factors should matter is to have what is called a **pre-mortem meeting**. The roots of the "postmortem" are "post" for "after" and "mortem" for "death." In murder mysteries, there is almost always a postmortem examination of the corpse to discover the causes of death. Doing a pre-mortem is, in somewhat ghoulish terms, examining the corpse before it is dead.

The idea of a pre-mortem meeting for a project or a decision is that everyone is told to come in the next day (or next week as appropriate) on the assumption that it is now a year later and the decision was an absolute disaster. The fictional point of the meeting is to provide explanations for what could go wrong. Just the process of thinking, ahead of time, about how you could avoid disaster/losses, often jiggles the creative process. This can get people thinking more sensibly of not only of whether or not to abandon the project, but also how to improve the project. One summary of what we should do in dynamic decision problems is to

Look forward and solve backwards.

In somewhat less aphoristic form, you should look forward to try to anticipate what may or may not happen, and on the basis of these considerations, work backwards to what you should be doing right now.

The analyses above suggest different kinds of opportunity costs/benefits that may have been overlooked. By asking "What would make us move up the launch?" "What would make us delay the launch?" and even "What would make us regret having started this project?" one can improve decisions.

E.6. Exercises.

PROBLEM E.1. *A project accumulates costs at a rate C for the interval $[0, T]$, measured in years, then accumulates benefits, B, in perpetuity, money is discounted continuously at rate r where $r = 0.12$ corresponds to an interest rate of 12% per annum. Fill in the 8 (eight) blank entries in the following table where "npv(r)" stands for the net present value at interest rate r.*

C	B	T	r	$npv(r)$
10	15	3	0.12	
10	15	3	0.18	
10	15	3	0.24	
10	15	3		0
20	75	8	0.12	
20	75	8	0.18	
20	75	8	0.24	
20	75	8		0

PROBLEM E.2. *You take out a loan for L agreeing to payback at a rate x per year over the course of T years. Interest is continuously compounded at rate r so that $L = \int_0^T xe^{-rt}\,dt$.*

a. *Find the payback rate, x, as a function of L, T, and r. Explain the intuitions for why x should depend in the fashion that it does on these three variables.*
b. *Find the necessary payback time T, as a function of x, L, and r. Explain the intuitions for why T should depend in the fashion that it does on these three variables, paying special attention to the case that there is no T solving the problem.*
c. *Now suppose that bank that is lending you the money believes that your business will fail with probability λdt in any given small interval of time $[t, t + dt)$. Let τ be the random time until you fail, i.e. $P(\tau \leq t) = 1 - e^{-\lambda t}$. If the bank wants to set x such that the expected valued of your repayments until you fail is L, i.e. $E \int_0^\tau x^{-rt}\,dt = L$, find the expected payback rate, x, as a function of L, T, r and λ. [This is one version of what are called risk premia, that is, the extra that someone in a riskier situation must pay.]*

PROBLEM E.3. *A project will cost 10 per year for the first three years, and then return benefits of 8 per year in perpetuity (i.e. for the foreseeable future). Throughout, suppose that interest is continuously compounded.*

a. *Find the internal rate of return (IRR) on this project, that is, find the r for which the net present value is equal to 0.*
b. *Find the payback period at interests rates of 10%, 20%, and 30%.*

PROBLEM E.4. *More problems on continuously compounded interest.*

a. *You pay back a principal P at an interest rate r over a period of T years. Give the ratio of P to the total dollars you pay back as a function of r for a fixed T. Explain why the answer should work this way.*
b. *You pay back a principal P at an interest rate r over a period of T years but you have a year with no repayments, that is, your payback period is from year 1 to year $T + 1$. Give the ratio of P to the total dollars you pay back as a function of r for a fixed T. Explain how and why your answer should differ from the one in previous part of this problem.*
c. *You pay back a principal P at an interest rate r over a period of T years. Give the ratio of P to the total dollars you pay back as a function of T for a fixed r. Explain why the answer should work this way.*
d. *You pay back a principal P at an interest rate r over a period of T years but you have a year with no repayments, that is, your payback period is from year 1 to*

year $T+1$. Give the ratio of P to the total dollars you pay back as a function of T for a fixed r. Explain how and why your answer should differ from the one in previous part of this problem.

F. Economies of Scale and Scope

When one doubles the diameter of a pipe, its cross-section increases by a factor of four, meaning that, at any given flow speed, four times as much can be pumped through. More generally, multiplying the diameter by x gives x^2 in capacity. However, the thickness of the pipe, a measure of the cost of material in the pipe, does not need to be increased by x^2, meaning that the average cost of pipe capacity goes down as a power of the diameter of the pipe.[8] Power laws appear in economics in many places, here we will look at them for simple inventory systems.

When one looks at statistics measuring the competence with which firms are run, after adjusting for the industry, one finds a weak effect in favor of firms with female CEO's, and a much stronger effect in favor of larger firms. We are now going to ivestigate a different advantage of being large, the decreasing average cost aspect of simple inventory systems. Decreasing average costs sometimes go by the name of economies of scale, and economies of scale are a crucial determinant of the horizontal boundary of a firm, and sometimes the vertical boundary.

Your firm needs Y units of, say, high grade cutting oil per year. Each time you order, you order an amount Q at an ordering cost of $F + pQ$, where F is the fixed cost of making an order (e.g. you wouldn't want just anybody to be able to write checks on the corporate account and such sytems are costly to implement), and p is the per unit cost of the cutting oil. This means that your yearly cost of ordering is $\frac{Y}{Q} \cdot (F + pQ)$ because $\frac{Y}{Q}$ is the number of orders per year of size Q that you make to fill a need of size Y.

Storing anything is expensive, and the costs include insurance, the opportunity costs of the space it takes up, the costs of keeping track of what you actually have, and so on. We suppose that these stockage costs are s per unit stored. Computerized records and practices like bar-coding have substantially reduced s over the last decades. Thus, when you order Q and draw it down at a rate of Y per year, over the course of the cycle that lasts Q/Y of a year, until you must re-order, you store, on average $Q/2$ units. This incurs a per year cost of $s \cdot \frac{Q}{2}$. Putting this together, the yearly cost of running an inventory system to keep you in cutting oil is

$$C(Y) = \min_Q \left[\frac{Y}{Q} \cdot (F + pQ) + s \cdot \frac{Q}{2} \right], \tag{F.1}$$

and the solution is $Q^*(Y, F, p, s)$.

a. Without actually solving the problem in equation (F.1), find out whether Q^* depends positively or negatively on the following variables, and explain, in each case, why your answers makes sense: Y; F; p; and s.

[8] One sees the same effect in sports where wind/water resistance is crucial. Going downhill on a bicycle, the force exerted by gravity is proportional to weight, that is, roughly, to the cube of the cross-sectional width of the cyclist. Wind resistance increases with the cross-sectional area, roughly, the square of the cross-sectional width of the cyclist. Lung surface area increases, again roughly, as a power less than three of the diameter of the cyclist, while the volume/weight increases, again roughly, as a cube of the diameter. Smaller cyclists rule on the climbs, larger cyclists on flat courses and downhills.

Ans: Let $f(Q;Y,F,p,s) = \left[\frac{Y}{Q}\cdot(F+pQ)+s\cdot\frac{Q}{2}\right]$ and check for increasing or decreasing differences in: Q and Y; Q and F; Q and p; and Q and s. In each case, isolate those parts of the function f that involve only Q and the variable of interest. Rewriting f is

$$f(Q;Y,F,p,s) = \left[\frac{FY}{Q}+pY+s\cdot\frac{Q}{2}\right],$$

Q and Y: only appear together in the $\frac{FY}{Q}$ term, easily seen to have decreasing differences, since this is minimization, $[Y\uparrow]\Rightarrow[Q^*\uparrow]$. This makes sense because needing to fill a larger yearly need should require larger orders.

Q and F: only appear together in the $\frac{FY}{Q}$ term, easily seen to have decreasing differences, since this is minimization, $[F\uparrow]\Rightarrow[Q^*\uparrow]$. This makes sense because higher fixed costs of ordering means that you want to do it less often, and this means that each order should be larger.

Q and p: do not appear together, cannot tell anything, but the economics of fixed costs should tell you that increases in p have no effect on Q^*.

Q and s: only appear together in the $s\frac{Q}{2}$ term, easily seen to have increasing differences, since this is minimization, $[s\uparrow]\Rightarrow[Q^*\downarrow]$. This makes sense because higher storage costs mean that you'd like to be storing less, and this requires smaller orders.

b. Now, using the result that $\frac{d}{dQ}(\frac{1}{Q}) = -\frac{1}{Q^2}$, explicitly solve for $Q^*(Y,F,p,s)$ and $C(Y)$.

Ans: Taking the derivative of $f(Q;Y,F,p,s)$ with respect to Q yields $f'(Q) = -\frac{FY}{Q^2}+\frac{s}{2}$, setting this equal to 0 gives the optimal tradeoff between the fixed costs ordering and inventory costs. Now, $-\frac{FY}{Q^2}+\frac{s}{2}=0$ requires $Q^*(Y,F,s)=\sqrt{Y}\cdot\sqrt{2F/s}$. Putting this value back into $f(Q;Y,F,p,s)$ yields, after simplification,

$$C(Y) = \sqrt{Y}\cdot\sqrt{2Fs}+pY.$$

c. Using the result that $\frac{d}{dY}(\sqrt{Y}) = \frac{1}{2\sqrt{Y}}$, find the marginal cost of an increase in Y. Verify that the average cost, $AC(Y)$, is decreasing and explain how your result about the marginal cost implies that this must be true.

Ans: $MC(Y) = C'(Y) = \frac{1}{2\sqrt{Y}}+p$ which is a decreasing function of Y. $AC(Y) = C(Y)/Y = \sqrt{2Fs}/\sqrt{Y}+p$, a decreasing function. Decreasing marginal costs of inventories imply that every unit of Y you need costs less than the previous unit, average costs include the average of all previous units, hence $AC(Y)$ must be decreasing.

d. With the advent and then lowering expenses of computerized inventory and accounting systems, the costs F and s have both been decreasing. Does this increase or decrease the advantage of being large?

Ans: The per unit cost advantage of being large enough that you need Y' rather than the smaller Y is $AC(Y)-AC(Y') = \sqrt{2Fs}\left[\frac{1}{\sqrt{Y}}-\frac{1}{\sqrt{Y'}}\right]$. If $Fs\downarrow$, this difference becomes smaller, making it easier for the smaller firm to compete with the larger one.

CHAPTER II

Decision Theory Under Uncertainty

A. Expected Utility Theory

THIS SECTION IS DRASTICALLY INCOMPLETE, IGNORE IT FOR NOW

A.1. Summary. Risky problems are one in which chance intervenes between the choices that are made and the eventual outcome, and in which the workings of chance are well enough understood that they can be described by a probability. When X is a random outcome taking the values x_i with probability p_i, $i = 1, \ldots, n$, the **expected utility of** X is $E\,u(X) = \sum_i u(x_i)p_i$. We assume that people choose between risky options on the basis of their expected utility. This is the same as assuming that people choose between risky options in a fashion that is independent of irrelevant alternatives.

A.2. A Simple Example. There is a probability $r > 0$ of your warehouse burning down. For a simple first time through, let us suppose that if it burns down, all that happens is that you have lost the value $V > 0$. By paying a price P, you can have the losses made up to you. Your wealth right now is W. You can choose to buy the insurance, in which case your wealth will be $(W - P)$ if the warehouse does not burn down, and it will be $(W - P) - V + V = (W - P)$ if it does burn down. You can choose not to buy the insurance, in which case your wealth will be W with probability $(1 - r)$ (if the warehouse does not burn down), and will be $W - V$ with probability r (if the warehouse does burn down). The choice is between two outcomes, certainty of $(W - P)$ and the random distribution over wealth just described.

To make the example more realistic and useful, we would want to consider how you react if you are fully insured, maybe you'll be less careful with the sprinkler system maintenance, how the insurace company will react to this kinds of problems, maybe by writing policies that only pay out if you have provably kept up your inspections.

A.3. Another Simple Example. Enter or not (details given in lecture).

A.4. Random Outcomes. Often, you will see these called "lotteries" in other textbooks. I dislike the connotations of that word, so will use the less euphonious "random outcomes." Throughout this section, we make our mathematical lives easier and assume that there is a finite set of possible outcomes, $\mathcal{X} = \{x_1, \ldots, x_n\}$.

Definition II.1. *A **random outcome** is a probability distribution on $\mathcal{X}$, that is, it is a vector $p = (p_1, \ldots, p_n)$, satisfying $p_i \geq 0$ and $\sum_i p_i = 1$.*

The set of random outcomes is denote $\Delta(\mathcal{X})$, sometimes just Δ. The outcomes x_i are equivalent to the vectors $e_i \in \Delta$ given by $e_i = (0, \ldots, 0, 1, 0, \ldots 0)$ where the 1 appears in the i'th position in the vector. Sometimes the probability e_i is denoted δ_{x_i}, which is called point mass on x_i.

A.5. Axioms and a Representation Theorem. We are after a representation theorem for preferences, $\succsim$, on Δ.

Axiom 1. *The preferences $\succsim$ are **rational**, that is, complete and transitive.*

Recall that completeness is the property that for every $p, q \in \Delta$, either $p \succ q$, or $q \succ p$ or $p \sim q$. Transitivity is the property that for every $p, q, r \in \Delta$, $[\,[p \succsim q] \wedge [q \succsim r]\,] \Rightarrow [p \succsim r]$.

Axiom 2. *Preferences are **continuous**, that is, $[p \succ q \succ r] \Rightarrow (\exists \alpha \in (0,1))[q \sim \alpha p + (1-\alpha) r]$.*

Rational and continuous preferences have the property that $[\,[p \succ q] \wedge [\alpha > \beta]\,] \Rightarrow [\alpha p + (1-\alpha) q \succ \beta p + (1-\beta) q]$. In other words, more weight on strictly better distributions strictly increases utility.

Axiom 3. *Preferences are **independent of irrelevant alternatives**, that is, for all $p, q, r \in \Delta$ and for all $\alpha \in (0,1)$,*

$$p \succsim q \text{ iff } (\alpha p + (1-\alpha) r) \succsim (\alpha q + (1-\alpha) r). \tag{A.1}$$

Rational, continuous preferences satisfying independence have the property that indifference surfaces are parallel: if $p \sim p'$, then for all $s \in \mathbb{R}$ and all $q \in \Delta$, if $q' := q + s(p - p') \in \Delta$, then $q' \sim q$.

Theorem II.1 (Von Neumann and Morgenstern). *Preferences on Δ are rational, continuous and satisfy indepedence iff there exists $u : \mathcal{X} \to \mathbb{R}$ with the property that $p \succsim q$ iff $\sum_{x \in \mathcal{X}} u(x) p(x) \geq \sum_{x \in \mathcal{X}} u(x) q(x)$.*

The function $u(\cdot)$ is called a **von Neumann-Morgenstern (vNM)** utility function or a **Bernoulli** utility function. If u has the property given in this result, then so does the function $v : \mathcal{X} \to \mathbb{R}$ defined by $v(x) = a + bu(x)$ for any $b > 0$.

Proof. If for all $x, x' \in \mathcal{X}$, $x \sim x'$, then set $u(x) \equiv 0$. Otherwise, let x a best point mass in Δ and x' a worst point mass. For any $p \in \Delta$, $x \succsim p \succsim x'$. By continuity, there exists a unique $\lambda_p \in [0,1]$ such that $p \sim \lambda_p x + (1-\lambda_p) x'$. Define $u(p) = \lambda_p$. $\square$

A.6. Expectations. For $p \in \Delta(\mathcal{X})$ and $u : \mathcal{X} \to \mathbb{R}$, $E^p u = \sum_x u(x) p(x)$ is the **expectation**.

Notation for $u(x) = x$ and $\mathcal{X} \subset \mathbb{R}$, is $E^p X$ where $P(X = x) = p(x)$. For $u(x) = x^2$, we use X^2, etc. Variance, standard deviation.

A.7. First and Second Order Stochastic Dominance.

A.8. Infinite Sets of Consequences$^{\mathbf{XAdv}}$. Let us first do this for $\mathcal{X} = [0, M] \subset \mathbb{R}$ or $\mathcal{X} = [0, \infty]$, then for yet more general spaces.

A **cumulative distribution function (cdf)** is a function $F : [0, M] \to \mathbb{R}$ with the interpretation that $P([0, x]) = F(x)$ where P is a probability distribution on $[0, M]$. From this, we have $P((a, b]) = F(b) - F(a)$, and for any finite disjoint union of intervals $(a_i, b_i]$, we have $P(\cup_i (a_i, b_i]) = \sum_i P((a_i, b_i])$.

For $x < x'$, we must have $F(x) \leq F(x')$, cdf's are non-decreasing. We assume that cdf's are **continuous from the right**, that is, for all $x \in [0, M]$, $F(x) = \lim_{x_n \downarrow x} F(x_n)$. If this were not true, then there would exist an $x \in [0, M]$ with $P([0, x]) < \inf_{x' > x} P([0, x'])$. For $\inf_{x' > x} P([0, x']) - P([0, x]) > 0$, there would have to exist points to the right of x yet strictly less than any $x' > x$. There are many models of $\mathbb{R}$ for which such points exist, but many people write down models of probabilities with this kind of property without saying that they are using such a model nor specifying which one they are using. Modeling without being explicit about what you are discussing can run you into all kinds of difficulties, and this particular lack of explicitness has generated a whole raft of non-sense.

If $F(x) > \lim_{x_n \uparrow x} F(x_n)$, then $F(\cdot)$ has a jump of size $p_x := \sup_{x' < x} F(x') - F(x)$ at x. The interpretation is that $P(\{x\}) = p_x$. For any $b_n \uparrow b > 0$, the sequence $F(b_n)$ is non-decreasing, bounded above by 1, hence converges to a limit. In this case, we define $P((a, b)) = \lim_n F(b_n) - F(a)$. The definition for $P([a, b]) = F(b) - \lim_n F(a_n)$ where $a_n \uparrow a < 0$. The definition of $P([a, b))$ should be clear from this.

If $F(\cdot)$ has a derivative, $F'(t) = f(t)$, then by the fundamental theorem of calculus, $P([0, x]) = \int_0^x f(t)\, dt$ and $P((a, b]) = P((a, b)) = P([a, b]) = P([a, b)) = \int_a^b f(t)\, dt$. The derivative of a cdf is called a **probability density function (pdf)**. Any non-negative $f : [0, M] \to \mathbb{R}$ that we can meaningfully integrate, e.g. the piece-wise continuous f's, that has the property that $\int_0^M f(t)\, dt = 1$ can serve as a pdf.

A.9. The Most Famous CDF in the World. For $\mathcal{X} = [0, 1]$, let $F(x) = x$ so that $F'(x) = f(x) \equiv 1$. This is called the uniform distribution, for any $0 \leq a < b \leq 1$, $P([a, b]) = (b - a)$ is just the length of the interval. Any interval of length s has a probability s.

A **random variable** on $[0, 1]$ is a function $X : [0, 1] \to \mathbb{R}$.

Special cases: simple rv's; increasing functions with inverses; general increasing functions; Skorokhod's representation for rv's in $\mathbb{R}$.

A.10. Hazard Rates. A random variable, $W \geq 0$, is **incomplete** if it has a mass point at ∞. For a possibly incomplete W with density on $[0, \infty)$ and $0 \leq t < \infty$, we have the following relations between the density, $f(t)$, the cumulative distribution function (cdf), $F(t)$, the reverse cdf, $G(t)$, the hazard rate, $h(t)$, the cumulative hazard, $H(t)$, and the mass at infinity, q:

$$\text{(A.2)} \quad F(t) = \int_0^t f(x)\, dx;\ G(t) = 1 - F(t);\ h(t) = \frac{f(t)}{G(t)} \text{ or } f(t) = h(t)G(t);$$
$$H(t) = \int_0^t h(x)\, dx;\ G(t) = e^{-H(t)};\ E\, W = \int_0^\infty G(t)\, dt; \text{and } q = e^{-H(\infty)}.$$

If the incompleteness parameter, q, is strictly positive, then $E\, W = \infty$, and, as time goes on, and one becomes surer and surer that the event will never happen, $P(W = \infty | W > t) \uparrow 1$ as $t \uparrow \infty$. For complete distributions, if the hazard rate is decreasing (resp. increasing), then the expected *future* waiting time, $E(W|W > t) - t$, is increasing (resp. decreasing).

A.11. Exercises.

PROBLEM A.1. *Suppose that preferences* $\succsim$ *on* Δ *are complete and transitive. Show that* $[\,[p \succ q] \wedge [\alpha > \beta]\,] \Rightarrow [\alpha p + (1-\alpha)q \succ \beta p + (1-\beta)q]$.

PROBLEM A.2. *Suppose that* $\succsim$ *are rational, continuous preferences on* Δ *satisfying independence. Show that if* $p \sim p'$*, then for all* $s \in \mathbb{R}$ *and all* $q \in \Delta$*, if* $q' := q + s(p - p') \in \Delta$*, then* $q' \sim q$.

PROBLEM A.3. *If* u *is a vNM utility function for* $\succsim$*, then for any* $b > 0$*, so is the function* $v : \mathcal{X} \to \mathbb{R}$ *defined by* $v(x) = a + bu(x)$.

PROBLEM A.4. *Finish the proof of Theorem II.1.*

PROBLEM A.5. *If* $f(x) = \frac{1}{M}$ *on the interval* $[0, M]$*, give the associated cdf and the hazard rate.*

PROBLEM A.6. *If* $f(x) = \kappa x^r$*,* $r > 0$*, on the interval* $[0, M]$*, give the* κ *that makes this a pdf, give the associated cdf and the hazard rate.*

PROBLEM A.7. *If* $f(x) = \kappa(M - x)^r$*,* $r > 0$*, on the interval* $[0, M]$*, give the* κ *that makes this a pdf, give the associated cdf and the hazard rate.*

PROBLEM A.8. *Show that* $F_\lambda(x) = 1 - e^{-\lambda x}$*,* $\lambda > 0$*,* $x \in [0, \infty)$*, is a cdf, give the associated pdf and hazard rate. If* X *is a random variable having cdf* F_λ*, show that for every* $t \geq 0$*,* $P(X \in (t + a, t + b]|X > t) = P(X \in (a, b])$*. This class of cdf's is called the exponential or the negative exponential class, the last property, and the property that the hazard rate is constant is called* ***memorylessness****.*

PROBLEM A.9. *If* X *is a random variable having cdf* F_λ*, and* $Y = X^\gamma$ *for* $\gamma > 0$*, give the cdf, the pdf, and hazard rate for* Y*. This class of random variables are called* ***Weibull distributions****.*

PROBLEM A.10. *If* $h : [0, \infty) \to \mathbb{R}_+$ *has the property that* $H(t) = \int_0^t h(x)\,dx < \infty$ *for all* $t < \infty$*, verify that defining the reverse cdf by* $G(t) = e^{-H(t)}$ *gives a random variable with hazard rate* h*.*

B. Comparative Statics II

This section contains some more advanced material, it provides a slightly more sophisticated coverage of monotone comparative statics in notation closer to what is typically used in choice problems in the presence of risk, and then gives some of the simpler results extending monotone comparative statics analyses to choice under uncertainty.

B.1. Basic Tools for Discussing Probabilities. For a finite set of outcomes, $\mathcal{X} = \{x_1, \ldots, x_n\}$, the probability distributions are given as $p = (p_1, \ldots, p_n)$ where each $p_i \geq 0$ and $\sum_i p_i = 1$.

B.2. More and Less Informative Signals. The central model in statistical decision theory can be given by a utility relevant random variable X taking value in a set $\mathfrak{C}$, a signal S that contains information about X, a set of possible actions, A, and a joint distribution for X and S given by $q(x, s)$. The associated time line has an observation of $S = s$, then the choice of an $a \in A$, and then the realization of the random utility $u(a, X)$.

B.2.1. *An Infrastructure Example.* In the following example, $X = G$ or $X = B$ corresponds to the future weather pattern, the actions are to Leave the infrastructure alone or to put in New infrastructure, and the signal, s, is the result of investigations and research into the distribution of future values of X. The utilities are

	Good	Bad
L	10	6
N	$(10-c)$	$(9-c)$

Beliefs/evidence yields $\beta_G = Prob(G)$, $\beta_B = (1-\beta_G) = Prob(B)$. One optimally leaves the infrastructure as it is if

$$10\beta_G + 6(1-\beta_G) > (10-c)\beta_G + (9-c)(1-\beta_G), \text{ that is, if} \tag{B.1}$$

$$c > 3(1-\beta_G). \tag{B.2}$$

Unsurprisingly, the criterion for not putting in the new infrastructure is that its costs are larger than the expected gains. Let us suppose that the new infrastructure costs 20% of the damages it prevents, that is, $c = 0.60$ so that one Leaves the infrastructure alone iff $\beta_G > 0.8$, i.e. iff there is less than one chance in five of the bad future weather patterns.

Suppose that the prior is $\beta_G = 0.75$ so that, without any extra information, one would put in the New infrastructure. Let us now think about signal structures. First let us suppose that we can run test/experiments that yield $S = s_G$ or $S = s_B$ with $P(S = s_G|G) = \alpha \geq \frac{1}{2}$ and $P(S = s_B|B) = \beta \geq \frac{1}{2}$. The joint distribution, $q(\cdot, \cdot)$, is

	Good	Bad
s_G	$\alpha \cdot 0.75$	$(1-\beta) \cdot 0.25$
s_B	$(1-\alpha) \cdot 0.75$	$\beta \cdot 0.25$

Beliefs or **posterior beliefs** are given by $\beta_{s_G}(G) = \frac{\alpha \cdot 0.75}{\alpha \cdot 0.75 + (1-\beta) \cdot 0.25}$ and $\beta_{s_B}(G) = \frac{(1-\alpha) \cdot 0.75}{(1-\alpha) \cdot 0.75 + \beta \cdot 0.25}$. Note that the average of the posterior beliefs is the prior, one has beliefs $\beta_{s_G}(\cdot)$ with probability $\alpha \cdot 0.75 + (1-\beta) \cdot 0.25$ and beliefs $\beta_{s_B}(\cdot)$ with probability $(1-\alpha) \cdot 0.75 + \beta \cdot 0.25$.

Problem B.1. *If $\alpha = \beta = \frac{1}{2}$, the signal structure is worthless. Give the set of $(\alpha, \beta) \geq (\frac{1}{2}, \frac{1}{2})$ for which the information structure strictly increases the expected utility of the decision maker. [You should find that what matters for increasing utility is having a positive probability of changing the decision.] Verify that the average of the posteriors is the prior.*

Problem B.2. *Now suppose that the test/experiment can be run twice and that the results are independent across the trials. Thus, $P(S = (s_G, s_G)|G) = \alpha^2$, $P(S = (s_G, s_B)|G) = P(S = (s_B, s_G)|G) = \alpha(1-\alpha)$, and $P(S = (s_B, s_B)|G) = (1-\alpha)^2$ with the parallel pattern for B. Fill in the probabilities in the following joint distribution $q(\cdot, \cdot)$ and verify that the average of posterior beliefs is the prior belief.*

	Good	*Bad*
(s_G, s_G)		
(s_G, s_B)		
(s_B, s_G)		
(s_B, s_B)		

If $\alpha = \beta = \frac{1}{2}$, the signal structure is worthless. Give the set of $(\alpha, \beta) \geq (\frac{1}{2}, \frac{1}{2})$ for which the information structure strictly increases the expected utility of the decision maker.

Problem B.3. *Now suppose that the test/experiment can be run twice but that the results are dependent across the trials. Thus, $P(S = (s_G, s_G)|G) = \alpha$, $P(S = (s_G, s_B)|G) = P(S = (s_B, s_G)|G) = 0$, and $P(S = (s_B, s_B)|G) = (1 - \alpha)$ with the parallel pattern for B. Fill in the probabilities in the following joint distribution $q(\cdot, \cdot)$.*

	Good	*Bad*
(s_G, s_G)		
(s_G, s_B)		
(s_B, s_G)		
(s_B, s_B)		

If $\alpha = \beta = \frac{1}{2}$, the signal structure is worthless. Give the set of $(\alpha, \beta) \geq (\frac{1}{2}, \frac{1}{2})$ for which the information structure strictly increases the expected utility of the decision maker.

Problem B.4. *Now suppose that the test/experiment can be run twice but that the results are γ-independent across the trials. That is, $P(S = (s_G, s_G)|G) = \gamma\alpha^2 + (1 - \gamma)\alpha$, $P(S = (s_G, s_B)|G) = P(S = (s_B, s_G)|G) = \gamma\alpha(1 - \alpha) + (1 - \gamma)0$, and $P(S = (s_B, s_B)|G) = \gamma(1 - \alpha)^2 + \gamma(1 - \alpha)$ with the parallel pattern for B. 1-independence is the independent signal structure given two problems above, 0-independence is the signal structure given in the previous problem. Fill in the probabilities in the following joint distribution $q(\cdot, \cdot)$ and verify that the average of the posterior beliefs is the prior belief.*

	Good	*Bad*
(s_G, s_G)		
(s_G, s_B)		
(s_B, s_G)		
(s_B, s_B)		

Suppose that $(\alpha, \beta) \gg (\frac{1}{2}, \frac{1}{2})$ has the property that the 1-independent structure is strictly valuable. Find the set of γ such that γ-independence is strictly valuable.

B.2.2. *Back to Generalities.* The problem is to pick a function from observations to actions, $s \mapsto a(s)$, so as to maximize

$$(\text{B.3}) \qquad (\ddagger) \qquad E\,u(a(S), X) = \sum_{x,s} u(a(s), x) q(x, s).$$

One can solve this problem by formulating a complete contingent plan, $s \mapsto a(s)$, or one can "cross that bridge when one comes to it," that is, wait until $S = s$ has been observed and figure out at that point what $a(s)$ should be.

Let $\pi(s) = \sum_x q(x, s)$, rewrite $E\,u = \sum_{x,s} u(a(s), x)q(x, s)$ as

$$(\text{B.4}) \qquad \sum_s \pi(s) \sum_x u(a(s), x) \frac{q(x, s)}{\pi(s)}.$$

$\beta(x|s) := \frac{q(x,s)}{\pi(s)}$ is the posterior probability that $X = x$ after $S = s$ has been observed. As is well-known and easy to verify, if $a^*(s)$ solves $\max_a \sum_x u(a, x)\,\beta(x|s)$ at each s with $\pi(s) > 0$, then $a^*(\cdot)$ solves the original maximization problem in ($\ddagger$). In other words, the dynamically consistent optimality of best responding to beliefs formed by Bayesian updating arises from the linearity of expected utility in probabilities. We will see that we have the same property for sets of priors.

The pieces of the basic statistical decision model are: $\beta(x) := \sum_s q(x, s)$, the marginal of q on X, the **prior** distribution; $\pi(s) := \sum_x q(x, s)$, the marginal of q on S, the **distribution of the signals**; and $\beta(\cdot|s) = \frac{q(\cdot,s)}{\pi(s)}$, the **posterior** distributions, i.e. the conditional distribution of X given that $S = s$. The martingale property of beliefs is the observation that the prior is the π-weighted convex combination of the posteriors, for all x,

$$(\text{B.5}) \qquad \sum_s \pi(s)\beta(x|s) = \sum_s \pi(s) \frac{q(x, s)}{\pi(s)} = \sum_s q(x, s) = \beta(x).$$

Beliefs at s, $\beta(\cdot|s)$ belong to $\Delta(\mathfrak{C})$ and have distribution π. Re-writing, information is a distribution, $\pi \in \Delta(\Delta(\mathfrak{C}))$ having mean equal to the prior. Blackwell (1950, 1951) showed that all signal structures are equivalent to such distributions.

Problem B.5. *Verify that if* $a^*(s)$ *solves* $\max_a \sum_x u(a, x)\,\beta(x|s)$ *at each* s *with* $\pi(s) > 0$*, then* $a^*(\cdot)$ *solves the original maximization problem in* ($\ddagger$).

B.2.3. *The Monotone Likelihood Ratio Property.* When $X \in \mathbb{R}$ and higher values of X are good news, then news that X is unambiguously higher is good news. We say that signal s' is better news than signal s if for all $r \in \mathbb{R}$, $P(X \geq r|s') \geq P(X \geq r|s)$. If the prior is β, then the prior odds ratio for any given value $x' > x$ is $\frac{\beta(x')}{\beta(x)}$.

Define $f(s|x) = P(S = s|X = x) = \frac{q(x,s)}{\beta(x)}$. In terms of $f(\cdot|\cdot)$, the odds ratio for $x' > x$ after seeing s' and s are given by

$$(\text{B.6}) \qquad \frac{\beta(x')f(s'|x')}{\beta(x)f(s'|x)} \text{ and } \frac{\beta(x')f(s|x')}{\beta(x)f(s|x)}.$$

This means that the posterior odds ratio increases if

$$(\text{B.7}) \qquad \frac{f(s'|x')}{f(s'|x)} > \frac{f(s|x')}{f(s|x)},$$

that is, if the ratio $\frac{f(\cdot|x')}{f(\cdot|x)}$ is increasing for every $x' > x$. This property has a suggestive name.

Definition II.2. *Any family of densities for signals s with the property that the ratio $\frac{f(\cdot|x')}{f(\cdot|x)}$ is non-decreasing for every $x' > x$ is said to have the **monotone likelihood ratio property (mlrp)**.*

Problem B.6. *Suppose that $u(\cdot, \cdot)$ has increasing differences in $a \in A$ and $x \in \mathcal{X}$ where $A \subset \mathbb{R}$ and $\mathcal{X} \subset \mathbb{R}$ and let $a^*(\cdot)$ solve $\max_{a \in A} u(a, x)$. Suppose that one observes a signal s before choosing a and that the densities for the signal s given $X = x \in \mathcal{X}$ has the mlrp. Show that $s \mapsto a^*(s)$ is increasing.*

C. Slightly More General Monotone Comparative Statics

!NOTATION CHANGES HERE!

Throughout, $x \in X$ will be a choice variable, and $\theta \in \Theta$ a parameter not under the control of the decision maker. We will study the dependence of the optimal x on θ, denoted $x^*(\theta)$, in two kinds of problems,

$$\max_{x \in X} f(x, \theta), \text{ and } \max_{x \in X} U(x, \theta) := \int u(x, s)\, dp_\theta(s). \tag{C.1}$$

Most often the probability p_θ will have a density $f(s; \theta)$ which gives the integral the form $U(x, \theta) = \int u(x, s) f(s; \theta) ds$.

C.1. Simplest Supermodular. The simplest and most common case has X and Θ being linearly ordered sets, and within this case, far and away the most common example has X and Θ being interval subsets or discrete subsets of $\mathbb{R}$ with the usual less-than-or-equal-to order.

Definition II.3. *For linearly ordered X and Θ, a function $f : X \times \Theta \to \mathbb{R}$ is **supermodular** if for all $x' \succ x$ and all $\theta' \succ \theta$,*

$$f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta), \tag{C.2}$$

equivalently

$$f(x', \theta') - f(x', \theta) \geq f(x, \theta') - f(x, \theta). \tag{C.3}$$

*It is **strictly supermodular** if the inequalities are strict.*

At t, the benefit of increasing from x to x' is $f(x', \theta) - f(x, \theta)$, at θ', it is $f(x', \theta') - f(x, \theta')$. This assumption asks that benefit of increasing x be increasing in θ. A good verbal shorthand for this is that f **has increasing differences in** x **and** θ. Three sufficient conditions in the differentiable case are: $\forall x$, $f_x(x, \cdot)$ is nondecreasing; $\forall t$, $f_t(\cdot, t)$ is nondecreasing; and $\forall x, \theta$, $f_{x\theta}(x, \theta) \geq 0$.

Theorem II.2. *If $f : X \times \Theta \to \mathbb{R}$ is supermodular and $x^*(\theta)$ is the largest (or the smallest) solution to $\max_{x \in X} f(x, \theta)$, then $[\theta' \succ \theta] \Rightarrow [x^*(\theta') \succsim x^*(\theta)]$. If f is strictly supermodular, then for any $x' \in x^*(\theta')$ and any $x \in x^*(\theta)$, $x' \geq x$.*

Proof. Suppose that $\theta' \succ \theta$ but that $x' = x^*(\theta') \prec x = x^*(\theta)$. Because $x^*(\theta)$ and $x^*(\theta')$ are maximizers, $f(x', \theta') \geq f(x, \theta')$ and $f(x, \theta) \geq f(x', \theta)$. Since x' is the largest of the maximizers at t' and $x \succ x'$, we know a bit more, that $f(x', \theta') > f(x, \theta')$. Adding the inequalities, we get $f(x', \theta') + f(x, \theta) > f(x, \theta') + f(x', \theta)$, or

$$f(x, \theta) - f(x', \theta) > f(x, \theta') - f(x', \theta'). \tag{C.4}$$

But $\theta' \succ \theta$ and $x \succ x'$ and supermodularity imply that this inequality must go the other way. The argument using strict supermodularity is similar. $\square$

A. The amount of a pollutant that can be emitted is regulated to be no more than $\theta \geq 0$. The cost function for a monopolist producing x is $c(x, \theta)$ with $c_\theta < 0$ and $c_{x\theta} < 0$. These derivative conditions means that increases is the allowed emission level lower costs and lower marginal costs, so that the firm will always choose θ. For a given θ, the monopolist's maximization problem is therefore

$$\max_{x \geq 0} f(x, \theta) = xp(x) - c(x, \theta) \tag{C.5}$$

where $p(x)$ is the (inverse) demand function. Show that output decreases as $\theta \downarrow$.

B. Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$ so that $CS(p) = \int_p^\infty x(r)\, dr$ is the consumer surplus when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. (From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.) The monopolist's profit when they produce x is $\pi(x) = x \cdot p(x) - c(x)$ where $c(x)$ is the cost of producing x. The maximization problem for the monopolist is

$$\max_{x \geq 0}\ \pi(x) + 0 \cdot CS(p(x)). \tag{C.6}$$

Society's surplus maximization problem is

$$\max_{x \geq 0}\ \pi(x) + 1 \cdot CS(p(x)). \tag{C.7}$$

Set $f(x, \theta) = \pi(x) + \theta CS(p(x))$, $\theta \in \{0, 1\}$, and verify that $f(x, \theta)$ is supermodular. What does this mean about monopolists output relative to the social optimum?

C. Suppose that a monopolist sells to N identical customers so their profit function is

$$\pi(x, N) = Nxp(x) - c(Nx) = N \cdot \left[xp(x) - \tfrac{c(Nx)}{N}\right]. \tag{C.8}$$

1. If $c(\cdot)$ is convex, how does $x^*(N)$ depend on N?
2. If $c(\cdot)$ is concave, how does $x^*(N)$ depend on N?

D. You start with an amount x, choose an amount, c, to consume in the first period, and have $f(x - c)$ to consume in the second period, and your utility is $u(c) + \beta u(f(x - c))$ where $u' > 0$ and $u'' < 0$. We suppose that $r \mapsto f(r)$ is increasing.

1. Consider the two-period consumption problem,

$$P_c(x) = \max\ \ u(c) + \beta u(f(x - c)) \text{ subject to } c \in [0, x]. \tag{C.9}$$

Prove that, because $f(\cdot)$ is increasing, this problem is equivalent to the two-period savings problem,

$$P_s(x) = \max\ \ u(x - s) + \beta u(f(s)) \text{ subject to } s \in [0, x]. \tag{C.10}$$

2. Prove that savings, $s^*(x)$, are weakly increasing in x.
3. Now define $V(y) = \max_{c_0, c_1, \cdots} \sum_{t=0}^\infty \beta^t u(c_t)$ subject to $x_0 = y$, $s_t = x_t - c_t$, $x_{t+1} = f(s_t)$, and $c_t \in [0, x_t]$, $t = 0, 1, \ldots$. Assuming that the maximization problem for V has a solution, show that $V(\cdot)$ is increasing. From this, prove

that the solution to the following infinite horizon savings problem is weakly increasing in x,

$$P(x) = \max \quad u(x-s) + \beta V(f(s) \text{ subject to } s \in [0, x]. \tag{C.11}$$

C.2. Choice Sets Also Move. Sometimes the set of available choices also shifts with θ.

THEOREM II.3. *Suppose that X and Θ are non-empty subsets of $\mathbb{R}$, that $\Gamma(\theta) = [g(\theta), h(\theta)] \cap X$ where g and h are weakly increasing functions with $g \leq h \leq \infty$, that $f : X \times \Theta \to \mathbb{R}$ is supermodular (i.e. has increasing differences in x and θ). Then the smallest and the largest solutions to the problem $P(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta)$ are weakly increasing functions. Further, if f is strictly supermodular, then every selection from $\Psi(\theta) := \arg\max_{x \in \Gamma(\theta)} f(x, \theta)$ is weakly increasing.*

E. Some theory and applications of this result follow.
1. Prove the part of the result before the word "Further".
2. Prove the part of the result after the word "Further".
3. Suppose that the function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is twice continuously differentiable and locally non-satiable. Let $x^*(\boldsymbol{p}, w)$ be the solution to the problem $\max\{u(\boldsymbol{x}) : \boldsymbol{x} \geq 0, \boldsymbol{px} \leq w\}$. We say that good 1 is **normal** if, for fixed $\boldsymbol{p}$, $x_1^*(\boldsymbol{p}, w)$ is a weakly increasing function of w.
   a. Prove that local non-satiability implies that that $\boldsymbol{p}x^*(\boldsymbol{p}, w) \equiv w$, i.e. that Walras's law holds.
   b. Prove that the problem $\max\{u(\boldsymbol{x}) : \boldsymbol{x} \geq 0, \boldsymbol{px} \leq w\}$ is equivalent to the problem $P(w) = \max\{u(x_1, (w - p_1 x_1)/p_2) : x_1 \in [0, w/p_1]\}$.
   c. Prove that $p_2 u_{21}(x_1, x_2) - p_1 u_{22}(x_1, x_2) \geq 0$ implies that good 1 is normal.
   d. Interpret the previous result geometrically in terms of tangencies and changes in slopes of indifference curves.
4. A monopolist with a constant unit cost c faces a demand curve $D(p)$ and charges a price $p \geq c$ to solve the problem

$$P(c) = \max \quad (p - c)D(p), \text{ subject to } p \in [c, \infty). \tag{C.12}$$

Prove that if the demand curve, $D(\cdot)$, is weakly decreasing in the price charged, then the largest and the smallest elements $p^*(c)$ are weakly increasing in c, and if it is strictly decreasing, then any selection is weakly increasing.

For the rest of this analysis of a monopolist, assume that $D'(p) < 0$.

We define the monopolist's **mark-up** as $m = p - c$. In terms of mark-up, the monopolist's problem is

$$\max \ m \cdot D(m + c), \text{ equivalently, } \max \ \log(m) + \log(D(m + c)). \tag{C.13}$$

Prove that if $r \mapsto \log(D(r))$ is strictly concave, then every selection from $m^*(c)$ is decreasing. Since $p^*(c) = m^*(c) + c$, show that there is always positive, but partial pass through to the consumers of increases in unit cost. Examine what happens if $r \mapsto \log(D(r))$ is strictly convex. In particular, what is the pass through of any increase in the unit cost?

C.3. Background for Choice Under Uncertainty. We now turn to some background useful for the choice under uncertainty problems,

$$\max_{x \in X} U(x,\theta) := \int u(x,s) f(s;\theta)\, ds. \tag{C.14}$$

to have $U(\cdot,\cdot)$ supermodular, we need, for $x' > x$ and $\theta' > \theta$, to have

$$\int [u(x',s) - u(x,s)]\ f(s;\theta')\, ds \geq \int [u(x',s) - u(x,s)]\ f(s,\theta)\, ds. \tag{C.15}$$

Definition II.4. *A family of densities on $\mathbb{R}$, $\{f(s;\theta) : \theta \in \Theta\}$, $\Theta \subset \mathbb{R}$, has the* ***monotone likelihood ratio property (MLRP)*** *if there exists a $s \mapsto T(s)$ such that for any $\theta' > \theta$, $f(s;\theta')$ and $f(s;\theta)$ are the densities of different distributions, and $\frac{f(s;\theta')}{f(s;\theta)}$ is a nondecreasing function of $T(s)$.*

Comment: if $s \mapsto T(s)$ is monotonic, as it often is, we can simplify the assumption to $\frac{f(s;\theta')}{f(s;\theta)}$ is a nondecreasing function of s.

F. An exponential distribution with parameter β has density $f(s;\beta) = \frac{1}{\beta} e^{-s/\beta}$ for $s \geq 0$, $\beta \in \Theta = (0,\infty)$.
   1. Does this class have the MLRP?
   2. For $\gamma > 0$, let $f(s;\gamma,\beta)$ be the density of X^γ where X has an exponential β.
      a. For fixed γ, does the class parametrized by β have the MLRP?
      b. For fixed β, does the class parametrized by γ have the MLRP?
   3. Let $Y = \sum_{i=1}^N X_i$ where the X_i are iid exponentials with parameter β. Does Y have the MLRP?

G. Consider the class of uniform distributions, $U[a,b]$.
   1. Does the class of uniform distributions $U[0,\theta]$ have the MLRP?
   2. What about the class of uniform distributions $U[-\theta,\theta]$?
   3. What about the class of uniform distributions $U[\theta - r, \theta + r]$?

H. Consider the class of distributions $f(s;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $s \in \mathbb{R}$.
   1. For fixed σ, does the class parametrized by μ have the MLRP?
   2. For fixed μ, does the class parametrized by σ have the MLRP?
   3. Show that for fixed σ, if $\mu' > \mu$, then the distribution $f(s;\mu',\sigma)$ first order stochastically dominates the distribution $f(s;\mu,\sigma)$.
   4. Show that for fixed μ, if $\sigma' > \sigma$, then the distribution $f(s;\mu,\sigma')$ is riskier than the distribution $f(s;\mu,\sigma)$.

I. Consider one-parameter families of distributions, $f(s;\theta) = C(\theta) e^{Q(\theta)T(s)} h(s)$ with $\theta \mapsto Q(\theta)$ a strictly increasing function. Verify that they have the MLRP. For this class of distributions, verify the existence of a uniformly most powerful test for $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$ by considering test of the form that they reject if $x > C$, accept if $x < C$ (and accept with probability γ if $x = C$).

MLRP classes are a special type of Pólya distribution, and more general results than the following are in Karlin's "Pólya Type Distributions, II," *Annals of Mathematical Statistics*, 28(2), 281-308 (1957).

Theorem II.4. *Let $f(s;\theta)$ be a family of densities on $\mathbb{R}$ with the MLRP.*

(a) *If $s \mapsto g(s)$ is nondecreasing, then $\theta \mapsto \int g(s)\, df(s;\theta)\, ds$ is nondecreasing.*

(b) *If $X_1, \ldots, X_n$ are iid $f(s;\theta)$ and $(s_1, \ldots, s_n) \mapsto g(s_1, \ldots, s_n)$ is nondecreasing in each argument, then $\theta \mapsto \int g(s_1, \ldots, s_n)\, f(s_1;\theta) \cdots f(s_n;\theta) ds_1 \cdots ds_n$ is nondecreasing.*

(c) *$\theta' > \theta$, then $f(\cdot;\theta')$ first order dominates $f(\cdot, \theta)$.*

(d) *If $s \mapsto g(s)$ crosses 0 from below at most once, i.e. for some s_0, $g(s) \leq 0$ for $s < s_0$ and $g(s) \geq 0$ for $s \geq s_0$, then either $\psi(\theta) := \int g(s) f(s;\theta)\, ds$ is everywhere positive or everywhere negative, or there exists θ_0 with $\psi(\theta) \leq 0$ for $\theta < \theta_0$ and $\psi(\theta) \geq 0$ for $\theta \geq \theta_0$.*

Proof. For part (a), define $A = \{s : f(s;\theta') < f(s;\theta)\}$, $B = \{s : f(s;\theta') > f(s;\theta)\}$, set $a = \sup_{s \in A} g(s)$, $b = \inf_{s \in A} g(s)$, and note that $a \leq b$. We must show that $\int g(s) \, [f(s;\theta') - f(s;\theta)] \, ds \geq 0$. Now,

$$\int g(s) \, [f(s;\theta') - f(s;\theta)] \, ds = \tag{C.16}$$

$$\int_A g(s) \, [f(s;\theta') - f(s;\theta)] \, ds + \int_B g(s) \, [f(s;\theta') - f(s;\theta)] \, ds \geq \tag{C.17}$$

$$a \int_A [f(s;\theta') - f(s;\theta)] \, ds + b \int_B [f(s;\theta') - f(s;\theta)] \, ds = \tag{C.18}$$

$$\left(a \int_A [f(s;\theta') - f(s;\theta)] \, ds + a \int_B [f(s;\theta') - f(s;\theta)] \, ds \right) + \tag{C.19}$$

$$\left(b \int_B [f(s;\theta') - f(s;\theta)] \, ds - a \int_B [f(s;\theta') - f(s;\theta)] \, ds \right) = \tag{C.20}$$

$$0 + (b - a) \int_B [f(s;\theta') - f(s;\theta)] \, ds \geq 0. \tag{C.21}$$

Part (b) follows by conditioning and induction, (c) by considering the functions $g(s) = 1_{(r,\infty)}(s)$.
For part (d), we shall show that the θ_0 we need is $\theta_0 := \inf\{\theta : \int g(s) f(s;\theta)\, ds > 0\}$. For this, it is sufficient to show that for any $\theta < \theta'$, $[\psi(\theta) > 0] \Rightarrow [\psi(\theta') \geq 0]$. There are two cases: (1) $\frac{f(s_0;\theta')}{f(s_0;\theta)} = \infty$, which requires $f(s_0;\theta) = 0$; and (2) $\frac{f(s_0;\theta')}{f(s_0;\theta)} = r$ for some $r \in \mathbb{R}_+$.
(1) Given the MLRP, $\frac{f(s_0;\theta')}{f(s_0;\theta)} = \infty$ and $f(s_0;\theta) = 0$ imply that $\psi(\theta) \leq 0$.
(2) Given that $\frac{f(s_0;\theta')}{f(s_0;\theta)} = r$, $g(s) \geq 0$ for all s in the set $C = \{s : f(s;\theta) = 0, f(s;\theta') > 0\}$. Integrating over the complement of C gives the first of the following inequalities (where we have avoided dividing by 0),

$$\int g(s) f(s;\theta')\, ds \geq \int_{C^c} g(s) \frac{f(s;\theta')}{f(s;\theta)} f(s;\theta)\, ds \tag{C.22}$$

$$\geq \int_{(-\infty, s_0)} r g(s) f(s;\theta)\, ds + \int_{[s_0, +\infty)} r g(s) f(s;\theta)\, ds, \tag{C.23}$$

and this last sum is equal to $r \int g(s) f(s;\theta)\, ds$. Since $r \geq 0$ and $\int g(s) f(s;\theta)\, ds > 0$, we conclude that $\int g(s) f(s;\theta')\, ds \geq 0$. □

J. If X has a distribution with density $s \mapsto f(s)$, then the class of densities $\{f(s - \theta) : \theta \in \mathbb{R}\}$ is a **location class**.
   1. For any location class, if $\theta' > \theta$, then $f(s - \theta')$ first order dominates $f(s - \theta)$.

2. Having the MLRP is sufficient for first order dominance. By considering the location class of Cauchy distributions, show that the reverse is not true.

C.4. Log Supermodularity.

DEFINITION II.5. *A non-negative function $h(x, \theta)$ is **log supermodular** if $\log h(x, \theta)$ is supermodular, that is, if for all $x' > x$ and $\theta' > \theta$), $h(x', \theta')h(x, \theta) \geq h(x', \theta)h(x, \theta')$.*

K. At various points, we will be using log supermodularity of density functions and of marginal utilities.
   1. Show that if the class $\{f(s; \theta) : \theta \in \Theta\}$ has the MLRP, then it is log supermodular in s and θ.
   2. Show that if the support set for each density in the class $\{f(s; \theta) : \theta \in \Theta\}$ is the same interval, then being log supermodular in s and θ implies that the class has the MLRP.
   3. Suppose that $u(\cdot)$ is a concave, increasing, twice continuously differentiable function. Show that $f(w, s) := u'(w + s)$ is log supermodular iff u has decreasing absolute risk aversion.

L. Suppose that $\{f(s; \theta) : \theta \in \Theta\}$ is a class of densities with the MLRP. Show that if $u(\cdot, \cdot)$ is supermodular, then the inequality in equation (C.15) holds.

M. Consider the following classes of portfolio choice problems,

$$\max_{x \in [0,w]} \int u(w - x + xs) f(s; \theta)\, ds \tag{C.24}$$

where $\{f(s; \theta) : \theta \in \Theta\}$, $\Theta \subset \mathbb{R}$, is a class of distributions on $[0, \infty)$ with the MLRP.
   1. If $u(r) = \log(r)$, does the supermodularity analysis tell us whether or not $x^*(\theta)$ is an increasing or decreasing function?
   2. If $u(r) = r^\gamma$, $0 < \gamma < 1$, does the supermodularity analysis tell us whether or not $x^*(\theta)$ is an increasing or decreasing function?
   3. If $u(r) = r^\gamma$, $\gamma \geq 1$, does the supermodularity analysis tell us whether or not $x^*(\theta)$ is an increasing or decreasing function?
   4. In the previous three problems, characterize, if possible, the set of θ for which x^* increases with w. [This is where the last part of Theorem II.4 may come in handy.]

D. Decision Trees

TO BE ADDED

CHAPTER III

Game Theory

Story telling is an old and honored tradition. If one takes a functionalist approach to social institutions, it is a tradition that is meant to inculcate[1] the values of a society in its members. In slightly less grand words, this is part of your indoctrination into thinking as economists think. Enough generalities, let us begin.

A. Defining a Game and its Equilibria

A game is a collection $\Gamma = (A_i, u_i)_{i \in I}$. This has three pieces:

(1) I is the (usually finite) set of agents/people/players,
(2) for each $i \in I$, A_i is the set of actions or strategies available to i, and, setting $A = \times_{i \in I} A_i$,
(3) for each $i \in I$, $u_i : A \to \mathbb{R}$ represents i's preferences (usually von Neumann-Morgenstern preferences) over the actions chosen by others.

Definition III.1. *Γ is **finite** if I and each A_i is a finite set.*

Having described who is involved in a strategic situation, the set I, and having described their available choices, the sets A_i, and their preferences over their own and everybody else's choices, we try to figure out what is going to happen. We have settled on a notion of equilibrium, due to John Nash, as our answer to the question of what will happen. The answer comes in two flavors: pure; and mixed.

Definition III.2. *A **pure strategy Nash equilibrium** is a vector $a^* \in A$, $a^* = (a_i^*, a_{-i}^*)$, of actions with the property that each a_i^* is a best response to a_{-i}^*.*

Δ_i or $\Delta(A_i)$ denotes the set of probability distributions on A_i, and it called the set of **mixed strategies**; $\Delta := \times_{i \in I} \Delta_i$ denotes the set of product measures on A; each u_i is extended to Δ by integration, $u_i(\sigma) := \int_A u_i(a_1, \ldots, a_I)\, d\sigma_1(a_1) \cdots \sigma_I(a_I)$.

Definition III.3. *A **mixed strategy Nash equilibrium** is a vector $\sigma^* \in \Delta$, $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$, of with the property that each σ_i^* is a best response to σ_{-i}^*.*

The following notation is very useful.

Notation III.1. *For $\sigma^\circ \in \Delta$, $Br_i(\sigma) = \{\sigma_i \in \Delta_i : u_i(\sigma_i, \sigma_{-i}^\circ) \geq u_i(\Delta_i, \sigma_{-i}^\circ)$ is i's **best response to** σ.*

Thus, σ^* is an equilibrium iff $(\forall i \in I)[\sigma_i^* \in Br_i(\sigma^*)]$.

Definition III.4. *An equilibrium σ^* is **strict** if $(\forall i \in I)[\#Br_i(\sigma^*) = 1]$.*

Through examples, we will start to see what is involved.

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[1]From the OED, to inculcate is to endeavour to force (a thing) into or impress (it) on the mind of another by emphatic admonition, or by persistent repetition; to urge on the mind, esp. as a principle, an opinion, or a matter of belief; to teach forcibly.

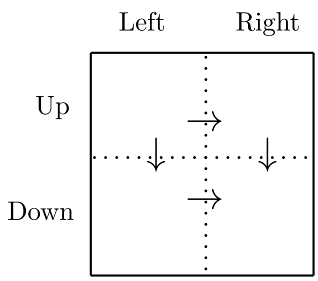
B. Some Finite Examples

We will start by looking at 2×2 games, said "two by two games." These are games with two players, and each player has two actions. The standard, abstract presentation of such games has $I = \{1, 2\}$, that is, the set of agents (or players, or people involved in the strategic interaction) is I and I contains two individuals, unimaginatively labelled 1 and 2. Each $i \in I$ has an action set, A_i. In the following matrix representation of the payoffs, A_1 will contain the actions labelled Up and Down, A_2 will contain the actions labelled Left and Right.

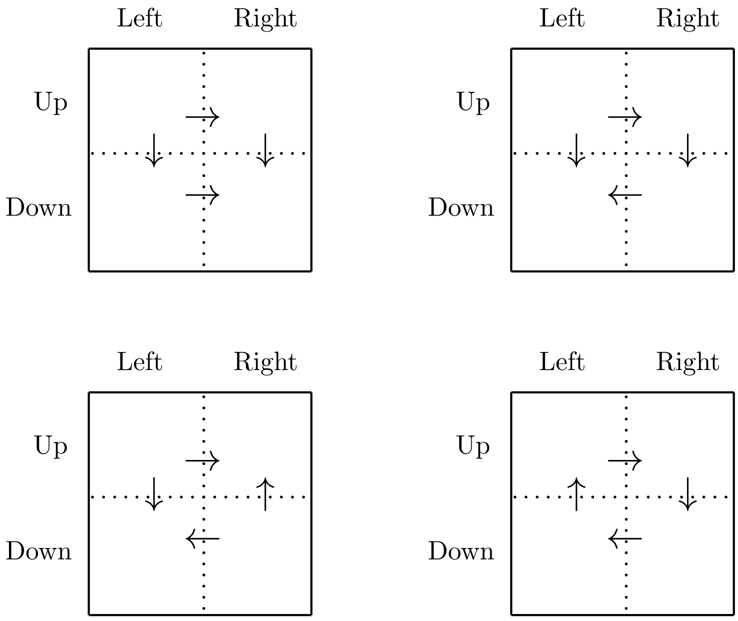
	Left	Right
Up	(a, b)	(c, d)
Down	(e, f)	(g, h)

Since $A := A_1 \times A_2$ has four elements, $A = \{$(Up,Left), (Up,Right), (Down,Left), (Down,Right)$\}$, we need to specify the payoffs for two agents for each of these four possibilities. This means that we need to specify 8 different payoff numbers. These are the entries (a, b), (c, d), (e, f), and (g, h). The convention is that the first entry, reading from left to right, is 1's payoff, the second entry is 2's payoff. For example a is 1's utility if 1 choose Up and 2 chooses Left, while b is 2's payoff to the same set of choices.

B.1. Classifying 2×2 Games. A pure strategy equilibrium is a choice of action, one for each agent, with the property that, give the other's choice, each agent is doing the best they can for themselves. For example, if $e > a$, then 1's best choice if 2 is picking Left is to pick Down. For another example, if $b > d$, then Left is 2's best choice if 1 is picking Up. A pair of choices that are mutually best for the players is the notion of equilibrium we are after. As we will see, sometimes we need to expand the notion of a choice to include randomized choices.
We will distinguish four types of 2×2 games on the basis of the strategic interests of the two players. We do this using boxes with **best response arrows**. If $e > a$ and $g > c$ and $d > b$ and $h > f$, we have the box representation



In this game, we see that Down is a better choice for 1 no matter what 2 does, and we say that Down is a **dominant strategy** for 1 and Up is a **dominated strategy** in this case. In the same way, Right is a dominant strategy for 2 and Left is dominated.
We are going to study the four classes of strategic interactions given by the following best response boxes.



• In the class of games represented by the top left box, both players have a dominant strategy. We will first see this under the name of the "Prisoners' Dilemma."
• In the class of games represented by the top right box, exactly one player has a dominant strategy. We will first see this under the name of "Rational Pigs."
• In the class of games represented by the bottom left box, neither player has a dominant strategy and there are two action pairs with arrows pointing into them. Games in this class are called coordination games, the named games in this class are the "Battle of the Sexes," the "Stag Hunt," and "Hawk-Dove."
• In the class of games represented by the bottom right box, neither player has a dominant strategy and there is no action pair with arrows pointing into it. Games in this class have no pure strategy equilibrium, the named games in this class are "Matching Pennies" or "Penalty Kick," and "Monitoring/Auditing Games."

Note that changing the labels on the actions or changing the numbering of the players has no effect on the strategic analysis. For example, in the last class of games, switching Left to Right or Up to Down changes the direction in which the arrows swirl, but, aside from relabelling, this makes no difference to the analysis of the game.

B.2. Dominance Solvable Games. A Nash equilibrium is defined as vector of mutual best responses. When a player has a dominated strategy, they will never play it as part of any Nash equilibrium. Once you remove the dominated strategy or strategies from the game, the remaining game is easier to solve. If the remaining game has dominated strategies, you then remove them. If after doing this as many times as possible, you end up at a single strategy for each player, the game is called **dominance solvable**. For dominance solvable games, Nash equilibria have a very firm behavioral foundation.

The classes of games represented by the top two boxes are dominance solvable. The first of these games that we will consider explains why it is sometimes advantageous to be small.[2]

B.2.1. *The Advantage of Being Small.* Here $I = \{\text{Big, Little}\}$, $A_1 = A_2 = \{\text{Push, Wait}\}$. This is called a 2×2 game because there are two players with two strategies apiece. The utilities are given in the following table.

Rational Pigs

	Push	Wait
Push	$(-1, 5)$	$(-1, 6)$
Wait	$(3, 2)$	$(0, 0)$

The numbers were generated from the following story: there are two pigs, one Big and one Little, and each has two actions. Little pig is player 1, Big pig player 2, the convention has 1's options being the rows, 2's the columns, payoffs (x, y) mean "x to 1, y to 2." The two pigs are in a long room. A lever at one end, when pushed, delivers food, worth 6 in utility terms, into a trough at the other end. Until the food has been emptied from the trough, the lever is non-functional, once the food has been emptied, it will again deliver food. Pushing the lever gives the pigs a shock on their sensitive snouts, causing a dis-utility of -1. The Big pig can move the Little pig out of the way and take all the food if they are both at the food trough together, the two pigs are equally fast getting across the room. During the time that it takes the Big pig to cross the room, Little can eat $\alpha = \frac{1}{2}$ of the food. Solve the game, note that the Little pig is getting a really good deal. There are situations where the largest person/firm has the most incentive to provide a public good, and the littler ones have an incentive to free ride. This game gives that in a pure form.

B.2.2. *The Prisoners' Dilemma.* Rational Pigs had a dominant strategy for one player, in this game, both players have a dominant strategy, and both are very very sorry to be using their dominant strategy. We give two versions of the payoffs for this game, one corresponding to the original story, one corresponding to a joint investment problem.

Prisoners' Dilemma

	Squeal	Silent
Squeal	$(-8, -8)$	$(0, -9)$
Silent	$(-9, 0)$	$(-1, -1)$

Joint Investment

	Don't invest	Invest
Don't invest	$(2, 2)$	$(12, 0)$
Invest	$(0, 12)$	$(9, 9)$

In the classical version of the Prisoners' Dilemma, two criminals have been caught, but it is after they have destroyed the evidence of serious wrongdoing. Without further evidence, the prosecuting attorney can charge them both for an offense carrying a term of 1 year. However, if the prosecuting attorney gets either prisoner to give evidence on the other (Squeal), they will get a term of 9 years. The prosecuting attorney makes a deal with the judge to reduce any term given to a prisoner who squeals by 1 year.

In the joint investment version of the game, two firms, a supplier and a manufacturer can invest in expensive, complementary technologies, and if they both do this, they will achieve the high quality output that will guarantee both high profits. The

[2]Even when we are not considering fitting through small openings.

problem is that if one of them has invested, the other firm would be better "free riding" on their investment, it's an expensive investment for both of them, and the improvements on just one side will improve profits somewhat, at no cost to the non-investor.
In both games, the non-cooperative action is the dominant strategy, and it is a disaster for both: 1 year in prison for both versus the equilibrium 8 years in the Prisoners' Dilemma; profits of 2 apiece when profits of 9 apiece are available.

A Detour

One useful way to view many economists is as apologists for the inequities of a moderately classist version of the political/economic system called, rather inaccurately, laissez faire capitalism. Perhaps this is the driving force behind the large literature trying to explain why we should expect cooperation in this situation. After all, if economists' models come to the conclusion that equilibria without outside intervention can be quite bad for all involved, they become an attack on some of the justifications used for laissez faire capitalism.
Another way to understand this literature is that we are, in many ways, a cooperative species, so a model predicting extremely harmful non-cooperation is very counter-intuitive. Yet another way to understand the lesson in this example is that, because the equilibrium is so obviously a disaster for both players, they will be willing to expend a great deal of resources to avoid the equilibrium. Elinor Ostrom's work has examined the many varied and ingenious methods that people have devised over the years to avoid bad equilibria.

B.2.3. *Contractual Respones to the Prisoners' Dilemma.* In the theory of the firm, we sometimes look to contracts to solve problems like these, problems which go by the general name of **holdup problems**. In particular, we look for contracts which, if broken, expose the violator to fines and look to structure the contracts and fines in such a way as to reach the better outcome. Returning to the supplier/manufacturer game above, let us suppose, just to make it a bit more interesting, that there are asymmetries in the payoffs,

	Don't invest	Invest
Don't invest	$(5, 7)$	$(32, 0)$
Invest	$(0, 22)$	$(28, 19)$

Consider people trying to promise their way out of this disaster with "Cross my heart and hope to die, oh, and by the way, if I fail to invest, I'll pay you the damages, 6 because 6 is enough that you have to take me seriously" If 1 signs the contractual part of that statement, and the contract is enforceable, but 2 hasn't, the payoffs become

	Don't invest	Invest
Don't invest	$(-1, 13)$	$(26, 6)$
Invest	$(0, 22)$	$(28, 19)$

This game has the same strategic structure as Rational Pigs, and the one and only equilibrium to it is in the lower left-hand corner, with payoffs $(0, 22)$. Firm 1's CEO is rather sorry to have started this process, and, being clever, she foresees the problems, and refuses to sign the contract.

Now consider contracts of the form: "I will invest, and if I do not invest while you have invested, I owe you damages of x. You will invest, and if you do not invest while I have invested, you owe me damages of x. Further, this contract is not valid unless both of us have signed it." If 1 has signed, 2 is faced with the choice between signing or not signing. If they do not sign, the game is the original one,

	Don't invest	Invest
Don't invest	$(5,7)$	$(32,0)$
Invest	$(0,22)$	$(28,19)$

If they do sign, it leaves them facing the happy situation given by

	Don't invest	Invest
Don't invest	$(5,7)$	$(32-x,x)$
Invest	$(x,22-x)$	$(28,19)$

Provided x is large enough, this game has the same *strategic* structure as the Prisoner's Dilemma, both players have a dominant strategy, Invest. Here, the equilibrium in dominant strategies has payoffs $(28,19)$. Specifically, if $x > 5$ $28 > 32-x$, $x > 7$ and $19 > 22-x$, then the strategy Invest dominates the strategy Don't Invest. Satisfying all of the inequalities simultaneously requires $x > 7$. The difference between this strategic situation and the original ones is that the equilibrium in dominant strategies is the socially optimal one. This is one version of what is called the "hold-up" problem, and in many cases finding a contract that solves the hold up problem is quite difficult, essentially because what you want to write is a contract specifying that both players will react constructively and cooperatively to whatever presently unforeseen circumstances arise. Writing out what constitutes "constructive and cooperative" behavior in legal language, which means 'language sufficiently precise that a court will understand breach of contract as the contractees understand it,' this is extremely difficult.

B.3. Games Having Only Mixed Equilibria. If you have played Hide and Seek with very young children, you may have noticed that they will always hide in the same place, and that you need to search, while loudly explaining your actions, in other places while they giggle helplessly. Once they actually understand hiding, they begin to *vary* where they hide, they *mix* it up, they *randomize*.[3] Randomizing where one hides is the only sensible strategy in games of hide and seek. One can either understand the randomizing as people picking according to some internal random number generator, or as observing some random phenomenom outside of themselves and conditioning what they do on that. Below, we will discuss, in some detail, a third understanding, due to Harsanyi, in which we can understand a mixed strategy as a description of the proportion of a population taking the different available actions. Whatever the understanding,

> ... the crucial assumption that we make for Nash equilibria is that when people randomize, they do it independently of each other.

[3]Once they are teenagers, you do not want to play this game with them.

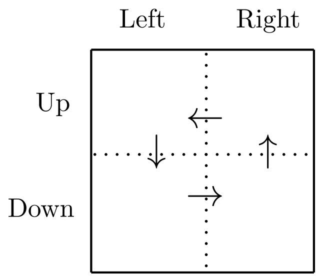
B.3.1. *Penalty Kick/Matching Pennies.* Another game in which randomization is the only sensible way to play, at least, the only sensible way to play if you play at all often, is the Penality Kick/Matching Pennies game. Let is look at the simplest version of the game, player 1 is the goal keeper, player 2 is taking the kick, the kick can go Right or Left, and the goal keeper can plan on covering the Right side or the Left side, if the goal keeper guesses correctly, the kick is blocked, if they guess incorrectly, the goal is scored.

	L	R
L	$(10, -10)$	$(-10, 10)$
R	$(-10, 10)$	$(10, -10)$

The matching pennies version of the game replaces L and R by Heads and Tails, the two players simultaneously reveal a coin showing either H or T, player 1 wins the matches, player 2 wins the mis-matches.
The unique Nash equilibrium for this game is for both players to independently randomize with probability $(\frac{1}{2}, \frac{1}{2})$: if the goal keeper plays, say, 0.4 of the time to the Left, then the kicker's best response is to shoot to the Right all of the time, thereby winning 60% of the time.
Remember, a Nash equilibrium involves mutual best responses. To check that strategies are mutual best responses requires supposing that 1 knows 2's strategy and check if 1's strategy is a best response, and *vice versa*. In the matching pennies version of the game, one way to understand the $(\frac{1}{2}, \frac{1}{2})$ randomization is to imagine you are playing against Benelock Cumberholmes or Sherdict Holmesbatch, who can, from the slightest twitches you make, perfectly infer whether you've chosen Heads or Tails. Playing against such a person, the best thing to do is to flip the coin, catch it without looking at it, and then present it to the inhumanly observant person you are playing against.

B.3.2. *Auditing/Inspection Games.* Since we have in mind applications from economics, we consider auditing or inspection games, which have the same essential structure. The idea in this class of games is that keeping someone honest is costly, so you don't want to spend effort to audit or otherwise monitoring their behavior. But if you don't monitor their behavior, they'll engage in self want to slack off. The mixed strategy equilibria that we find balance these forces. We'll give two versions of this game, a very basic one, and a more complicated one.
In this game, the CEO or CFO can Fiddle the Books (accounts) in a fashion that makes their stock options or restricted stock awards or phantom stock plans or stock appreciation rights are more valuable, or else they can prepare the quarterly report in a fashion that respects the letter and the spirit of the GAAP (Generally Accepted Accounting Practices) rules. The accounting firm that the CEO/CFO hires to audit the accounts can do an in-depth audit or they can let it slide. We are going to suppose that the auditing firm's reputation for probity is valuable enough to them that they would prefer to catch Fiddles when they occur. Putting symbols into the game, we have

	Audit	Let it slide
Fiddle	$(s-p, B+d)$	$(s+b, B)$
GAAP	$(s, B-c)$	(s, B)



Here s represents the base salary, p the penalty for being caught Fiddling the Books, c is the cost of an in-depth audit, d is the reputation benefit, net of costs, to the auditor from deterring other auditees from Fiddling, and B is the auditing firm's baseline payoff. The circling pattern of the arrows implies that, just as in the childrens' game of hide-and-seek, there cannot be an equilibrium in which the two players always choose one particular strategy, there is no pure strategy equilibrium. For there to be an equilibrium, there must be randomization. An equilibrium involving randomization is called a **mixed (strategy) equilibrium** or a **randomized equilibrium**, one not involving randomization a **pure (strategy) equilibrium**.

Let α be the probability that 1, the CEO/CFO, Fiddles, β the probability that the auditor performs an in-depth audit. From the considerations above, the only way that $0 < \alpha < 1$ is a best response iff Fiddling and GAAPing are indifferent. Whether or not they are indifferent depends on β. Specifically,

$$(\text{Fiddle}, \beta) \mapsto \beta(s-p) + (1-\beta)(s+b), \text{ and} \tag{B.1}$$

$$(\text{GAAP}, \beta) \mapsto \beta s + (1-\beta)s. \tag{B.2}$$

(B.3)

These are equal iff $\beta(-p) + (1-\beta)b = 0$, that is, iff $\beta^* = \frac{b}{b+p}$, that is, iff the probability of an in-depth audit is $\frac{b}{b+p}$. In just the same way, $0 < \beta < 1$ is a best response iff Auditing and Letting it Slide are indifferent. Whether or not they are indifferent depends on α. Specifically,

$$(\text{Audit}, \alpha) \mapsto \alpha(B+d) + (1-\alpha)(B-c), \text{ and} \tag{B.4}$$

$$(\text{Slide}, \alpha) \mapsto \alpha B + (1-\alpha)B. \tag{B.5}$$

(B.6)

We have indifference iff $\alpha^* = \frac{c}{c+d}$, that is, iff the probability of Fiddles is $\frac{c}{c+d}$. Thus the unique equilibrium for this game is

$$\sigma^* = ((\alpha^*, (1-\alpha^*)), (\beta^*, (1-\beta^*))) = \left(\left(\frac{c}{c+d}, \frac{d}{c+d}\right), \left(\frac{b}{b+p}, \frac{p}{b+p}\right)\right). \tag{B.7}$$

We have less Fiddling if c is small and d is large, we have less Auditing if b is small and p is large.

B.4. Games with Pure and Mixed Equilibria. So far we have seen games that have only a pure equilibrium or only a mixed equilibrium. This does not exhaust the possibilities.

B.4.1. *The Stag Hunt.* Here is another 2×2 game,

Stag Hunt

	Stag	Rabbit
Stag	(S, S)	$(0, R)$
Rabbit	$(R, 0)$	(R, R)

As before, there is a story for this game: there are two hunters who live in villages at some distance from each other in the era before telephones; they need to decide whether to hunt for Stag or for Rabbit; hunting a stag requires that both hunters have their stag equipment with them, and one hunter with stag equipment will not catch anything; hunting for rabbits requires only one hunter with rabbit hunting equipment. The payoffs have $S > R > 0$, e.g. $S = 20$, $R = 1$, which gives

	Stag	Rabbit
Stag	$(20, 20)$	$(0, 1)$
Rabbit	$(1, 0)$	$(1, 1)$

This is a **coordination game**, if the players' coordinate their actions they can both achieve higher payoffs. There are two obvious Nash equilibria for this game. There is a role then, for some agent to act as a coordinator.

It is tempting to look for social roles and institutions that coordinate actions: matchmakers; advertisers; publishers of schedules e.g. of trains and planes. Sometimes we might imagine a tradition that serves as coordinator — something like we hunt stags on days following full moons except during the spring time.

Macroeconomists, well, some macroeconomists anyway, tell stories like this but use the code word "sunspots" to talk about coordination. This may be because overt reference to our intellectual debt to Keynes is out of fashion. In any case, any signals that are correlated and observed by the agents can serve to coordinate the peoples' actions.

B.4.2. *Correlating Behavior in the Stag Hunt.* One version of this is that on sunny days, which happen γ of the time, the hunters go for stag, and on the other days, they go for rabbit. If both hunters always observe the same "signal," that is, the weather is the same at both villages, this gives the following distribution over outcomes:

	Stag	Rabbit
Stag	γ	0
Rabbit	0	$(1 - \gamma)$

This distribution over A is our first **correlated equilibrium**. It is important to notice that this notion of equilibrium involves there being a stage prior to actual play of the game. We will not study correlated equilibria in this class.

B.4.3. *Mixed Equilibria for the Stag Hunt.* A correlated equilibrium that: 1) is not point mass on some action, and 2) has the actions of the players stochastically independent is a **mixed strategy Nash equilibrium**, a mixed eq'm for short.

Let $\alpha \in [0, 1]$ be the probability that 1 goes for Stag, $\beta \in [0, 1]$ be the probability that 2 goes for Stag, the independence gives the distribution

	Stag	Rabbit
Stag	$a = \alpha\beta$	$b = \alpha(1-\beta)$
Rabbit	$c = (1-\alpha)\beta$	$d = (1-\alpha)(1-\beta)$

Observation: Given the independence, the only way for those numbers to be a correlated equilibrium and have $0 < \alpha < 1$ is to have

$$20\beta + 0(1-\beta) = 1\beta + 1(1-\beta), \text{ i.e. } \beta = 1/20.$$

By symmetry, $\alpha = 1/20$ is the only way to have $0 < \beta < 1$ in equilibrium. Combining, this game has 3 Nash equilibria, the two pure strategy eq'a, (S, S) and (R, R), and the mixed equilibrium, $((1/20, 19/20), (1/20, 19/20))$. It also has infinitely many correlated equilibria.
Coordination problems often turn out to be deeply tied to complimentarities in the players' strategies. 1's expected utility from play of $((\alpha, (1-\alpha)), (\beta, (1-\beta)))$ is

$$U_1(\alpha, \beta) := E\, u_1 = 20\alpha\beta + 1(1-\alpha),$$

and

$$\frac{\partial^2 U_1}{\partial\alpha\partial\beta} = 20 > 0.$$

This means that increases in β increase 1's marginal utility of increases in α. By symmetry, both players' best responses have this property. It is this that opens the possibility of multiple eq'a.

B.4.4. *Battle of the Sexes.*

	Opera	Rodeo
Opera	$(3, 5)$	$(1, 0)$
Rodeo	$(0, 1)$	$(5, 3)$

Two pure and one mixed Nash equilibrium.

B.4.5. *Hawk/Dove.*

	Hawk	Dove
Hawk	$(-1, -1)$	$(5, 0)$
Dove	$(0, 5)$	$(1, 1)$

Two pure and one mixed Nash equilibrium.

B.5. Exercises.

Problem B.1. *Two firms, a supplier and a manufacturer can invest in expensive, complementary technologies, and if they both do this, they will achieve the high quality output that will guarantee both high profits. The problem is that if one of them has invested, the other firm would be better "free riding" on their investment, it's an expensive investment for both of them, and the improvements on just one side will improve profits somewhat, at no cost to the non-investor. Putting numbers on the payoffs, let us suppose they are*

	Don't invest	*Invest*
Don't invest	$(5, 7)$	$(32, 0)$
Invest	$(0, 22)$	$(28, 19)$

a. *Suppose that a vertical merger or acquisition is arranged and that the joint firm receives the sum of the payoffs to the two firms. What is the optimal investment pattern for the joint firm?*
b. *Suppose that one of the firms hasn't had the advantage of your experience with the idea of solving dynamic interactions by "looking forward and solving backwards." Not knowing this cardinal principle, they decide that they will move first, invest, and give the other firm every incentive to invest. What will be the result?*
c. *Consider contracts of the form: "I will invest, and if I do not invest while you have invested, I owe you damages of x. You will invest, and if you do not invest while I have invested, you owe me damages of x. Further, this contract is not valid unless both of us have signed it." For what values of x will the contract have the property that signing the contract and then investing becomes the dominant strategy?*

PROBLEM B.2. *If a chicken packing firm leaves the fire escape doors operable, they will lose c in chickens that disappear to the families and friends of the chicken packers. If they nail or bolt the doors shut, which is highly illegal, they will no longer lose the c, but, if they are inspected (by say OSHA), they will be fined f. Further, if the firedoor is locked, there is a risk, ρ, that they will face civil fines or criminal worth F if there is a fire in the plant that kills many of the workers because they cannot escape.*[4] *Inspecting a plant costs the inspectors k, not inspecting an unsafe plant costs B in terms of damage done to the inspectors' reputations and careers. Filling in the other terms, we get the game*

		Inspectors	
		Inspect	*Not inspect*
Imperial	*unlocked*	$(\pi - c, -k)$	$(\pi - c, 0)$
	locked	$(\pi - f - \rho F, f - k)$	$(\pi - \rho F, -B)$

If f and ρF are too low, specifically, if $c > f + \rho F$, then Imperial has a dominant strategy, and the game is, strategically, another version of Rational Pigs.
If $f + \rho F > c > \rho F$ and $f - k > -B$, neither player has a dominant strategy, and there is only a mixed Nash equilibrium. In this case, we have another instance of a game like the inspection game.
Assume that $f + \rho F > c > \rho F$ and $f - k > -B$ in the inspection game. Show that the equilibrium is unchanged as π grows. How does it change as a function of c?

PROBLEM B.3. [*Cournot equilibrium*] *Two firms compete by producing quantities $q_i \geq 0$ and $q_j \geq 0$ of a homogeneous good, and receiving profits of the form*

$$\pi_i(q_i, q_j) = [p(q_i + q_j) - c]q_i,$$

where $p(\cdot)$ is the inverse market demand function for the good in question. Assume that $p(q) = 1 - q$ and that $0 \leq c \ll 1$.

[4] White collar decisions that kill blue collar workers rarely result in criminal prosecutions, and much more rarely in criminal convictions. See Mokhiber for some rather depressing statistics. Emmett Roe, the owner of the Imperial chicken processing plant that locked the doors killed 25 workers and injured 56 more on September 3, 1991. He plea-bargained to 25 counts of involuntary manslaughter, was sentenced to 20 years in prison, and was eligible for early release after 3 years, and was released after 4 and a half years, that is, 65 days for each of the dead. The surviving injured workers and families of the dead only won the right to sue the state for failure to enforce safety codes on February 4, 1997, after a five-year battle that went to the state Court of Appeals. Damage claims will be limited to $100,000 per victim.

a. *For each value of q_j, find i's best response, that is, find $Br_i(q_j)$.*
b. *Find the unique point at which the best response curves cross. This is called the* ***Cournot equilibrium****.*
c. *Show that there is no mixed strategy equilibrium for this game.*
d. *Show that social surplus is inefficiently small in the Cournot equilibrium.*
e. *Though this is perhaps a bit artificial, show that, by appropriately including consumer surplus into the payoffs of the firms, one can increase the social welfare of the equilibrium.*

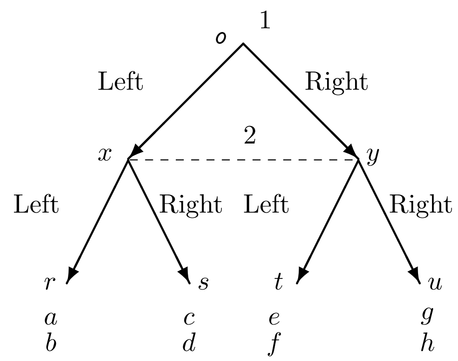
C. Commitment Power and First/Second Mover Advantages

There are two competing intuitions about being able to commit to an action: by forcing other people to react to me, I gain an advantage; by keeping my options free until they have committed, I gain an advantage. In situations where the first is true, I have a first mover advantage, in situations where the second is true, I have a second mover advantage. We study these by drastically changing the strategic situation of simultaneous 2×2 games to a sequential move game in which first one player moves, then the other observes this and reacts to it.

C.1. Simple Game Trees for Commitment. Start with the basic 2×2 game,

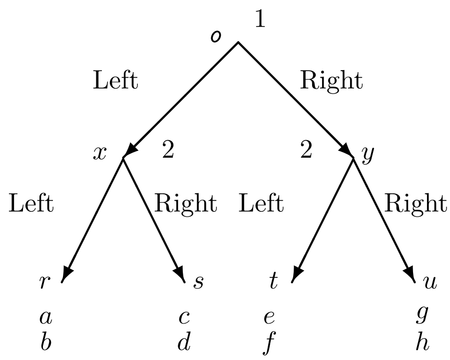
	Left	Right
Up	(a, b)	(c, d)
Down	(e, f)	(g, h)

With some extra notations for the nodes, o, x, y, r, s, t, and u, this simultaneous move game has the following game tree representation.



The dotted line joining nodes x and y represents an information set for 2. We interpret information sets as follows: at that information set, 2 must pick between Left and Right, but 2 does not know whether they are at x or at y. The claim is that this is the same as simultaneous choice by the players: it doesn't matter if 1 has actually chosen when 2 makes their choice so long as 2 does not know what 1 has chosen; it doesn't matter if 2 has not chosen when 1 maker his/her choice so long as 2 will not learn what 1 has chosen by the time 2 chooses.

C.2. Adding Commitment. If 2 does know what 1 has picked, we have a different game, it has the same tree as above with the huge and glaring exception that 2 knows 1's action when he/she picks. In other words, 1 can commit to an action before 2 has any say in the matter. The associated game tree is



Mutatis mutandi, one can change this game so that 2 moves first, commits to an action before 1 can do anything.
Note that there are two choice nodes for 2 at this point, hence 2 has a total of 4 strategies in this game. We solve this game using the "Look forward, solve backward" advice. At node x, 2 will choose the larger of the payoffs b and d, at node y, 2 will choose the larger of the two payoffs f and h. Moving backwards through the tree, upwards in this graphical formulation, knowing that 2 will act in this fashion, 1 picks the best of their options.

C.3. Examples of Commitment. Consider the Stag Hunt,

	Stag	Rabbit
Stag	$(20, 20)$	$(0, 1)$
Rabbit	$(1, 0)$	$(1, 1)$

The unique "look forward, solve backward" equilibrium outcome when either 1 or 2 moves first has payoffs $(20, 20)$. In a coordination game with a Pareto dominant pure strategy equilibrium, both players want someone, anyone to coordinate their actions.
By contrast, consider the Battle of the Sexes,

	Opera	Rodeo
Opera	$(3, 5)$	$(1, 0)$
Rodeo	$(0, 1)$	$(5, 3)$

Here if 1 moves first, the unique "look forward, solve backwards" equilibrium has payoffs $(5, 3)$, if 2 moves first, the unique "look forward, solve backwards" equilibrium has payoffs $(3, 5)$. Both players would prefer to have the commitment power in this game, both have a first mover advantage.
Consider the joint investment/hold up game,

	Don't invest	Invest
Don't invest	$(5, 7)$	$(32, 0)$
Invest	$(0, 22)$	$(28, 19)$

Whoever moves first, the unique "look forward, solve backwards" equilibrium has payoffs $(5, 7)$. This kind of commitment power cannot solve this problem.
Consider the penalty kick game, if the goal keeper must commit to going one way and the penalty kicker knows this, the outcome is easy to see.

	L	R
L	$(10, -10)$	$(-10, 10)$
R	$(-10, 10)$	$(10, -10)$

If 1 moves first, the unique "look forward, solve backwards" equilibrium has payoffs $(-10, 10)$, if 2 moves first, the unique "look forward, solve backwards" equilibrium has payoffs $(10, -10)$. Here there is a strong second mover advantage.

C.4. A Little Bit of Decision Theory. We assume that people act so as to maximize their expected utility taking others' actions/choices as given. In other words, assuming that what they choose in their optimization does not affect what others choose. Here is a useful Lemma. It may seem trivial, but it turns out to have strong implications for our interpretations of equilibria in game theory.

Lemma III.1 (Rescaling). *Suppose that $u : A \times \Omega \to \mathbb{R}$ is bounded and measurable. $\forall Q_a, Q_b \in \Delta(A)$, $\forall P \in \Delta(\mathcal{F})$,*

$$\int_A \left[\int_\Omega u(x, \omega)\, dP(\omega) \right] dQ_a(x) \geq \int_A \left[\int_\Omega u(y, \omega)\, dP(\omega) \right] dQ_b(y)$$

iff

$$\int_A \left[\int_\Omega [\alpha \cdot u(x, \omega) + f(\omega)]\, dP(\omega) \right] dQ_a(x) \geq \int_A \left[\int_\Omega [\alpha \cdot u(y, \omega) + f(\omega)]\, dP(\omega) \right] dQ_b(y)$$

for all $\alpha > 0$ and P-integrable functions f.

Remember how you learned that Bernoulli utility functions were immune to multiplication by a positive number and the addition of a constant? Here the constant is being played by $F := \int_\Omega f(\omega)\, dP(\omega)$.

Proof. Suppose that $\alpha > 0$ and $F = \int f\, dP$. Define $V(x) = \int_\Omega u(x, \omega)\, dP(\omega)$. The Lemma is saying that $\int_A V(x)\, dQ_a(x) \geq \int_A V(y)\, dQ_b(y)$ iff $\alpha \left[\int_A V(x)\, dQ_a(x)\right] + F \geq \alpha \left[\int_A V(y)\, dQ_b(y)\right] + F$, which is immediate. □

C.5. Back to Commitment. After adding a function of 2's choice to 1's payoffs and *vice versa*, we do not change the strategic structure of the simultaneous move game, but we can drastically change the outcome of the games with commitment power. For example, in the following game, 1 would like to commit to playing the strategy that is strictly dominated in the simultaneous move game, giving payoffs $(11, 2)$, while 2 would like to commit to L, giving payoffs $(1, 4)$.

	L	R
U	$(0, 0)$	$(11, 2)$
D	$(1, 4)$	$(12, 3)$

C.6. Utility Rankings in 2×2 Games. We are going to focus on games where there are no ties — for each $i \in I$ and a_{-i}, $u_i(a_i, a_{-i}) \neq u_i(b_i, a_{-i})$ for $a_i \neq b_i$. Within this class of 2×2 games, we've seen four types:

(1) Games in which both players have a dominant strategy, e.g. Prisoners' Dilemma;
(2) Games in which exactly one player has a dominant strategy, e.g. Rational Pigs;
(3) Games in which neither player has a dominant strategy and there are three equilibria, e.g. Stag Hunt, Battle of the Partners, Deadly Force, Chicken;
(4) Games in which neither player has a dominant strategy and there is only a mixed strategy equilibrium, e.g. Hide and Seek, Matching Pennies, Inspection.

The basic result for 2×2 games with no ties is that these four types of games exhaust the possibilities.

C.7. Rescaling and the Strategic Equivalence of Games. Consider the 2×2 game

	Left	Right
Up	$(\boldsymbol{a}, e)$	$(\boldsymbol{b}, f)$
Down	$(\boldsymbol{c}, g)$	$(\boldsymbol{d}, h)$

where we've put 1's payoffs in bold for emphasis. Since we're assuming there are no ties for player 1, $\boldsymbol{a} \neq \boldsymbol{c}$ and $\boldsymbol{b} \neq \boldsymbol{d}$. Consider the function $f_1(a_2)$ given by $f_1(\text{Left}) = -\boldsymbol{c}$ and $f_1(\text{Right}) = -\boldsymbol{b}$. Lemma III.1 tells us that adding f_1 to 1's payoffs cannot change either CEq or Eq. When we do this we get the game

	Left	Right
Up	$(a - c, \boldsymbol{e})$	$(0, \boldsymbol{f})$
Down	$(0, \boldsymbol{g})$	$(d - b, \boldsymbol{h})$

where we've now put 2's payoffs in bold for emphasis. Since we're assuming there are no ties for player 2, $\boldsymbol{e} \neq \boldsymbol{f}$ and $\boldsymbol{g} \neq \boldsymbol{h}$. Consider the function $f_2(a_1)$ given by $f_2(\text{Up}) = -\boldsymbol{f}$ and $f_1(\text{Down}) = -\boldsymbol{g}$. Lemma III.1 tells us that adding f_2 to 2's payoffs cannot change either CEq or Eq. When we do this we get the game

	Left	Right
Up	(x, y)	$(0, 0)$
Down	$(0, 0)$	(r, s)

where $x = a - c$, $y = e - f$, $r = d - b$, $s = h - g$, and $x, y, r, s \neq 0$. We've just proved that all 2×2 games with no ties are equivalent to 2×2 games with $(0, 0)$'s in the off-diagonal positions.
Applying this procedure to the six of the 2×2 games we've seen yields

Prisoners' Dilemma

	Squeal	Silent
Squeal	$(1,1)$	$(0,0)$
Silent	$(0,0)$	$(-1,-1)$

Rational Pigs

	Push	Wait
Push	$(-3.9,-1.1)$	$(0,0)$
Wait	$(0,0)$	$(1,-2)$

Matching Coins

	H	T
H	$(2,-2)$	$(0,0)$
T	$(0,0)$	$(2,-2)$

Stag Hunt

	Stag	Rabbit
Stag	$(9,9)$	$(0,0)$
Rabbit	$(0,0)$	$(1,1)$

Battle of the Partners

	Dance	Picnic
Dance	$(12,8)$	$(0,0)$
Picnic	$(0,0)$	$(8,12)$

Chicken

	Chicken	Thru
Chicken	$(-10,-10)$	$(0,0)$
Thru	$(0,0)$	$(-7,-7)$

Once games are in this form, what matters for strategic analysis are the signs of the utilities x, y, r, s, e.g. the sign patterns for the first two games are

Prisoners' Dilemma

	Squeal	Silent
Squeal	$(+,+)$	$(0,0)$
Silent	$(0,0)$	$(-,-)$

Rational Pigs

	Push	Wait
Push	$(-,-)$	$(0,0)$
Wait	$(0,0)$	$(+,-)$

There are 2^4 possible sign patterns, but there are not 2^4 strategically distinct 2×2 games.

Definition III.5. *If Γ and Γ' are both 2×2 games, we say that are* ***strategically equivalent*** *if they have the same sign pattern after any finite sequence of*

(a) *applying Lemma III.1 to arrive at 0's off the diagonal,*
(b) *relabeling a player's actions, or*
(c) *relabeling the players.*

For example, in Rational Pigs, Little Pig was player 1 and Big Pig was player 2. If we relabeled them as 2 and 1 respectively, we would not have changed the strategic situation at all. We would have changed how we represent the game, but that should make no difference to the pigs. This would give a game with the sign pattern

	Push	Wait
Push	$(-,-)$	$(0,0)$
Wait	$(0,0)$	$(-,+)$

If we were to relabel the actions of one player in Chicken, we'd have the game

	a_2	b_2
a_1	$(0,0)$	$(-10,-10)$
b_1	$(-7,-7)$	$(0,0)$

which is equivalent, via Lemma III.1, to a game with the sign pattern

	a_2	b_2
a_1	$(+,+)$	$(0,0)$
b_1	$(0,0)$	$(+,+)$

In just the same way, the following two sign patterns, from games like Matching Coins are equivalent,

Matching Coins

	H	T
H	$(+,-)$	$(0,0)$
T	$(0,0)$	$(+,-)$

Coins Matching

	H	T
H	$(-,+)$	$(0,0)$
T	$(0,0)$	$(-,+)$

Problem C.1. *Show that all* 2×2 *games without ties are equivalent to one of the four categories identified at the beginning of this section* (p. 91).

C.8. The gap between equilibrium and Pareto rankings. The defining charateristic of an equilibrium is the mutual best response property. Pareto optimality arguments are very peculiar from the mutual best response point of view.

C.8.1. *Stag Hunt reconsidered.* An implication of Lemma III.1 is that the following two versions of the Stag Hunt are strategically equivalent.

Stag Hunt

	Stag	Rabbit
Stag	(S,S)	$(0,R)$
Rabbit	$(R,0)$	(R,R)

Hunting Stag

	Stag	Rabbit
Stag	$(S-R,S-R)$	$(0,0)$
Rabbit	$(0,0)$	(R,R)

Remember that $S > R > 0$, which makes the Pareto ranking of the pure strategy equilibria in the first version of the game easy and clear. However, the Pareto rankings of the two pure strategy equilibria agree across the two versions of the game only if $S > 2R$. If $R < S < 2R$, then the Pareto criterion would pick differently between the equilibria in the two strategically equivalent games.

C.8.2. *Prisoners' Dilemma reconsidered.* An implication of Lemma III.1 is that the following two versions of the Prisoners' Dilemma are strategically equivalent.

Prisoners' Dilemma

	Squeal	Silent
Squeal	$(1,1)$	$(0,0)$
Silent	$(0,0)$	$(-1,-1)$

Dilemma of the Prisoners

	Squeal	Silent
Squeal	$(-14,-14)$	$(0,-15)$
Silent	$(-15,0)$	$(-1,-1)$

If we take the Pareto criterion seriously, we feel very differently about the equilibria of these two games. In the first one, the unique equilibrium is the Pareto optimal feasible point, in the second, the unique equilibrium is (very) Pareto dominated.

C.9. Minimizing $\sum_i v_i(a)$ for Equivalent Utility Functions. We would, perhaps, never imagine that equilibria are the actions that solve $\min_{a \in A} \sum_i u_i(a)$.

Definition III.6. *For any metric space M and and $\mu \in \Delta(M)$, the **support of a** μ is* $\text{supp}\,(\mu) = \bigcap\{F : \mu(F) = 1,\ F \text{ a closed set}\}$.

Problem C.2. *Suppose that (M,d) is separable. Show that* $\text{supp}\,(\mu)$ *can be expressed as a countable intersection and that* $\mu(\text{supp}\,(\mu)) = 1$.

Recall that a $\mathbb{R}$-valued function is Lipschitz continuous with Lipschitz constant L if $|f(x) - f(y)| \leq L \cdot d(x,y)$.

For the following, we say that utility functions u_i and v_i are equivalent if $v_i(a) = \beta u_i(a) + f(a_{-i})$ for some $\beta > 0$ and function $a_{-i} \mapsto f(a_{-i})$.

Theorem III.1. *If* $\Gamma = (A_i, u_i)_{i\in I}$ *is a compact and Lipschitz continuous game and* $a^* = (a_i^*)_{i\in I} \in Eq(\Gamma)$ *is a pure strategy equilibrium, then there is an equivalent set of utility functions,* $(v_i)_{i\in I}$*, such that* a^* *solves* $\min_{a\in A} \sum_{i\in I} v_i(a)$ *and the game* $\Gamma_v = (A_i, v_i)_{i\in I}$ *is also Lipschitz continuous.*

To put it another way, for any pure strategy of a Lipschitz game, there is an equivalent set of utility functions, also Lipschitz, for which the equilibrium minimizes the sum of utilities.

Problem C.3. *Prove Theorem* III.1. *If possible, prove the following generalizations:*

(a) *Suppose that the utility functions are jointly continuous but are not Lipschitz in a compact and continuous game.* [*Hint: continuous functions on compact sets are uniformly continuous.*]
(b) *Suppose that the utility functions are jointly continuous and the* A_i *are metric spaces.*
(c) *Suppose that the utility functions are bounded, but otherwise unrestricted.*

C.10. Conclusions about Equilibrium and Pareto rankings. From these examples, we should conclude that the Pareto criterion and equilibrium have little to do with each other. This does not mean that we should abandon the Pareto criterion — the two versions of the Prisoners' Dilemma are equivalent only if we allow player i's choice to add 15 years of freedom to player $j \neq i$. Such a change does not change the strategic considerations, but it drastically changes the social situation being analyzed.

In other words: the difference between the two versions of the Prisoners' Dilemma is that we have stopped making one person's action, Squeal, have so bad an effect on the other person's welfare. One might argue that we have made the game less interesting by doing this. In particular, if you are interested in (say) understanding how people become socialized to pick the cooperative action when non-cooperation is individually rational but socially disastrous, the new version of the Prisoners' Dilemma seems to be no help whatsoever. The new version has synchronized social welfare and individual optimization.

My argument about socialization would be phrased in terms of changes to the utility functions, though not necessarily the changes given in Lemma III.1. Utility functions are meant to represent preferences, and preferences are essentially indistinguishable from revealed preferences, that is, from choice behavior. If one thinks that both being Silent is the right outcome, then you need to change the preferences so that the players prefer being Silent.

Socialization is one very effective way to change preferences. Many people feel badly if their actions harm others, even others they do not personally know, and make choices so as to not do harm. I take this as partial evidence that they have preferences that include consequences to others.[5] The reason that it is only partial evidence runs as follows: if your choices are being watched, then doing harm to others can earn you a reputation as someone who does harm; who will be friends with you/engage in potentially mutually profitable activities if this is your reputation?

[5]There is some direct evidence, FMRI readings on the brains of people playing a repeated prisoners' dilemma showed increased blood flow to the pleasure centers when they did the nice thing, and this effect was somewhat independent of payoffs.

To study socialization via utility functions/preferences to cooperative actions, one needs to study how preferences are changed in a fashion somehow independent of the situations that people are involved in. This requires observing people when they think their actions are observed and when they do not think they are observed, not the easiest thing to arrange, but a staple of melodramas and morality fables.

C.11. Risk dominance and Pareto rankings. One possible reaction to the previous section is "Yeah, yeah, that's all fine so far as the mathematics of equilibrium is concerned, but when I write down a game with specified payoffs, I really mean that those payoffs represent preferences, they are not merely devices for specifying best responses." If you take this point of view (or many points of view like it), analyzing Pareto optimality again makes sense.[6] However, if you take this point of view, you are stuck when you come to games in which the players disagree about which equilibrium is better. One way to try to resolve this is using the idea of risk dominance.
In some coordination games, we (might have) favored one equilibrium outcome over another because it was better for everyone. In the following game (with the same best response pattern as the Stag Hunt), Pareto ranking does not work,

	L	R
T	$(5,6)$	$(3,2)$
B	$(0,2)$	$(6,4)$

One idea that does work to pick a unique equilibrium for this game is called **risk dominance**. The two pure strategy equilibria for this game are $e^1 = (T, L)$ and $e^2 = (B, R)$. The set of σ_2 for which T, 1's part of e^1, is a best response for player 1 is $S_1^{e^1} = \{\sigma_2 : \sigma_2(L) \geq 3/8\}$. The set of σ_2 for which B, 1's part of e^2, is a best response for player 1 is $S_1^{e^2} = \{\sigma_2 : \sigma_2(L) \leq 3/8\}$. Geometrically, S_{e^1} is a larger set than S_{e^2}. One way to interpret this is to say that the set of beliefs that 1 might hold that make 1's part of e^1 a best response is larger that the set that make 1's part of e^2 a best response. In this sense, it is "more likely" that 1 plays his/her part of e^1 than his/her part of e^2. Similarly, $S_2^{e^1} = \{\sigma_1 : \sigma_1(T) \geq 1/3\}$ is geometrically larger than the set $S_2^{e^2} = \{\sigma_1 : \sigma_1(B) \leq 1/3\}$, so that it is "more likely" that 2 plays his/her part of e^1 than his/her part of e^2. This serves as a definition of **risk dominance**, e^1 risk dominates e^2.
What we have just seen is that it is possible to invent a principle that takes over when Pareto ranking does not pick between equilibria. There are at least two more problems to overcome before we can reach an argument for systematically picking a single equilibrium, even in the set of 2×2 games that we have been looking at.

(1) The two players may disagree about which equilibrium risk dominates as in

	L	R
T	$(5,6)$	$(3,5)$
B	$(0,2)$	$(6,4)$

which is the same as the previous game, except that 2's payoff to (T, R) has been changed from 2 to 5. The sets $S_1^{e^1}$ and $S_1^{e^2}$ are unchanged,

[6] The previous section is just a bad dream to be ignored while you get on with the serious business of proving that all works out for the best in this best of all possible worlds.

but $S_2^{e^1} = \{\sigma_1 : \sigma_1(T) \geq 2/3\}$ and $S_2^{e^2} = \{\sigma_1 : \sigma_1(B) \leq 2/3\}$. Now e^1 risk dominates e^2 for 1 but e^2 risk dominates e^1 for 2.

(2) Risk dominance may disagree with the Pareto ranking, so we actually need to decide whether we believe more strongly in risk dominance than in Pareto ranking. Return to the Stag Hunt,

	Stag	Rabbit
Stag	(S, S)	$(0, R)$
Rabbit	$(R, 0)$	(R, R)

where $S > R > 0$. While $(S, S)^T \gg (R, R)^T$, making S look good, for each hunter the Rabbit strategy looks less 'risky' in the sense that they are less dependent on the actions of the other. Arguing directly in terms of the risk dominance criterion, the Stag equilibrium risk dominates if $S > 2R$, while Rabbit risk dominates of $2R > S > R$. However, Stag always Pareto dominates.

Even if Pareto rankings do not survive the utility transformations of Lemma III.1, risk dominance rankings do.

PROBLEM C.4. *Suppose that a 2×2 game without ties, $\Gamma = (A_i, u_i)$, has two pure strategy equilibria, e^1 and e^2 and that e^1 risk dominates e^2. Suppose that $\Gamma' = (A_i, v_i)$ where the v_i are derived from the u_i using any of the transformations allowed in Lemma* III.1. *We know that e^1 and e^2 are equilibria of Γ'. Show that e^1 risk dominates e^2 in Γ'.*

D. Background on Bayesian Information Structures

From Micro II, or other sources ([**5**] on Blackwell orderings of information, [**3**] and [**2**]), we have the following model of and results about information structures.

Model:

1. utility depends on an action, $a \in A$, and a state $\omega \in \Omega$, $u(a, \omega)$;
2. there is a random signal that will take a value $s \in S$, and the signal has a joint distribution $Q \in \Delta(S \times \Omega)$;
3. the marginal of Q on Ω is the **prior**, P, defined by $P(E) := Q(S \times E)$ for $E \subset \Omega$;
4. one picks $a^*(s)$ to maximizes

(D.1)
$$V_u(Q) := \int_{S\times\Omega} u(a^*(s), \omega)\, dQ(s, \omega).$$

Results about an equivalent formulation of the optimization problem:

1. for moderately well-behaved spaces S and Ω, the problem in (D.1) can be replaced by solving

(D.2)
$$v_u(\beta) = \max_{a\in A} \int_{\Omega} u(a, \omega)\, d\beta(\omega|s)$$

   where $\beta(\cdot|s)$ is the posterior distribution on Ω after the signal s has been observed;
2. with $s \sim \text{marg}_S(Q)$, the mapping $\beta(\cdot|s)$ induces a distribution, $\mathcal{Q}$, on $\Delta(\Delta(\Omega))$;
3. [iterated expectations] for all $E \subset \Omega$, $\int \beta(E)\, d\mathcal{Q}(\beta) = P(E)$;

4. for all u,

$$V_u(Q) = V_u(\mathcal{Q}) := \int_{\Delta(\Omega)} v_u(\beta)\, d\mathcal{Q}(\beta). \tag{D.3}$$

Results about ranking information structures:
1. either Q or $\mathcal{Q}$ is an **information structure**;
2. if your utility function is $u : A \times \Omega \to \mathbb{R}$, your value is
3. for all u, $V_u(Q)$, equivalently, $V_u(\mathcal{Q})$ is the value of your information;
4. we can rank information structures by $\mathcal{Q}' \succsim \mathcal{Q}$ if for all u, $V_u(\mathcal{Q}') \geq V_u(\mathcal{Q})$;
5. let Q' and Q be two joint distributions on $S \times \Omega$ with the same prior, and let s' and s denote their two signals, from Blackwell ([**3**] and [**2**]) we have $Q' \succsim Q$ iff (s, ω) has the distribution of some $(f(s', \omega'), \omega)$ where $\omega' \in \Omega'$ is distributed independently of Ω and s'.

The functions $s = f(s', \omega')$ are sometimes called "scrambles" or "Markov scrambles" of s'.

Problem D.1. *For the following information structures, if $Q' \succsim Q$ verify that, if not, then give two decision problems, u' and u, such that $V_{u'}(Q') > V_u(Q)$ and $V_u(Q) > V_{u'}(Q')$.*

a. $Q =$

s_2	1/12	2/12
s_1	2/12	7/12
	ω_1	ω_2

$Q' =$

s_2	2/12	3/12
s_1	1/12	6/12
	ω_1	ω_2

b. $Q =$

s_2	3/12	5/12
s_1	3/12	1/12
	ω_1	ω_2

$Q' =$

s_3	2/12	1/12
s_2	2/12	2/12
s_1	2/12	3/12
	ω_1	ω_2

c. $Q =$

s_2	2/12	5/12	2/12
s_1	1/12	1/12	1/12
	ω_1	ω_2	ω_3

$Q' =$

s_2	0/12	0/12	1/12
s_2	0/12	3/12	0/12
s_1	3/12	3/12	2/12
	ω_1	ω_2	ω_3

Problem D.2. *The following are basic.*

a. If $Q \in \Delta(S \times \Omega)$ is the product of some $p \in \Delta(S)$ and the prior, P, then for all Q', $Q' \succsim Q$.

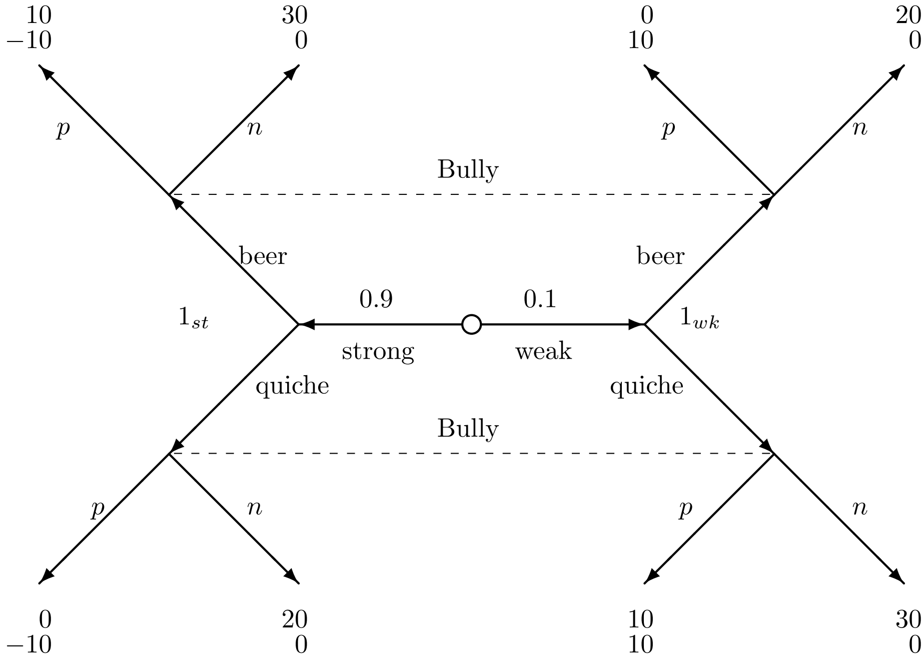
b. If $S = \Omega$, and $Q'' \in (S \times \Omega)$ puts all its mass on the diagonal, $D = \{(s, \omega) : s = \omega\}$, then for all Q', $Q'' \succsim Q'$.

c. $\mathcal{Q}' \succsim \mathcal{Q}$ iff for all convex $V : \Delta(\Omega) \to \mathbb{R}$, $\int V(\beta)\, d\mathcal{Q}'(\beta) \geq \int V(\beta)\, d\mathcal{Q}(\beta)$.

E. Some Material on Signaling and Other Dynamic Games

More information is better in single person decision problems. It is not in strategic situations. For a rather militaristic example, someone captured by the U.S. army on a battle field would rather have it be known that I have no information worth being tortured for.

In less militaristic contexts, if I have information which, if you knew it, would determine your best course of action, then potentially at least, I have incentives to lie to you. However, you know this, so whether or not you believe which of my statements is a more complicated issue. Here is a specific entry-deterrence game, due to Cho and Kreps (1987) that starts us on these issues.



There is a fellow who, on 9 out of every 10 days on average, rolls out of bed like Popeye on spinach. When he does this we call him "strong." When strong, this fellow likes nothing better than Beer for breakfast. On the other days he rolls out of bed like a graduate student recovering from a comprehensive exam. When he does this we call him "weak." When weak, this fellow likes nothing better than Quiche for breakfast. In the town where this schizoid personality lives, there is also a swaggering bully. Swaggering bullies are cowards who, as a result of deep childhood traumas, need to impress their peers by picking on others. Being a coward, he would rather pick on someone weak. He makes his decision about whether to pick, p, or not, n, after having observed what the schizoid had for breakfast.
We are going to solve this game in two different fashions, the first works with the normal form, the second uses what is called the agent normal form. The agent normal form corresponds to a different version of the motivating story above, and solving it by iterated deletion requires the use of self-referential tests.

E.1. Working in the normal form. Taking expectations over Nature's move, the 4×4 normal form for this game is

(st,wk)\(b,q)	(p,p)	(p,n)	(n,p)	(n,n)
(b,b)	$(9,-8)$	$(9,-8)$	$(29,0)$	$(29,0)$
(b,q)	$(10,-8)$	$(12,-9)$	$(28,1)$	$(30,0)$
(q,b)	$(0,-8)$	$(18,1)$	$(2,-8)$	$(20,0)$
(q,q)	$(1,-8)$	$(21,0)$	$(1,-8)$	$(21,0)$

The equilibrium set for this game can be partitioned into E_1 and E_2 where

$$E_1 = \{((q,q),(0,\beta,0,1-\beta)) : 21 \geq 12\beta + 30(1-\beta), \text{ i.e. } \beta \geq \frac{1}{2}\},$$

and

$$E_2 = \{((b,b),(0,0,\beta,1-\beta)) : 29 \geq 28\beta + 30(1-\beta) \text{ i.e. } \beta \geq \frac{1}{2}\}.$$

Note that $\mathbb{O}(\cdot)$ is constant on the two closed and connected sets E_1 and E_2, so that once again, the players are indifferent to points within the sets E_k. The quiche-eating set of equilibria, E_1, is not intuitive. Any $\sigma \in E_1$ corresponds to the weak type hiding behind the shadow of the strong type, but the strong type not getting what they want. Iterated deletion of weakly dominated strategies kills all of E_1 and all but one point in E_2.

The strategy (p,p) is strictly dominated by (n,n) for the bully, the strategy (q,b) is strictly dominated by $\frac{3}{4}$ on (b,q), $\frac{1}{4}$ on (q,q). Eliminating these gives the game

(st,wk)\(b,q)	(p,n)	(n,p)	(n,n)
(b,b)	$(9,-8)$	$(29,0)$	$(29,0)$
(b,q)	$(12,-9)$	$(28,1)$	$(30,0)$
(q,q)	$(21,0)$	$(1,-8)$	$(21,0)$

In this game, (n,n) weakly dominates (p,n) for 2, and once (p,n) is eliminated, (b,b) and (b,q) strongly dominated (q,q) for 1. Eliminating these gives the game

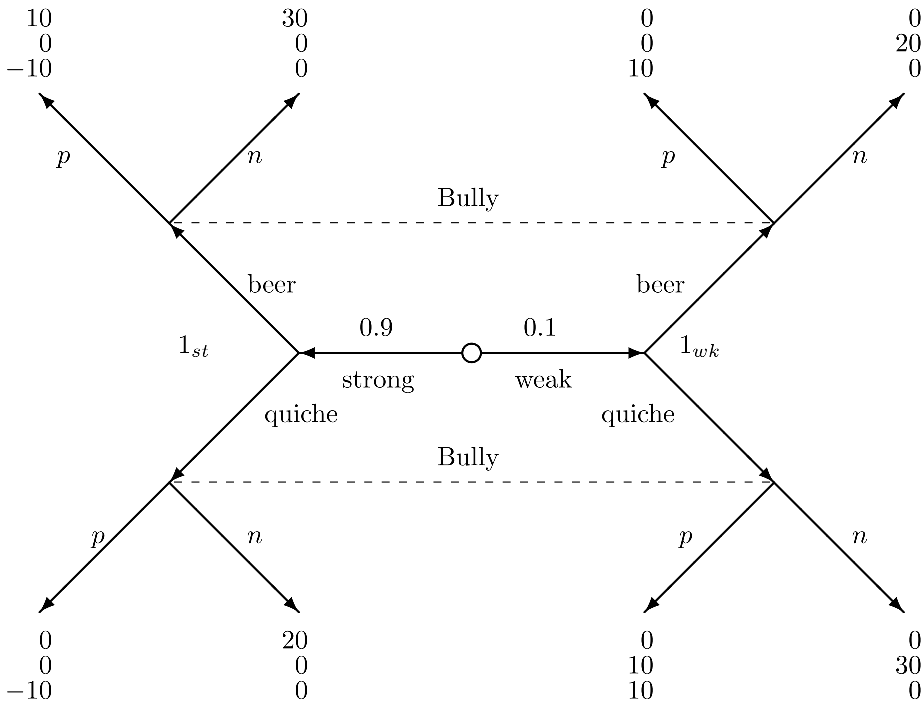
(st,wk)\(b,q)	(n,p)	(n,n)
(b,b)	$(29,0)$	$(29,0)$
(b,q)	$(28,1)$	$(30,0)$

In this game, (n,p) weakly dominates (n,n), once (n,n) is removed, (b,b) is 1's strict best response, so the only equilibrium to survive iterated deletion of weakly dominated strategies is $((b,b),(n,p))$, i.e. both types have Beer for breakfast and the Bully leaves anyone having Beer alone, but picks a fight with anyone having Quiche for breakfast.

E.2. Working in the agent normal form. Consider the following, rather different version of the same basic story: 9 out of every 10 days on average, a stranger who feels like Glint Westwood[7] comes into town. We call such strangers "strong." Strong strangers like nothing better than Beer (and a vile cigar) for breakfast. On the other days, a different kind of stranger comes to town, one who feels like a graduate student recovering from a comprehensive exam. We call such strangers "weak." Weak strangers like nothing better than Quiche for breakfast. Strong and weak strangers are not distinguishable to anyone but themselves. In

[7]A mythical Hollywood quasi-hero, who, by strength, trickiness and vile cigars, single-handedly overcomes huge obstacles, up to and including bands of 20 heavily armed professional killers.

the town frequented by breakfast-eating strangers, there is also a swaggering bully. Swaggering bullies are cowards who, as a result of deep childhood traumas, need to impress their peers by picking on others. Being a coward, he would rather pick on someone weak. He makes his decision about whether to pick, p, or not, n, after having observed what the stranger had for breakfast. With payoffs listed in the order 1_{st}, 1_{wk}, 2 and normalizing strangers' payoffs to 0 when they are not breakfasting in this town, the game tree is



This game has three players (four if you include Nature), 1_{st} (aka Glint), 1_{wk} (aka the weak stranger), and 2 (aka the Bully). In principle, we could also split the Bully into two different people depending on whether or not they observed Beer or Quiche being eaten. The logic is that we are the sum of our experiences, and if our experiences are different, then we are different people. If we did this second agent splitting, we would have the game in what is called **agent normal form**. In this game, instead of putting 0's as the utilities for the strangers' when they are not breakfasting in this town, we could have made 1_{st}'s utility equal to 1_{wk}'s even when they are out of town. Since we are changing utilities by adding a function that depends only on what someone else is doing, this cannot change anything about the equilibrium set.

More generally, to give the agent normal form for an extensive form game, 1) for each $H \in U_i \in P_i$ (go back and look at the notation for extensive form games if you have already forgotten it), we invent a new agent i_H, 2) we assign all "copies" of each i the same utility at each terminal node. This is a bit confusing — we are acting as if these are different people, but they are different people with exactly the same preferences. There are two reasons for this confusing choice:

(1) It demonstrates Kuhn's theorem, that extremely important result that we will get to (hopefully not for your first time) in a while.
(2) We would like the sets of equilibrium outcomes to be the same in the original game and in the agent normal form version of the game. In the signaling game above, since no players' information sets ever precede each other, the different copies having different utilities didn't matter. It would matter if 1_{st} made a choice that impacted 1_{wk}'s utility and subsequent choices.

2 still has a dominated strategy, (p,p). By varying 2's strategy amongst the remaining 3, we can make either Beer or Quiche be a strict best response for both strangers. This means that no strategies are dominated for the strangers, and iterated deletion of dominated strategies stops after one round.
Again, the equilibrium set for this game can be partitioned into two sets, E_q and E_b, but note that we must now specify 3 strategies,

$$E_q = \left\{((q),(q),(0,\beta,0,1-\beta)) : 21 \geq 12\beta + 30(1-\beta), \text{ i.e. } \beta \geq \frac{1}{2}\right\},$$

and

$$E_b = \left\{ ((b),(b),(0,0,\beta,1-\beta)) : 29 \geq 28\beta + 30(1-\beta) \text{ i.e. } \beta \geq \frac{1}{2}\right\}.$$

Again, $\mathbb{O}(\cdot)$ is constant on the two closed and connected sets E_1 and E_2.
However, the self-referential tests do eliminate E_1 — for 1_{wk}, Beer is dominated relative to E_1, after removing Beer for 1_{wk}, (p,n) is weakly dominated for 2, implying that no $\sigma \in E_1$ survives the iterative steps. It is fairly easy to check (and you should do it) that E_2 does survive the iterative steps of the self-referential tests.

E.3. About the difference between a game and its agent normal form. First, whether or not we should use a particular game, its agent normal form, or some hybrid (as above where we did not split player 2) depends on what story we are telling. Games boil away a great deal of contextual detail, this is the source of their power as story-telling devices. Trying to make a blanket pronouncement about which form is generally correct is like trying to decide, on the basis of the game matrix, which of the Partners (in the Battle of the Partners coordination game) is dominant in the sphere of week-end entertainment. This is a ridiculous exercise: any answer must be intellectually bankrupt; and any answer would lessen our ability to explain.
Second, even though the set of equilibrium outcomes is the same in a game and in the agent normal form of the game, splitting agents makes a huge difference to the power of iterated deletion arguments, even in games where the copies of an agent do not play after each other.
The following demonstrates some of the subtleties that arise when we try to make dominance arguments with different representations of the same strategic situation.

Problem E.1 (Kohlberg & Mertens, Figure 9). *Consider the two player game*

	Left	Right
X	$(2,2)$	$(1,1)$
Y	$(2,2)$	$(0,0)$
Z	$(0,0)$	$(0,0)$

a. *Show that (X, L) is the only undominated equilibrium of this game.*
b. *Give an extensive form representation of the following game: Split agent 1 into 1_a who picks whether or not to play Y and 1_b who, if s/he has chosen not to play Y, picks between X and Z; have 2 pick L or R after 1_a and 1_b and in ignorance of those choices.*
c. *In the extensive form game you have given, show that R is dominated for 2. Show that in the extensive form game with R deleted, only Y is undominated for 1_a.*

F. Iterated Deletion Procedures

We will begin with a brief coverage of rationalizability, which provides a simple introduction to iterative deletion of strategies that cannot be best responses to any beliefs about what other people are doing. After this we turn to a much more subtle and powerful technique, iterated deletion of equilibrium dominated strategies.

F.1. Rationalizability. In the setting where one has beliefs β_s about ω, and maximizes $\int u(a,\omega)\, d\beta_s(\omega)$, an action $a \in A$ is **potentially Rational (**pR**)** if there exists some β_s such that $a \in a^*(\beta_s)$. An action a **dominates action** b if $\forall \omega\, u(a,\omega) > u(b,\omega)$. The following example shows that an action b can be dominated by a random choice.

Example III.2. *$\Omega = \{L, R\}$, $A = \{a, b, c\}$, and $u(a,\omega)$ is given in the table*

$A \downarrow, \Omega \rightarrow$	L	R
a	5	9
b	6	6
c	9	5

Whether or not a is better than c or vice versa depends on beliefs about ω, but $\frac{1}{2}\delta_a + \frac{1}{2}\delta_c$ dominates b. Indeed, for all $\alpha \in (\frac{1}{4}, \frac{3}{4})$, $\alpha\delta_a + (1-\alpha)\delta_c$ dominates b.

Let $R_i^0 = \mathrm{pR}_i \subset A_i$ denote the set of potentially rational actions for i when they believe that others' actions have some distribution. Define $R^0 := \times_{i \in I} R_i^0$ so that $\Delta(R^0)$ is the largest possible set of outcomes that are at all consistent with rationality. (In Rational Pigs, this is the set $\delta_{\mathrm{Wait}} \times \Delta(A_2)$.) As we argued above, it is too large a set. Now we'll start to whittle it down.

Define R_i^1 to be the set of maximizers for i when i's beliefs β_i have the property that $\beta_i(\times_{j \neq i} R_j^0) = 1$. Since R_i^1 is the set of maximizers against a smaller set of possible beliefs, $R_i^1 \subset R_i^0$. Define $R^1 = \times_{i \in I} R_i^1$, so that $\Delta(R^1)$ is a candidate for the set of outcomes consistent with rationality. (In Rational Pigs, you should figure out what this set is.)

Given R_i^n has been define, inductively, define R_i^{n+1} to be the set of maximizers for i when i's beliefs β_i have the property that $\beta_i(\times_{j \neq i} R_j^n) = 1$. Since R_i^n is the set of maximizers against a smaller set of possible beliefs, $R_i^{n+1} \subset R_i^n$. Define

$R^{n+1} = \times_{i \in I} R_i^{n+1}$, so that $\Delta(R^n)$ is a candidate for the set of outcomes consistent with rationality.

LEMMA III.2. *For finite games,* $\exists N \forall n \geq N\ R^n = R^N$.

We call $R^\infty := \bigcap_{n \in \mathbb{N}} R^n$ the set of **rationalizable strategies**. $\Delta(R^\infty)$ is then the set of **signal rationalizable outcomes**.[8]

There is (at least) one odd thing to note about $\Delta(R^\infty)$ — suppose the game has more than one player, player i can be optimizing given their beliefs about what player $j \neq i$ is doing, so long as the beliefs put mass 1 on R_j^∞. There is no assumption that this is actually what j is doing. In Rational Pigs, this was not an issue because R_j^∞ had only one point, and there is only one probability on a one point space. The next pair of games illustrate the problem.

F.2. Variants on iterated deletion of dominated sets. We begin with the following.

DEFINITION III.7. *A strategy* $\sigma_i \in \Delta_i$ ***dominates (or strongly dominates)*** $t_i \in A_i$ ***relative to*** $T \subset \Delta$ *if*

$$(\forall \sigma^\circ \in T)[u_i(\sigma^\circ \backslash \sigma_i) > u_i(\sigma^\circ \backslash t_i)].$$

If $T = \Delta$, this is the previous definition of dominance. Let $D_i(T)$ denote the set of $t_i \in A_i$ that are dominated relative to T. Smaller T's make the condition easier to satisfy.

In a similar fashion we have the following.

DEFINITION III.8. *a strategy* $\sigma_i \in \Delta_i$ ***weakly dominates*** $t_i \in A_i$ ***relative to*** $T \subset \Delta$ *if*

$$(\forall \sigma^\circ \in T)[u_i(\sigma^\circ \backslash \sigma_i) \geq u_i(\sigma^\circ \backslash t_i)], \text{ and}$$
$$(\exists \sigma' \in T)[u_i(\sigma' \backslash \sigma_i) > u_i(\sigma' \backslash t_i)].$$

Let $WD_i(T)$ *denote the set of* $t_i \in A_i$ *that are weakly dominated relative to* T.

LEMMA III.3. *If* Γ *is finite, then for all* $T \subset \Delta$, $A_i \setminus D_i(T) \neq \emptyset$ *and* $A_i \setminus WD_i(T) \neq \emptyset$.

This is not true when Γ is infinite.

PROBLEM F.1. Two variants of 'pick the largest integer'.

(1) $\Gamma = (A_i, u_i)_{i \in I}$ *where* $I = \{1, 2\}$, $A_i = \mathbb{N}$, $u_i(n_i, n_j) = 1$ *if* $n_i > n_j$, *and* $u_i(n_i, n_j) = 0$ *otherwise. Every strategy is weakly dominated, and the game has no equilibrium.*

(2) $\Gamma = (A_i, v_i)_{i \in I}$ *where* $I = \{1, 2\}$, $A_i = \mathbb{N}$, *and* $v_i(n_i, n_j) = \Phi(n_i - n_j)$, $\Phi(\cdot)$ *being the cdf of a non-degenerate Gaussian distribution, every strategy is strongly dominated (hence the game has no equilibrium).*

Iteration sets $S_i^1 = A_i$, defines $\Delta^n = \times_{i \in I} \Delta(S_i^n)$, and if S^n has been defined, set $S_i^{n+1} = S_i^n \setminus D_i(\Delta^n)$. If Γ is finite, then Lemma III.3 implies

$$(\exists N)(\forall n, n' \geq N)[S_i^n = S_i^{n'} \neq \emptyset].$$

There are many variations on this iterative-deletion-of-dominated-strategies theme. In all of them, $A_i^1 = \Delta_i$.

[8] I say "signal rationalizable" advisedly. **Rationalizable outcomes** involve play of rationalizable strategies, just as above, but the randomization by the players is assumed to be stochastically independent.

(1) **serially undominated** $S_i^{n+1} = S_i^n \setminus D_i(\Delta^n)$. If this reduces the strategy sets to singletons, then the game is **dominance solvable** (a term due to Herve Moulin).
(2) **serially weakly undominated** $S_i^{n+1} = S_i^n \setminus WD_i(\Delta^n)$ where $WD_i(T)$ is the set of strategies weakly dominated with respect to T.
(3) Set $S_i^2 = S_i^1 \setminus WD_i(\Delta^1)$, and for $n \geq 2$, set $S_i^{n+1} = S_i^n \setminus D_i(\Delta^n)$. [**6**], [**4**] show that the most that can be justified by appealing to common knowledge of the structure of the game and common knowledge of expected utility maximization is this kind of iterated deletion procedure.

F.3. The basic result. Every element of a rationalizable strategy set is serially undominated, and if there exists a correlated equilibrium in which $a \in A$ receives positive mass, then a is serially undominated.

F.4. Equilibrium Dominated Sets. A starting observation is that for almost all assignments of utilities to terminal nodes, the outcome function is constant on the connected components of the Nash equilibria. This makes analysis much easier, and we start with an example showing how genericity can fail.

F.4.1. *Nongeneric Failure of Outcome Constancy.* **Important to note**: E_1 and E_2 are connected sets of equilibria, and the outcome function, and hence payoffs, is constant on them. We will see this pattern in all the games that we look at. For some really silly non-generic games, we may not see this.

EXAMPLE III.3. $I = \{1, 2, 3\}$, 1 *chooses which of the following two matrix games are played between* 2 *and* 3, *so* $A_1 = \{$*Left Box, Right Box*$\}$, $A_2 = \{$*Up, Down*$\}$, *and* $A_3 = \{$*Left, Right*$\}$, *and the payoffs are*

	Left	Right
Up	$(0,0,1)$	$(0,0,1)$
Down	$(0,0,2)$	$(0,0,2)$

	Left	Right
Up	$(0,0,3)$	$(0,0,3)$
Down	$(0,0,4)$	$(0,0,4)$

Notice that for all $\sigma \in \Delta$, *all* $i \in I$, *and all* $a_i \neq b_i \in A_i$, $U_i(\sigma\backslash a_i) = U_i(\sigma\backslash b_i)$. *Thus,* $Eq = \Delta$, *which is a nice closed connected set. However, the outcome function is* ***not*** *constant on this set, nor are the utilities, which are anyplace in the line segment* $[(0,0,1),(0,0,4)]$.

Returning to beer-quiche, there are dominance relations in this game, a mixed strategy dominates a pure strategy for 1, and after iterated elimination of dominated normal form strategies, only E_2 survives.

F.4.2. *Behavioral strategies and an agent normal form analysis.* Consider a mixed strategy $(\alpha, \beta, \gamma, \delta)$ for player 1. In the agent normal form, we take extraordinarily seriously the idea that every is the sum total of their experiences, and that different experiences make different people. This turns 1 into two people, having independent randomization at each information set. Give the Kuhn reduction to behavioral strategies, give Kuhn's Theorem.

There are no dominated strategies in the agent normal form of the game. However, there is something else, something that we will spend a great deal of time with.

F.4.3. *Variants on iterated deletion of dominated sets.* Repeating what we had above, we begin with the following.

Definition III.9. *A strategy* $\sigma_i \in \Delta_i$ ***dominates (or strongly dominates)*** $t_i \in A_i$ ***relative to*** $T \subset \Delta$ *if*

$$(\forall \sigma^\circ \in T)[u_i(\sigma^\circ \backslash \sigma_i) > u_i(\sigma^\circ \backslash t_i)].$$

$D_i(T)$ *denotes the set of* $t_i \in A_i$ *that are dominated relative to* T.

If $T = \Delta$, this is the previous definition of dominance. It is important to realize that smaller T's make the condition easier to satisfy. We are going to go after as small a set of T's as we can, in this fashion eliminating as many strategies as possible. In a similar fashion, we have the following

Definition III.10. *a strategy* $\sigma_i \in \Delta_i$ ***weakly dominates*** $t_i \in A_i$ ***relative to*** $T \subset \Delta$ *if*

$$(\forall \sigma^\circ \in T)[u_i(\sigma^\circ \backslash \sigma_i) \geq u_i(\sigma^\circ \backslash t_i)], \text{ and}$$
$$(\exists \sigma' \in T)[u_i(\sigma' \backslash \sigma_i) > u_i(\sigma' \backslash t_i)].$$

Let $WD_i(T)$ *denote the set of* $t_i \in A_i$ *that are weakly dominated relative to* T.

Lemma III.4. *If* Γ *is finite, then for all* $T \subset \Delta$, $A_i \backslash D_i(T) \neq \emptyset$ *and* $A_i \backslash WD_i(T) \neq \emptyset$.

This is not true when Γ is infinite.

Problem F.2. Two variants of 'pick the largest integer'.

(1) $\Gamma = (A_i, u_i)_{i \in I}$ *where* $I = \{1, 2\}$, $A_i = \mathbb{N}$, $u_i(n_i, n_j) = 1$ *if* $n_i > n_j$, *and* $u_i(n_i, n_j) = 0$ *otherwise. Every strategy is weakly dominated, and the game has no equilibrium.*

(2) $\Gamma = (A_i, v_i)_{i \in I}$ *where* $I = \{1, 2\}$, $A_i = \mathbb{N}$, *and* $v_i(n_i, n_j) = \Phi(n_i - n_j)$, $\Phi(\cdot)$ *being the cdf of a non-degenerate Gaussian distribution, every strategy is strongly dominated (hence the game has no equilibrium).*

F.4.4. *Self-referential tests.* The iterated procedures become really powerful when we make them self-referential. Let us ask if a set of equilibria, $E \subset Eq(\Gamma)$, is "sensible" or "internally consistent" are "stable" by asking if it passes an E-test. This kind of self-referential test is (sometimes) called an **equilibrium dominance test**. Verbally, this makes (some kind of) sense because, if everyone knows that only equilibria in a set E are possible, then everyone knows that no-one will play any strategy that is either weakly dominated or that is strongly dominated *relative to E itself.* That is, E should survive an E-test.

There is a problem with this idea, one that can be solved by restricting attention to a class $\mathcal{E}$ of subsets of $Eq(\Gamma)$. The class $\mathcal{E}$ is the class of closed and connected[9] subsets of $Eq(\Gamma)$.

Formally, fix a set $E \subset Eq(\Gamma)$, set $S_i^1 = A_i$, $E^1 = E$, given A_i^n for each $i \in I$, set $\Delta^n = \times_{i \in I} \Delta(S_i^n)$, and iteratively define S_i^{n+1} by

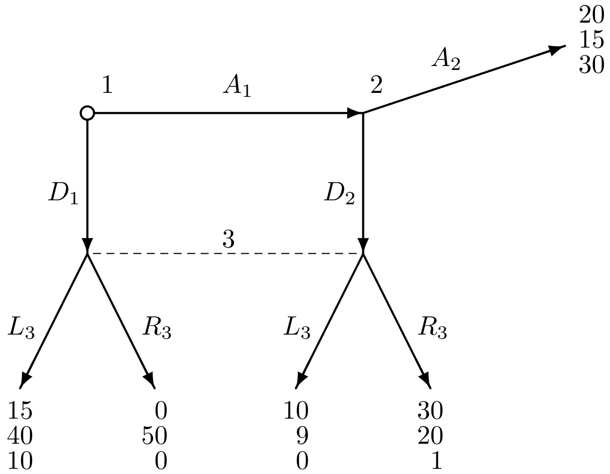
$$S_i^{n+1} = S_i^n \setminus \{WD_i(\Delta^n) \cup D_i(E^n)\}.$$

$E \in \mathcal{E}$ **passes the iterated equilibrium dominance test** if at each stage in the iterative process, there exists a non-empty $E^{n+1} \in \mathcal{E}$, $E^{n+1} \subset E^n$, such that for all $\sigma \in E^{n+1}$ and for all $i \in I$, $\sigma_i(\{WD_i(\Delta^n) \cup D_i(E^n)\}) = 0$. This means that something in E^n must be playable in the game with strategy sets S^{n+1}.

[9] If you've had a reasonable amount of real analysis or topology, you will know what the terms "closed" and "connected" mean. We will talk about them in more detail later. Intuitively, you can draw a connected set (in our context) without taking your pencil off of the paper.

We will examine this workings of this logic first in a "horse" game, then return to beer-quiche, which belongs to a class of games known as signaling games.

F.4.5. *A Horse Game.* These games are called horse games because the game tree looks like a stick figure horse, not because they were inspired by stories about the Wild West.



There are three sets of equilibria for this game, Listing 1's and 2's probabilities of playing D_1 and D_2 first, and listing 3's probability of playing L_3 first, the equilibrium set can be partitioned into $Eq(\Gamma) = E_A \cup E_B \cup E_C$,

$$E_A = \left\{((0,1),(0,1),(\gamma,1-\gamma)) : \gamma \geq \frac{5}{11}\right\}$$

where the condition on γ comes from $15 \geq 9\gamma + 20(1-\gamma)$,

$$E_B = \left\{((1,0),(\beta,1-\beta),(1,0)) : \beta \geq \frac{1}{2}\right\}$$

where the condition on β comes from $15 \geq 10\beta + 20(1-\beta)$, and

$$E_C = \{((0,1),(1,0),(0,1))\}.$$

Note that $\mathbb{O}(\cdot)$ is constant on the sets E_A, E_B, and E_C. In particular, this means that for any $\sigma, \sigma' \in E_k$, $u(\sigma) = u(\sigma')$. I assert without proof that the E_k are closed connected sets.[10]

There are no weakly dominated strategies for this game:

(1) $u_1(s\backslash D_1) = (15, 15, 0, 0)$ while $u_1(s\backslash A_1) = (10, 20, 30, 20)$ so no weakly dominated strategies for 1,
(2) $u_2(s\backslash D_2) = (40, 9, 50, 20)$ while $u_2(s\backslash A_2) = (40, 15, 50, 15)$ so no weakly dominated strategies for 2,
(3) $u_3(s\backslash L_3) = (10, 0, 10, 30)$ while $u_3(s\backslash R_3) = (0, 1, 0, 3)$ so no weakly dominated strategies for 3.

[10]Intuitively, the sets are closed because they are defined by weak inequalities, and they are connected because, if you were to draw them, you could move between any pair of points in any of the E_k without lifting your pencil.

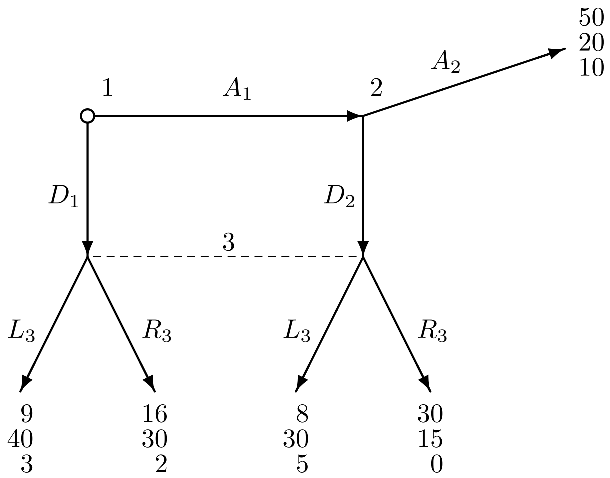
Each E_k survives iterated deletion of weakly dominated strategies. However, E_A and E_B do not survive self-referential tests, while E_C does.

(1) E_A — the strategy D_1 is dominated for 1 **relative to** E_A. Removing D_1 makes L_3 weakly dominated for 3, but every $\sigma \in E_A$ puts mass on the deleted strategy, violating the iterative condition for self-referential tests. (We could go further, removing L_3 make A_2 dominated for 2, and every $\sigma \in E_A$ puts mass on A_2.)

(2) E_B — the strategy R_3 is dominated for 3 relative to E_B, removing R_3 make D_2 weakly dominated for 2, meaning that every $\sigma \in E_B$ puts mass on the deleted strategy, violating the iterative condition for self-referential tests.

The set E_C contains only one point, and it is easy to check that 1 point survives iterated deletion of strategies that are either weakly dominated or weakly dominated relative to E_C.

Problem F.3. *For the following horse game, partition $Eq(\Gamma)$ into closed and connected sets on which the outcome function, $\mathbb{O}(\cdot)$, is constant and find which of the elements of the partition survive the iterative condition for self-referential tests.*



F.4.6. *Back to Beer and Quiche.* Return to quiche equilibria in the beer-quiche game,

$$E_q = \left\{ ((q), (q), (0, \beta, 0, 1-\beta)) : 21 \geq 12\beta + 30(1-\beta), \text{ i.e. } \beta \geq \frac{1}{2} \right\}.$$

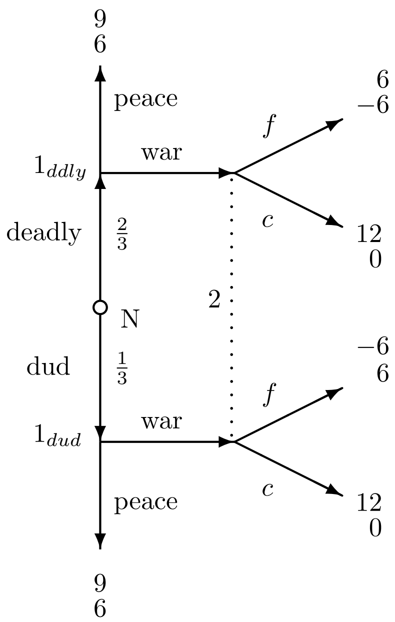
We will apply the iterated self-referential test to this set of equilibria.

- Step 1: In the agent normal form, there are no weakly dominated strategies. Relative to the set of quiche equilibria, 1_{str} has no dominated strategies because they are receiving a utility of 20, and their possible utilities to having beer for breakfast against the set of equilibrium strategies is the interval $[10, 20]$. However, relative to the set of quiche equilibria, 1_{wk} has a dominated strategy, beer, because they are receiving a utility of 30, and their possible utilities to having beer for breakfast against the set of equilibrium strategies is the interval $[0, 10]$.

- Step 2: In the new game, with beer removed for 1_{wk}, p after beer is weakly dominated for 2, hence is deleted. But this makes all of the strategies in the quiche set of equilibria unplayable.

In terms of beliefs, it is dominated for 1_{wk} to have beer for breakfast, and if we remove this strategy for 1_{wk}, 2's only possible belief after seeing someone have beer for breakfast is that it was 1_{str}.

F.4.7. *War, Peace, and Spies.* With probability $\rho = 2/3$, country 1's secret military research program makes their armies deadlier (i.e. giving higher expected utility in case of war through higher probability of winning and lower losses), and with probability 1/3 the research project is a dud (i.e. making no change in the army's capacities). Knowing whether or not the research program has succeeded, country 1 decides whether or not to declare war on or to remain at peace with country 2. Country 2 must decide how to respond to the invasion, either $f$ighting or $c$eding territory, all of this without knowing the outcome of 1's research program. With payoffs, one version of the game tree is:



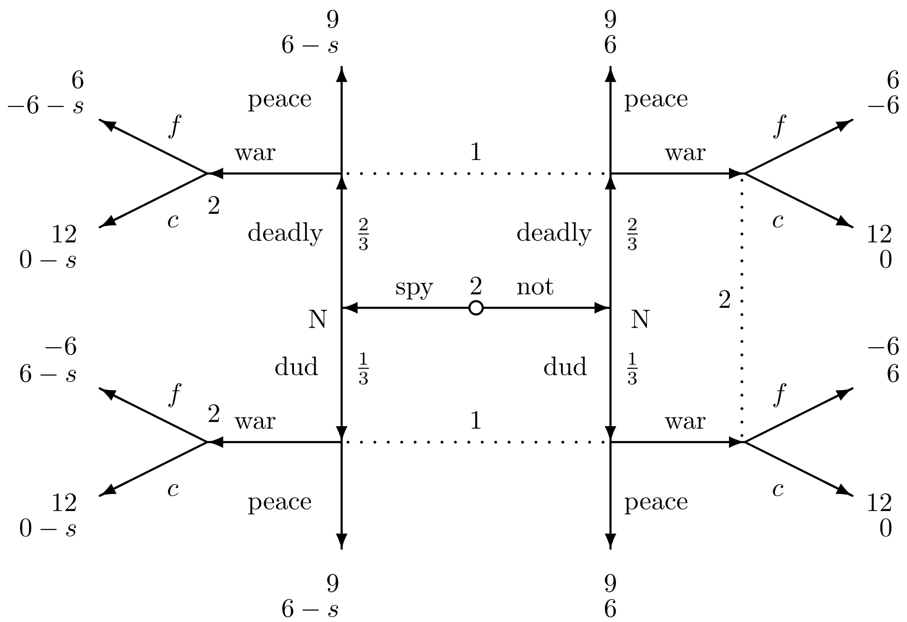
Analysis of the equilibrium set for this game can proceed along the following lines.

- It is easy to show that there are no separating equilibria.
- There are no equilibria in which 1_{ddly} and 1_{dud} both randomize. To see why let γ_f be country 2's probability of fighting an invasion: if $\gamma_f = \frac{1}{2}$, then 1_{ddly} is just indifferent between war and peace, and 1_{dud} strictly

prefers peace; if $\gamma_f = \frac{1}{6}$, then 1_{dud} is just indifferent between war and peace, and 1_{ddly} strictly prefers war. Thus both cannot be indifferent.

- There are no equilibria in which 1_{ddly} strictly randomizes. To see why, note that if they are indifferent, then 1_{dud} is peaceful, which means that 2 knows that all invaders are deadly, and will cede with probability 1, which means that 1_{dud} being peaceful is not a best response.
- There are no equilibria in which 1_{dud} strictly randomizes (for similar reasons).
- There are two sets of pooling equilibria: the singleton set $E_w = ((w, w), c)$; and the set $E_p = \{(p, p), \gamma_f) : \gamma_f \geq \frac{1}{2}\}$.
- There are no weakly dominated strategies. Relative to E_p, war is dominated for 1_{dud}. Deleting this makes f weakly dominated for 2, which means that none of the strategies in E_p can be played, and E_p fails the iterated self-referential test. Leaving E_w as the stable equilibrium set.
- The Cho-Kreps intuitive criterion does not arrive at this result — 1_{ddly} cannot claim that 1_{dud} would not want to convince 2 that he is deadly.

Let us now modify the previous game by adding an earlier move for country 2. Before country 1 starts its research, country 2 can, at a cost $s > 0$, insert sleepers (spies who will not act for years) into country 1. Country 1 does not know whether or not sleepers have been inserted, and if sleepers are inserted, country 2 will know whether or not 1's military research has made them deadlier. One version of the game tree is:



Before starting analysis of the game, note if you were to do a normal form analysis, it would be a 4×16 game. I, at least, would need to be extraordinarily strongly motivated to be willing to analyze this game in such a form. Though, to be fair to the normal form analyses, weak dominance arguments would reduce this to a 4×4 game, which is more manageable.

We are going to work with small(ish) values for s, the societal cost of having a spy agency is fairly large, at least if you consider secrecy a danger to democracy, but still, it is arguably of an order smaller than the cost difference between war and peace. Analysis of the equilibrium set for this game can proceed along the following lines.

- The strategy of $f$ighting after spying and seeing a deadly army coming their way is weakly dominated for country 2, as is the strategy of ceding after spying and seeing a dud army coming their way. Eliminating both of those strategies makes the left-hand side of the game tree a good bit simpler.
- In any stable equilibrium set, γ_s, 2's probability of spying, must belong to the open interval $(0, 1)$. To see why, if $\gamma_s = 1$, then 1's best response is to war if deadly, to be peaceful otherwise. This means that 2 is incurring the expense s for no reason, not a best response. On the other hand, if $\gamma_s = 0$, then the previous analysis applies, and war whether deadly or dudly is country 1's stable equilibrium strategy. By spying, 2 would receive payoffs of $\frac{2}{3}(0 - s) + \frac{1}{3}(6 - s) = 2 - s$ against this strategy, by not spying they receive the payoff 0. As long as $0 < s < 2$, the definition of "small(ish)," spying is a best response.
- One might guess that 1_{ddly} and 1_{dud} both going to war is part of an equilibrium. As we just argued, this makes spying a strict best response for 2, and $\gamma_s = 1$ is not part of a stable equilibrium set.
- The previous arguments about 1_{ddly} being indifferent between war and peace implying that 1_{dud} strictly prefers peace are now strengthened by a strictly positive probability of spying. Thus, there will be no equilibrium in which both are randomizing.
- 1_{ddly} strictly randomizing and 1_{dud} not randomizing cannot be an equilibrium because 2 can then infer that anyone invading must be deadly, hence spying is an expensive waste of effort, which is not part of any equilbrium.
- Finally, 1_{ddly} going to war, 1_{dud} going to war with probability $s/2$, 2 spying with probability $\frac{1}{6}$, and $c$eding to any invader after not spying yields the stable equilibrium outcome.

Problem F.4. *Find the stable equilibrium outcome or outcomes for these last two games when the probability of a deadly breakthrough in military technology is $\rho \ll \frac{1}{2}$.*

F.5. No News is Bad News. This problem takes you through the basics of what is called "unraveling." The setting involves a persuader who cannot directly lie, anything that they say must be true, but they can lie by emphasis and omission, i.e. they could tell you that what they are trying to sell you easily exceeds an older, obsolete standard. You are the DM, you know that they want to sell you as much as they can. What, if anything, do you believe of what they tell you? The following gets at the main results about persuasion in Milgrom's paper [**13**] on the monotone likelihood ratio property (mlrp) (see the discussion in §B.2.3 and especially Definition II.2).

Problem F.5. *The quality of a good is a random variable, ω, which takes on any value in the set $\Omega = \{\omega_1, \ldots, \omega_K\}$, $\omega_k < \omega_{k+1}$, according to a prior distribution*

ρ that is common knowledge. The seller of the good knows the realization of ω, but the potential buyer does not. The seller can remain silent, or make any true statement about the value of ω, but cannot lie (e.g. there are costless, convincing demonstrations of quality, or that there are truth-in-advertising laws). Formally, the set of actions available to a seller with goods of quality ω_k is $A(\omega_k) = \{\emptyset\} \cup \{E \subset \Omega : \omega_k \in E\}$.

After hearing silence, $\{\emptyset\}$, or whatever the seller chooses to say, the potential buyer of the good chooses a quantity, q. If $\omega = \omega_k$, the buyer's utility is $u_B(q, \omega_k) = \omega_k F(q) - pq$ where $p > 0$ is the price of the good, which is fixed, $F : \mathbb{R}_+ \to \mathbb{R}$ is bounded, strictly increasing, concave, continuously differentiable on $\mathbb{R}_{++}$, and satisfies $\lim_{q \downarrow 0} F'(q) = \infty$. When the true value of ω is not known, the buyer maximizes expected utility. The sellers utility is $u_S(q)$, a strictly increasing function of q.

- a. *Show that a first order stochastically dominating shift in the buyer's beliefs, $\mu \in \Delta(\Omega)$, leads to a strict increase in their optimal choice of quantity, $q^*(\mu)$.*
- b. *Show that there is at least one Nash equilibrium with the outcome that the buyer always buys $q^*(\rho)$.*
- c. *Show that in any perfect equilibrium, the seller will always say $\{\omega_K\}$ when $\Theta = \omega_K$.* [*Hint: weak dominance.*]
- d. *Show that the previous step implies that any perfect equilibrium outcome involves full disclosure, meaning that the outcome is that with probability ρ_k, the quality is ω_k and the buyer buys $q^*(\delta_{\omega_k})$ (where δ_{ω_k} is point mass on ω_k).*
- e. *Now suppose that the seller can now costlessly lie about the quality of the good. The model is otherwise unchanged. What is the set of perfect (or sequential) equilibrium outcomes?*
- f. *Now suppose that it hurts to lie. Specifically, let $u_S(q) = q - c$ if the seller lies and $u_S(q) = q$ if the seller does not lie. As a function of c, what are the perfect (or sequential) equilibrium outcomes?* [*This has a pretty straightforward generalization the the cost, c, being a function of the size of the lie.*]

F.6. Signaling Game Exercises in Refinement. Here are a variety of signaling games to practice with. The presentation of the games is a bit different than the extensive form games we gave above, part of your job is to draw extensive forms. Recall that a pooling equilibrium in a signaling game is an equilibrium in which all the different types send the same message, a separating equilibrium is one in which each types sends a different message (and can thereby be separated from each other), a hybrid equilibrium has aspects of both behaviors.

The presentation method is taken directly from Banks and Sobel's (1987) treatment of signaling games. Signaling games have two players, a Sender S and a Receiver R. The Sender has private information, summarized by his type, t, an element of a finite set T. There is a strictly positive probability distribution ρ on T; $\rho(t)$, which is common knowledge, is the ex ante probability that S's type is t. After S learns his type, he sends a message, m, to R; m is an element of a finite set M. In response to m, R selects an action, a, from a finite set $A(m)$; S and R have von Neumann-Morgenstern utility functions $u(t, m, a)$ and $v(t, m, a)$ respectively. Behavioral strategies are $q(m|t)$, the probability that S sends the message m given that his type is t, and $r(a|m)$, the probability that R uses the pure strategy a when message m is received. R's set of strategies after seeing m is the $\#A(m) - 1$ dimensional simplex Δ_m, and utilities are extended to $r \in \Delta_m$ in the usual fashion.

For each distribution λ over T, the receiver's best response to seeing m with prior λ is

$$Br(\lambda, m) = \arg\max_{r(m)\in\Delta_m} \sum_{t\in T} v(t, m, r(m))\lambda(t). \tag{F.1}$$

Examples are represented with a bi-matrix $B(m)$ for each $m \in M$. There is one column in $B(m)$ for each strategy in $A(m)$ and one row for each type. The (t, a)'th entry in $B(m)$ is $(u(t, m, a), v(t, m, a))$. With t_1 being the strong type, t_2 the weak, m_1 being beer, m_2 being quiche, a_1 being *p*ick a fight, and a_2 being *n*ot, the Beer-Quiche game is

$B(m_1)$	a_1	a_2
t_1	$10, -10$	$30, 0$
t_2	$0, 10$	$20, 0$

$B(m_2)$	a_1	a_2
t_1	$0, -10$	$20, 0$
t_2	$10, 10$	$30, 0$

You should carefully match up the parts of this game and the extensive form of B-Q given above.
Here is a simple example to start on:

$B(m_1)$	a_1
t_1	$2, 2$
t_2	$2, 2$

$B(m_2)$	a_1	a_2
t_1	$3, 3$	$0, 0$
t_2	$0, 0$	$3, 3$

Problem F.6. *Draw the extensive form for the game just specified. Find the* 3 *connected sets of equilibria. Show that all equilibria for this game are both perfect and proper. Show that the* 3 *connected sets of equilibria satisfy the iterated self-referential tests described above.*

The following is a sequential settlement game of a type analyzed by Sobel (1989): There are two types of defendants, S: type t_2 defendants are negligent, type t_1 defendants are not, $\rho(t_1) = 1/2$. S offers a low settlement, m_1, or a high settlement, m_2. R, the plaintiff, either accepts, a_1, or rejects a_2. If R accepts, S pays R an amount that depends on the offer but not S's type. If R rejects the offer, S must pay court costs and a transfer depending only on whether or not S is negligent. With payoffs, the game is

$B(m_1)$	a_1	a_2
t_1	$-3, 3$	$-6, 0$
t_2	$-3, 3$	$-11, 5$

$B(m_2)$	a_1	a_2
t_1	$-5, 5$	$-6, 0$
t_2	$-5, 5$	$-11, 5$

Problem F.7. *Draw the extensive form for the game just specified. Analyze the equilibria of the above game, picking out the perfect, the proper, the sequential, and the sets satisfying the self-referential tests.*

This game has $\rho(t_1) = 0.4$.

$B(m_1)$	a_1
t_1	$0, 0$
t_2	$0, 0$

$B(m_2)$	a_1	a_2	a_3	a_4
t_1	$-1, 3$	$-1, 2$	$1, 0$	$-1, -2$
t_2	$-1, -2$	$1, 0$	$1, 2$	$-2, 3$

Problem F.8. *Draw the extensive form for the game just specified. Find the pooling and the separating equilibria, if any, check the perfection and properness of any equilibria you find, and find the Hillas stable sets.*

G. Correlated Equilibria$^{\mathbf{XAdv}}$

G.1. Returning to the Stag Hunt. It is possible that the weather is different at the hunters' villages. Suppose the joint distribution of sun/rain at the two villages is

	Sun	Rain
Sun	a	b
Rain	c	d

Suppose both follow the strategy "Stag if it's sunny at my village, Rabbit else." If we find conditions for these strategies to be mutual best responses, we've found another correlated equilibrium. If all row and column sums are positive, the conditions for player 1 are

$$20\frac{a}{a+b} + 0\frac{b}{a+b} \geq 1\frac{a}{a+b} + 1\frac{b}{a+b} \text{ if } (a+b) > 0, \tag{G.1}$$

and

$$1\frac{c}{c+d} + 1\frac{d}{c+d} \geq 20\frac{c}{c+d} + 0\frac{d}{c+d}. \text{ if } (c+d) > 0, \tag{G.2}$$

These are sensible if you think about conditional probabilities and suppose that the players maximize the expected value of the utility numbers we write down.

Problem G.1. *Write down the inequalities for player* 2 *that correspond to* (G.1) *and* (G.2). *To avoid the potential embarassment of dividing by* 0*, show that the conditional inequalities in* (G.1) *and* (G.2) *are satisfied iff*

$$20a + 0b \geq 1a + 1b, \text{ and } 1c + 1d \geq 20c + 0d. \tag{G.3}$$

G.2. More General Correlated Equilibria.

Notation III.4. *For any finite set* S, $\Delta(S) := \{p \in \mathbb{R}^S_+ : \sum_a p_a = 1\}$.

Definition III.11. *A distribution,* $\nu \in \Delta(A)$*, is a* ***correlated equilibrium*** *if, for all* i*, for all* $a, b \in A_i$, $\sum_{a_{-i}} u(a, a_{-i})\nu(a, a_{-i}) \geq \sum_{a_{-i}} u(b, a_{-i})\nu(a, a_{-i})$*, equivalently, if* $\sum_{a_{-i}}[u(a, a_{-i}) - u(b, a_{-i})]\nu(a, a_{-i}) \geq 0$.

If you want to think in terms of conditional probabilities, then the inequalities in the Definition are $\sum_{a_{-i}} u(a, a_{-i})\frac{\nu(a,a_{-i})}{\sum_{c\in A}\nu(c,a_{-i})} \geq \sum_{a_{-i}} u(b, a_{-i})\frac{\nu(a,a_{-i})}{\sum_{c\in A}\nu(c,a_{-i})}$ because $\frac{\nu(a,a_{-i})}{\sum_{c\in A}\nu(c,a_{-i})}$ is the conditional distribution over A_{-i} given that a was drawn for player i.

Problem G.2. *Show that the only correlated equilibrium of the Rational Pigs game is the Nash equilibrium.*

Problem G.3. *In the Stag Hunt game, what is the maximum correlated equilibrium probability of mis-coordination? If, instead of* 1 *being the payoff to* R*, it is* $x > 0$*, how does the maximum depend on* x*, and how does the dependence vary across different ranges of* x*, e.g.* $x > 20$*?*

For each $i \in I$, let $(S_i, \mathcal{S}_i)$ be a signal space and a σ-field of subsets. To make somethings more convenient, we suppose each is a standard probability space. Let $(S, \mathcal{S}) = (\times_{i\in I} S_i, \otimes_{i\in I}\mathcal{S}_i)$ be the joint space of signals, let μ be a probability distribution on $\otimes_{i\in I}\mathcal{S}_i$, and let $(\mu_i)_{i\in I}$ be the marginals.

We assume that $s \sim \mu$ is drawn, each $i \in I$ sees s_i, then picks $\tau_i(s_i) \in \Delta(A_i)$ where $\tau_i : S_i \to \Delta(A_i)$ is a measurable function, and $\tau = (\tau_i)_{i\in I}$. Essentially, the joint distribution of the s_i allows the players to coordinate/correlate their choices of actions.
The payoffs to τ with information structure $(S, \mathcal{S}, \mu)$ are

$$U_i(\tau) = \int_S \langle u_i, \times_{j\in I}\tau_j(s_j)\rangle \, d\mu(s). \tag{G.4}$$

As usual, τ^* is an **equilibrium for the game Γ with information structure** $(S, \mathcal{S}, \mu)$ if, $(\forall i \in I)(\forall \tau_i')[U_i(\tau^*) \geq U_i(\tau^*\backslash\tau_i')$. An easy observation: if $\sigma^* \in Eq(\Gamma)$, then $\tau^* \simeq \sigma^*$ is an equilibrium.
A vector τ of strategies induces a distribution over A given by

$$\mathbb{O}(\tau)(\times_{j\in I}E_j) = \int_S \times_{j\in I}\tau_j(s_j)(E_j)\, d\mu(s). \tag{G.5}$$

Definition III.12. *If τ^* is an equilibrium for a signal structure $(S, \mathcal{S}, \mu)$, then $\mathbb{O}(\tau^*)$ is a **correlated equilibrium outcome**.*

The question is how does on characterize the set of correlated equilibrium outcomes? Let $(A, \mathcal{A})$ denote A with its Borel σ field. Let $\iota : X \to X$ denote the identity function whatever X is, i.e. $\iota(x) \simeq x$.

Lemma III.5. *If τ^* is an equilibrium for the information structure $(S, \mathcal{S}, \mu)$, then $(\iota_i)_{i\in I}$ is an equilibrium for the information structure $(A, \mathcal{A}, \mathbb{O}(\tau^*))$.*

Proof. Conditioning on $\tau_i^*(s_i) = \sigma_i$, use convexity of the set of beliefs consistent with σ_i being a best response. □

Theorem III.2. *$(\iota_i)_{i\in I}$ is an equilibrium for the information structure $(A, \mathcal{A}, \mu)$ iff for all $i \in I$ and all measurable $\varphi_i : A_i \to A_i$, $\int u_i(\iota(a))\, d\mu(a) \geq \int u_i(\iota\backslash\varphi_i(a))\, d\mu(a)$.*

Problem G.4. *Show that for finite games, the condition just given is equivalent to the definition given earlier.*

G.3. Exercises.

H. Some Infinite Examples

One of the general results in this class of examples is that if we have an equilibrium in a game with differentiable utility functions, and the equilibrium is in the interior of action spaces, then it will be inefficient, except by very rare accident.

H.1. Collapsing and Underfunded Commons.

TO BE ADDED

H.2. Cournot, Bertrand and Stackelberg. Two firms compete by producing quantities q_i and q_j of a homogeneous good, and receiving profits of the form

$$\pi_i(q_i, q_j) = [p(q_i + q_j) - c]q_i,$$

where $p(\cdot)$ is the inverse market demand function for the good in question. Assume that $p(q) = 1 - q$ and that $0 \leq c \ll 1$.

a. (Cournot competition) Suppose that the firms pick quantities simultaneously. Find the unique equilibrium.

b. (Stackelberg competition) Suppose that firm i picks q_i, which is then observed by firm j before they pick q_j. Find the set of equilibria for this game. Find the unique subgame perfect equilibrium (called the Stackelberg equilibrium).
c. Find the profit rankings of the Cournot and the Stackelberg equilibrium.

H.3. Partnerships and Timing. A software designer, s, and a marketer, m, form a partnership to which they contribute their efforts, respectively $x \geq 0$ and $y \geq 0$. Both have quasi-linear utility functions, $u_s = \$_s - x^2$ and $u_m = \$_m - y^2$, where $\$_s$ and $\$_m$ are monies received by s and m respectively.
The twice continuously differentiable, strictly concave profit function π satisfies

$$(\dagger) \qquad \pi(0,0) = 0, \text{ and } (\forall x^\circ, y^\circ > 0)[\partial\pi(x^\circ, y^\circ)/\partial x > 0, \ \partial\pi(x^\circ, y^\circ)/\partial y > 0].$$

The profit function need not be symmetric, that is, it may happen that $\pi(x,y) \neq \pi(y,x)$.

Consider the following scenarios: neither effort nor sidepayments are legally enforceable, the partners choose their efforts simultaneously, and share the profits equally; neither effort nor sidepayments are legally enforceable, and the partners share the profits equally. However, the software designer chooses her effort before the marketer chooses hers, and the marketer observes the designer's effort before choosing her own. Compare the payoffs of these two scenarios with the efficient payoffs.

H.4. Bertrand Competition.

TO BE ADDED

I. Bargaining

I.1. Summary. The word "bargaining" covers a range of phenomena. It can be about dividing a surplus between two or more people, e.g. a seller who wants to sell a particular object as dearly as possible a buyer who wants to buy as cheaply as possible. In this case, one person's gain is another's loss, there may be two-sided uncertainty about the value of the object to the other party. The value of each sides walk-away-utility and what each side knows about each other will be crucial to determining the likely outcome. It can be about choosing a joint course of action where the different choices determine not only the future streams of joint benefits but can also change future bargaining positions. In this case, attitudes toward future payoffs can be part of determining the likely outcome.
The academic study of the essential indeterminacy in bargaining goes back to Schelling [**16**]. There is typically a large range of mutually acceptable agreements, even a large range of mutually acceptable efficient agreements. If we are dividing (say) a pile of money and I believe that you will accept any amount more than $\alpha = \frac{1}{3}$ of a it, and you believe that I will balk unless I receive $(1-\alpha) = \frac{2}{3}$ of it, then the $(\frac{1}{3}, \frac{2}{3})$ division is an equilibrium outcome. But if you and I believe a different α is the relevant one, then that α is the relevant outcome.
We are going to start by looking at three different models of bargaining that make sensible choices from large ranges of outcomes when both sides know the value to the other side of the various divisions: the Nash bargaining solution; the Kalai-Smorodinsky solution; and the Rubinstein-Ståhl model. The different models emphasize different determinants of the bargaining situation as being determining for the outcome: walk-away utility with the idea that having good outside options is

good for you; maximal utility with the idea that people bargaing harder when they can end up with more; and patience, with the idea that being under time pressure to come to an agreement is bad for you.

We will then turn to a mechanism design approach to thinking about what are the sets of possible outcomes when there is uncertainty on one or both sides of the bargaining process. This is a somewhat less ambitious approach, looking for the set of possibilities, but it is an approach to a much harder problem.

More than 30 years after its publication, Raiffa's [**15**] book is still worth consulting. He divides bargaining situations along the following dimensions: whether or not there are more than two parties to the bargaining; whether or not the parties are monolithic; whether or not the interactions are repeated; whether or not there are possible linkages between the issue being bargained over and other issues; whether or not the bargaining is about more than one issue; whether or not the bargainers can actually walk away without an agreement; whether or not their are time constraints on one or more of the bargainers; whether or not the agreements reached are binding; whether or not there are relevant group norms such as truthfulness, forthcomingness, willingness to use threats; whether or not third party intervention is possible, or will result if agreement is not reached. Throughout Raiffa studies what he calls *cooperative antagonists*, in modern terms, he was studying what we now call *coopetition*.

I.2. Schelling's Indeterminacy Lemma. Consider the problem of dividing a pie between two players, let x and y denote 1 and 2's payoffs, (x, y) is **feasible** if $x \geq e_1$ and $y \geq e_2$ and $g(x, y) \leq 1$ where g is a smooth function w/ everywhere strictly positive partial derivatives. Let V denote the set of feasible utility levels and assume that V is convex. An allocation $(x, y) \in V$ is **efficient** if $g(x, y) = 1$. Consider the simultaneous move game where the players suggest a division, if their suggestions are feasible, that's what happens, if they are not feasible, e is what happens.

LEMMA III.6 (Schelling). *An allocation is efficient and feasible if and only if it is an equilibrium.*

This is Schelling's basic insight, for something to be an equilbrium in a game of division, both have to believe that it is an equilibrium, but not much else needs to happen. (In this part of Schelling's analysis he lays the framework for much of the later work on common knowledge analyses of games.) Especially in a game like this, the conditions for a Nash equilibrium seem to need something more before you want to believe in them.

Missing from this analysis is any sense of what might affect agreements. For example, if I am selling my house and bargaining about the price with potential buyer A, having another offer come from potential buyer B strengthens my position *vis à vis* and A and probably means that the eventual price will be higher — provided I do not bargain so intransigently that A goes away, perhaps because I am demand more than it is worth to them.

I.3. The Nash Bargaining Solution. Let X be the set of options available to a pair of people engaged in bargaining, perhaps two people in a household, or two people considering a joint venture of some other kind. A point $x \in X$ may specify an allocation of the rights and duties to the two people. Let $u_i(x)$ be i's

utility to the option x, $i = 1, 2$. Let $V = \{(u_1(x), u_2(x)) : x \in X\} \subset \mathbb{R}^2$ be the set of possible utility levels V. This focus on the utilities abstracts from Raiffa's question of whether or not a single issue is being bargained over, but it requires that both players know each others' utilities to the various $x \in X$.
Let $e = (e_1, e_2)$ be a point in $\mathbb{R}^2$. For $v \in V$, let $L_i(v)$ be the line $L_i(v) = \{v + \lambda \mathbf{1}_i : \lambda \in \mathbb{R}\}$, $\mathbf{1}_i$ the unit vector in the i'th direction. The idea is that player i can guarantee him/herself e_i if the negotiation fails. Sometimes this is thought of as the *walk-away utility* or the *reservation utility.* One could also understand the vector e as the expected utilities of e.g. binding arbitration if the bargaining breaks down.

Definition III.13. *A **bargaining situation** (V, e) is a set $V \subset \mathbb{R}^2$ and a point e satisfying*

(1) V is closed,
(2) V is convex,
(3) $V = V + \mathbb{R}^2_-$, and
(4) for all $v \in V$, $L_1(v) \not\subset V$, $L_2(v) \not\subset V$, and
(5) e is in the interior of V

What we really want is that the set of possible utilities for the two players involve tradeoffs. The following result gets at this idea.

Lemma III.7. *If $V \subset \mathbb{R}^2$ is convex and $V = V + \mathbb{R}^2_-$, then if there exists $v' \in V$ and $L_i(v') \not\subset V$, then for all $v \in V$, $L_i(v) \not\subset V$.*

The interpretation of $e = (e_1, e_2)$ is that e_i is i's reservation utility level, the utility they would get by breaking off the bargaining. This gives a lower bound to what i must get out of the bargaining situation in order to keep them in it. By assuming that e is in the interior of V, we are assuming that there is something to bargain about.

Definition III.14. *The **Nash bargaining solution** is the utility allocation that solves*

$$\max\ (v_1 - e_1) \cdot (v_2 - e_2) \ \ \textit{subject to} \ \ (v_1, v_2) \in V,\ v \geq e.$$

Equivalently,

$$\max_{x \in X}\ (u_1(x) - e_1)(u_2(x) - e_2) \ \textit{subject to} \ \ (u_1(x), u_2(x)) \geq e.$$

It is worthwhile drawing a couple of pictures to see what happens as you move e around. Also check that the solution is invariant to affine positive rescaling of the players' utilities. It is remarkable that this solution is the only one that satisfies some rather innocuous-looking axioms. We'll need

Definition III.15. *A **bargaining solution** is a mapping $(V, e) \mapsto s(V, e)$, $s \in V$, $s \geq e$. The solution is **efficient** if there is no $v' \in V$ such that $v' > s$.*

We'll also need

Definition III.16. *For $(x_1, x_2) \in \mathbb{R}^2$, a **positive affine rescaling** is a function $A(x_1, x_2) = (a_1 x_1 + b_1, a_2 x_2 + b_2)$ where $a_1, a_2 > 0$.*

Here are some reasonable looking axioms for efficient bargaining solutions:

(1) Affine rescaling axiom: The solution should be independent of positive affine rescalings of the utilities. That is, $s(AV, Ae) = A(s(V, e))$ for all positive affine rescalings A.
(2) Midpoint axiom: If $V \cap \mathbb{R}^2_+ = \{(u_1, u_2) : u_1 + u_2 \leq 1\}$ and $e = (0, 0)$, then the midpoint of the line, $(1/2, 1/2)$, is the solution.
(3) Independence of irrelevant alternatives axiom: If $s(V, e) \in V' \subseteq V$, then $s(V', e) = s(V, e)$.

THEOREM III.3 (Nash). *There is only one efficient bargaining solution that satisfies these three axioms, and it is the solution to the problem*

$$\max\ (v_1 - e_1) \cdot (v_2 - e_2) \quad \textit{subject to} \quad (v_1, v_2) \in V.$$

Let us apply this result to a question in property rights. Suppose that a household is modeled as solving the following problem when facing a set of options X,

$$\max_{x \in X,\ u_i(x) > e_i,\ i=1,2} (u_1(x) - e_1)(u_2(x) - e_2). \tag{I.1}$$

In effect, $w(x) = (u_1(x) - e_1)(u_2(x) - e_2)$ is the household utility function. Khan [**8**] argues, in the context of patenting activity as married women in the U.S. gained the right to sign legally binding contracts, that changing the property laws does not change X. Therefore, changes in the property laws can only affect the optimal behavior in the above problem if they change the e_i. This may be a reasonable way to understand the legal changes – they gave women a better set of outside options, which is captured by increasing the womens' reservation utility level.

I.4. Approximating the Nash Solution with Noncooperative Games. Binmore gives the details of a version of the Nash bargaining solution that was suggested by Nash. Let $g(\cdot, \cdot)$ be smooth, increasing and convex. Suppose that Z is a random variable and the random feasible set when $Z = z$ is $V = \{(x, y) : (x, y) \geq (e_1, e_2),\ g(x, y) \leq z\}$. Two agents pick their offers, (x, y), then the value of Z is realized. The offers are infeasible when $g(x, y) > z$ and if this happens, the agents receive (e_1, e_2). The distribution of Z has a cdf with $F(\underline{z}) = 0$, $F(\bar{z}) = 1$ and $\underline{z} < 1 < \bar{z}$. Binmore's observation is

LEMMA III.8. *If F^n is a sequence of full support distributions on $[\underline{z}, \overline{z}]$ converging weakly to point mass on 1, then the equilibrium outcomes of this game converge Nash's bargaining solution.*

PROOF. Easy. □

I.5. The Kalai-Smorodinsky Bargaining Solution. For more detail on this see [**1**]. For one of our bargaining problems (V, e), let ∂V denote the (upper) boundary of V, and let $\bar{u}_i^V = \max\{u_i : (u_i, e_i) \in V\}$.

(1) Affine rescaling axiom: The solution should be independent of affine rescalings of the utilities, that is, $s(AV, Ae) = A(s(V, e))$ for all positive affine rescalings A.
(2) Box axiom: If $V \cap \mathbb{R}^2_+ = \{(u_1, u_2) : u_i \leq u_i^\circ\}$, then $s(V, e) = (u_1^\circ, u_2^\circ)$.
(3) Proportional increases axiom: Suppose that $s(V, e) \in \partial V'$ and that $(\bar{u}_1^V, \bar{u}_2^V)$ and $(\bar{u}_1^{V'}, \bar{u}_2^{V'})$ are proportional. Then $s(V, e) = s(V', e)$.

Geometrically, to find the Kalai-Smorodinsky bargaining solution, one shifts so that $e = (0, 0)$, solves the problem $\max\{\lambda : \lambda \geq 0,\ \lambda(\bar{u}_1^V, \bar{u}_2^V) \in V\}$ for λ^*, and set $s(V, e) = \lambda^*(\bar{u}_1^V, \bar{u}_2^V)$.

THEOREM III.4 (Kalai-Smorodinsky). *The Kalai-Smorodinsky solution is the unique efficient solution concept satisfying the affine rescaling axiom, the box axiom, and the proportional increases axiom.*

Nash's bargaining solution "explains" the effect of changes in property laws as increases in womens' reservation utility levels. There is a complementary "explanation" for the Kalai-Smorodinsky solution, by letting people realize more of their potential, their maximal utility, $\bar{u}_i^V$, increases.

I.6. Rubinstein-Ståhl bargaining. Two people, 1 and 2, are bargaining about the division of a cake of size 1. They bargain by taking turns, one turn per period. If it is i's turn to make an offer, she does so at the beginning of the period. The offer is α where α is the share of the cake to 1 and $(1 - \alpha)$ is the share to 2. After an offer α is made, it may be accepted or rejected in that period. If accepted, the cake is divided forthwith. If it rejected, the cake shrinks to δ times its size at the beginning of the period, and it becomes the next period. In the next period it is j's turn to make an offer. Things continue in this vein either until some final period T, or else indefinitely.

Suppose that person 1 gets to make the final offer. Find the unique subgame perfect equilibrium. Suppose that 2 is going to make the next to last offer, find the unique subgame perfect equilibrium. Suppose that 1 is going to make the next to next last offer, find the subgame perfect equilibrium. Note the contraction mapping aspect and find the unique solution for the infinite length game in which 1 makes the first offer.

Now suppose that the utility to the players shrinks future rewards differently, that is suppose that there is a δ_i and a δ_j that shrinks the utility to the two players.

The solution here emphasizes that bargaining power comes from being more patient than the person you are bargaining with. The flip side of this patience is the ability to impose waiting times on the other player. For example, it is an advantage, in this game, to have a slower turn-around time for accepting/rejecting an offer. Delegating the bargaining to someone with limited authority to accept/reject offers, to someone who must go back and explain the offer in detail and get authorization before anything will happen becomes a source of bargaining advantage in this game. For this and many related aspects of bargaining, [**15**] is, more than 30 years after publication, still a very good resource.

I.7. A Mechanism Design Approach to Two-Sided Uncertainty. The value of a used car to a Buyer is v_B, the value to a Seller v_S, $v = (v_B, v_S) \sim Q$. Values are private information, and it is very intuitive to those who have ever haggled over a price that they should stay that way.

Buyers are interested in getting what they want at minimal cost and sellers are interested in selling so dearly as possible. The most the buyer is willing to pay and the least the seller is willing to accept are private information. When a buyer and seller get together, they go through posturing of various (culture dependent) types until they either strike a deal or walk away. The equilibrium of the game provides a map from their private information to the final outcome. This suppression of the strategies is very useful for analysis. It means that we do not need to understand

anything about how the bargaining actually works, we can just study the possible equilibrium associations between outcomes and private information.
We're going to go through the revelation principle logic for this specific context: If the mapping from private info to the final outcome was the result of an equilibrium, then we could just enter the private info into the mapping and ask about the utility properties of the final outcome at the end. In particular, both the buyer and the seller, if they knew they were submitting their private info to such a mapping, would be perfectly happy to reveal their true private info. After all, if they would be happier revealing some other value of their private info and thereby getting some other outcome, then they could have acted that way in the original game, and gotten that other outcome. But we had an equilibrium, so they cannot like the other outcome better. This is called the **revelation principle**. It is widely used.
Also, since neither can be forced to trade, the outcomes must be at least as good as the walk-away utility. You need to be careful about this constraint, mostly it is sensible, sometimes it is not.

I.7.1. *Is Efficient Trade Possible?* Using the revelation principle, we can get at whether or not efficient trade is possible. Efficient trade and no coercion require that whenever $v_B > v_S$, that the ownership of the car be transfered from the Seller to the Buyer for some price (aka transfer of value) $t \in [v_S, v_B]$. One question we are after is "Can efficiency be achieved by any type of game?" that is, "Is some efficient allocation implementable?" To answer this, we ask for a mapping from the vector of values to a (trade,transfer) pair with the property that each agent, after learning their own private information but before learning other's information, has expected utility from the allocation function being implemented higher (or at least, no lower) reporting their true type than in lying about it.
Akerlof's lemons model: used cars have random qualities, q, distributed according to a distribution μ; the seller of a car knows its quality q, the buyer does not. The seller's value if the sell a car of quality q at a price p is $u_s = p - q$, the buyer's value if they buy a car of quality q at price p is $u_b = r \cdot q - p$ where $r > 1$. Efficiency requires the car to be sold at a price $p(q) \in [q, r \cdot q]$. However, a revelation principle analysis tells us that only one price can prevail in equilibrium. When a price p prevails in the market, only sellers with cars of quality $q < p$ will be in the market. This means that the value to buyers when p is the price is $r \cdot E\,(Q|Q < p) - p$. Examining different distributions for Q at different values of r gives conditions under which efficiency is and is not possible.
Adverse selection for insurance companies is essentially the same story.
If quality will be revealed, say by whether or not the car breaks down, a Seller with a high value car who can credibly commit to making any repairs can usefully differentiate her/himself from the Seller of a low value car. If the gain to doing so is high enough, we expect that they will do it, and efficiency can be restored.
This has started to get us back into strategic considerations of information transmission, the idea that we can order the people on one side of a transaction according to their information in a fashion that is correlated with the cost of taking some action.
The essential problem with the idea is that signaling activities can be costly.[11]

I.7.2. *Ex-Post Implementability and Efficiency.* The following is a question of academic interest rather than substantive interest. The short version of what we

[11] Schudson's aphorism here.

are about to see is that the two criteria, ex post implementability and efficiency, are mutually exclusive in the interesting cases.

A related question, which we will consider first is whether it is *ex post* implementable. This asks that, for each agent, after ALL the private information has been revealed, does the agent like the allocation being implemented more than any one I could have ended up by lying about my private information? This is a MUCH stronger condition, hence much harder to satisfy. Why would we ask for such a thing? Well, it has the advantage that implementing in this fashion gives one a mechanism that does not depend on the mechanism designer having, for example, a complete a description of the environment, e.g. the joint distribution of the Buyer's and Seller's values.

Suppose that Q is a joint distribution over the four Buyer-Seller valuations $a = (6,3)$, $b = (12,3)$, $c = (12,9)$, and $d = (6,9)$. Trade should happen at a, b, c, but not at d. Let $t(s)$ be the transfer from the Buyer to the Seller at state s, $s = a, b, c, d$. We know that, because of the no coercion condition, $t(d) = 0$. What can we figure out about the others?

Inequalities for ex post implementability:

(1) First, inequalities from the Buyer, assuming truthfulness by the Seller:
   (a) If the Seller truthfully says their value is 3, then
      (i) the low value Buyer, 6, must prefer $6-t(a)$ to 0, the no coercion inequality, and must prefer $6 - t(a)$ to $6 - t(b)$.
      (ii) the high value Buyer, 12, must prefer $12 - t(b)$ to 0, the no coercion inequality, and must prefer $12 - t(b)$ to $12 - t(a)$.
      (iii) Combining, $t(a) \leq 6$, and $t(a) = t(b)$.
   (b) If the Seller truthfully says their value is 9, then there are some more inequalities that reduce to $t(c) \in [6, 12]$.

(2) Now, inequalities from the Seller, assuming truthfulness by the Buyer:
   (a) If the Buyer truthfully says their value is 12, then
      (i) the low value Seller, 3, must prefer $t(b)$ to 3, the no coercion inequality, and must prefer $t(b)$ to $t(c)$.
      (ii) the high value Seller, 9, must prefer $t(c)$ to 9, the no coercion inequality, and must prefer $t(c)$ to $t(b)$.
      (iii) Combining, $t(c) \geq 9$, and $t(b) = t(c)$.

Combining all of these, $t(a) = t(b) = t(c)$, i.e. a posted price, and $t(a) \leq 6$ while $t(c) \geq 9$. Oooops.

The posted price intuition for ex post implementability is pretty clear. It gives a great deal of inefficiency for the interesting Q's. Elaborate on this.

I.7.3. *Implementability and Efficiency.* An advantage of examining ex post implementability is that we need make no assumptions about Q. Now let us suppose that the joint distribution of $v_B = 6, 12$ and $v_S = 3, 9$ is, for $x \in [0, \frac{1}{2}]$, given by

	3	9
6	x	$\frac{1}{2} - x$
12	$\frac{1}{2} - x$	x

Using the revelation principle, we can, as a function of x, find when efficient trade is implementable.

I.8. Exercises.

Problem I.1. *Prove Lemma III.7.*

Problem I.2. *Let* $[e_1, \overline{v}_1]$ *be the interval of* v_1 *such that* $(v_1, e_2) \in V$*. Suppose that for each* $v_1 \in [e_1, \overline{v}_1]$*, the maximal possible* v_2 *such that* $(v_1, v_2) \in V$ *is given by* $g(v_1)$ *where* $g(\cdot)$ *is a decreasing, concave function. Let* $s^*(e) = (s_1^*(e_1, e_2), s_2^*(e_1, e_2))$ *be the Nash bargaining solution, i.e. the point that solves*

$$\max_{v_1 \in [e_1, \overline{v}_1]} (v_1 - e_1) \cdot (g(v_1) - e_2). \tag{I.2}$$

(1) *Find the dependence, positive or negative, of* s_1^* *on* e_1 *and* e_2*, and the dependence, positive or negative, of* s_2^* *on* e_1 *and* e_2*.*
(2) *Assuming that* $g(\cdot)$ *is smooth, where possible, find whether the following partial derivatives are positive or negative:*
$$\frac{\partial^2 s_1^*}{\partial e_1^2}, \quad \frac{\partial^2 s_1^*}{\partial e_1 \partial e_2}, \quad \frac{\partial^2 s_2^*}{\partial e_2}.$$
(3) *Consider the following variant of the Nash maximization problem,*
$$\max \; ((av_1 + b) - (ae_1 + b)) \cdot (v_2 - e_2) \quad \text{subject to} \quad (v_1, v_2) \in V \tag{I.3}$$
where $a > 0$*. Show that the solution to this problem is* $(as_1^* + b, s_2^*)$ *where* (s_1^*, s_2^*) *is the Nash bargaining solution we started with. In other words, show that the Nash bargaining solution is independent of affine rescalings. (You might want to avoid using calculus arguments for this problem.)*

Problem I.3. *Prove Theorem III.3.*

Problem I.4. *This is a directed compare and contrast problem:*

(1) *Give two* (V, e) *where the Nash solution is the same as the Kalai-Smorodinsky solution.*
(2) *Give two* (V, e) *where the Nash solution is different than the Kalai-Smorodinsky solution.*
(3) *Let* $s^{KS}(V, e)$ *denote the Kalai-Smorodinsky solution. If possible, find whether or not* s_i^{KS} *is increasing or decreasing in* e_j*,* $i, j \in \{1, 2\}$*.*
(4) *Let* $s^{KS}(V, e)$ *denote the Kalai-Smorodinsky solution. If possible, find whether or not* s_i^{KS} *is increasing or decreasing in* $\bar{u}_j^V$*,* $i, j \in \{1, 2\}$*.*

Problem I.5. *The Joker and the Penguin have stolen* 3 *diamond eggs from the Gotham museum. If an egg is divided, it loses all value. The Joker and the Penguin split the eggs by making alternating offers, if an offer is refused, the refuser gets to make the next offer. Each offer and refusal or acceptance uses up* 2 *minutes. During each such* 2 *minute period, there is an independent, probability* r*,* $r \in (0, 1)$*, event. The event is Batman swooping in to rescue the eggs, leaving the two arch-villains with no eggs (eggsept the egg on their faces, what a yolk). However, if the villains agree on a division before Batman finds them, they escape and enjoy their ill-gotten gains.*
Question: What does the set of subgame perfect equilibria look like? [*Hint: it is not the Rubinstein bargaining model answer. That model assumed that what was being divided was continuously divisible.*]

CHAPTER IV

Repeated Games

The chapter covers some dynamic decision theory for single agents, then uses these tools to study a subclass of repeated games. Background on discounting and hazard rates is a good starting point.

A. Review of Discounting and Hazard Rates

We are interested in $E\sum_{t=0}^{T} u_t$ when $T \in \{0, 1, 2, \ldots\}$ is a random time representing the last period in which benefits, u_t, will accrue to the decision maker. Several interpretations are possible: in the context of repeated games, the interpretation of T will be the random time until the relations/interactions between the players ends; in social planning contexts, the interpretation will be the time at which society comes to an end; in single person decision problems, the interpretation can be the last period of life.

The essential observation is that one receives u_0 if $T \geq 0$, that is, with probability $P(T \geq 0)$, one receives u_1 so long as $T \geq 1$, that is, with probability $P(T \geq 1)$, ..., one receives u_t with probability $P(T \geq t)$. Therefore,

$$E\sum_{t=0}^{T} u_t = \sum_{t=0}^{\infty} u_t P(T \geq t). \tag{A.1}$$

Let $G(t) = P(T \geq t)$ denote the reverse cdf for T. $G(\cdot)$ is decreasing, after any times t where $G(\cdot)$ drops, all utilities $u_{t'}$ at later times, $t' > t$, are downweighted, during times when $G(\cdot)$ is approximately flat, all the $u(t')$ receive approximately equal weight. One can see the same pattern a different way, using hazard rates.

Given that one has waited until t and the random time T has not arrived, what is the probability that this period, t, is the last one? That is what the **discrete hazard rate at** t answers. It is $h(t) := P(T = t|T \geq t)$ which is equal to $\frac{p_t}{G(t)}$ where $p_t = P(T = t)$. Note that

$$G(t) = (1 - h(0)) \cdot (1 - h(1)) \cdots (1 - h(t-1)). \tag{A.2}$$

Taking logarithms gives $\log(G(t)) = \sum_{s<t} \log(1 - h(s))$. Since $d\log(x)/dx_{|x=1} = 1$, this is approximately $-\sum_{s<t} h(s)$ so that $G(t) \simeq e^{-\sum_{s<t} h(s)}$ and the approximation is better when the hazard rates are smaller. Thus, increases in the hazard rate drive $G(\cdot)$ downward thereby downweighting all future u_t's, decreases in the hazard rate can leave $G(\cdot)$ almost flat, giving future u_t's approximately equal weights.

One special case is worth examining, that of the random T's having a constant hazard rate. Recall that for $|r| < 1$, $\sum_{t=0}^{\infty} r^t = \frac{1}{1-r}$, and taking derivatives on both sides with respect to r yields

$$\tfrac{d}{dr}\left(\sum_{t=0}^{\infty} r^t\right) = \left(\sum_{t=0}^{\infty} t r^{t-1}\right) = \tfrac{1}{r}\left(\sum_{t=0}^{\infty} t r^t\right) = \tfrac{1}{(1-r)^2}. \tag{A.3}$$

Rearranging, $\sum_{t=0}^{\infty} tr^t = \frac{r}{(1-r)^2}$.
We say that T has a **geometric distribution with parameter** $\lambda \in (0,1)$ if $P(T=t) = (1-\lambda)^t\lambda$ for $t = 0, 1, \ldots$. In this case, $E\,T = \lambda \cdot \sum_{t\geq 0} t(1-\lambda)^t = \frac{(1-\lambda)}{\lambda}$, or $E\,T = \frac{1}{\lambda} - 1$. This increases to ∞ as $\lambda \downarrow 0$. It is worthwhile working out the yearly interest rates used as discount factors $\delta = \frac{1}{1+r}$ corresponding to the different expectations of T.
The reverse cdf's for this class of distributions is

$$G(t) = (1-\lambda)^t \sum_{s=0}^{\infty}(1-\lambda)^s\lambda = (1-\lambda)^t \cdot \frac{\lambda}{1-(1-\lambda)} = (1-\lambda)^t. \tag{A.4}$$

This means that the discrete hazard rate is constant, $h(t) = \frac{(1-\lambda)^t\lambda}{(1-\lambda)^t}$, and that $E\sum_{t=0}^{T} u_t = \sum_{t=0}^{\infty} u_t(1-\lambda)^t$. In particular, when the hazard rate is low, that is, when the probability that any given period is the last one given that one has survived so long, $\lambda \simeq 0$, and this corresponds to very patient preferences, that is, to the u_t being multiplied by δ^t where $\delta := (1-\lambda) \simeq 1$.
Notice how the prospect of T ending the stream of rewards interacts with regular discounting: if $u_t = x_t \cdot \beta^t$, then $E\sum_{t\leq T} u_t = \sum_{t\geq 0} x_t\beta^t(1-\lambda)^t$ when T is geometric; more generally $E\sum_{t\leq T} u_t = \sum_{t\geq 0} x_t\beta^t G(t)$ when $G(\cdot)$ is the reverse cdf. A good name for $\beta(1-\lambda)$ is the **risk adjusted discount factor**.
Let $u = (u_0, u_1, \ldots)$ be a sequence of utilities. We often normalize the weights in $v(u,\delta) := \sum_{t\geq 0} u_t\delta^t$ to sum to 1, replacing $v(u,\delta)$ with $V(u,\delta) := (1-\delta)\sum_{t\geq 0} u_t\delta^t$ because $\sum_{t\geq 0}(1-\delta)\delta^t = 1$. This allows us to compare discounted utilities with long-run average utilities, that is, with the utilities $A(u,T) := \frac{1}{T}\sum_{t=0}^{T-1} u_t$.
For δ close to 1 and T very large, the weights $w_t(1-\delta)\delta^t$, $t = 0, 1, 2, \ldots$ and the weights $w'_t = \frac{1}{T}$, $t = 0, 1, \ldots, (T-1)$ are different implementations of the idea that the decision maker is patient. They really are different implementations: there are uniformly bounded sequences $u = (u_0, u_1, \ldots)$ with $\liminf_T A(u,T) < \limsup_T A(u,T)$; and there are sequences for which $\lim_T A(u,T)$ exists but is not equal to $\lim_{\delta\uparrow 1} V(u,\delta)$. Fortunately for the ease of analysis, it is "typical" of optima that the limits exist and are equal.

B. Simple Irreversibility

As the poets tell us, all choices of actions are irreversible.

The Moving Finger writes; and, having writ,
Moves on: nor all thy Piety nor Wit
Shall lure it back to cancel half a Line,
Nor all thy Tears wash out a Word of it.
(Omar Khayyam)

We are here interested in the timing of decisions that are either impossible to reverse or are so very expensive to reverse that we would never consider it. This is a huge topic, we content ourselves with a very simple example.
Suppose that prices start at time $t = 0$ and $p_0 = p > 0$ and that $p_{t+1} = p_t \cdot \eta_t$ where the $\eta_t > 0$ are independent have mean 1, and (for convenience) have continuous densities $h_t(\cdot)$. At each t where the dm has not acted, he/she can choose to act, $a = 1$, or not to act, $a = 0$. If the dm has acted, he/she can only choose not to act. (Start at $A_0 = 0$, $A_{t+1} = A_t + a_t$, $a_t, A_t \in \{0,1\}$.) The instantaneous payoff to not acting is 0, the payoff to choosing to act at t at p_t is $\beta^t u(p_t)$. Assume that

$u(\cdot)$ is increasing, smooth and concave and that there exists a $p^m > 0$ such that for all $p < p^m$, $u(p) < 0$ while for all $p > p^m$, $u(p) > 0$.
Because $\beta E\, u(p \cdot \eta) < u(E\, p \cdot \eta)$, the optimal strategy involves waiting until p_t is large enough and then acting. We wish to understand properties of the optimal value of "large enough."
To see that one waits until a price higher than p^m is to examine properties of the value function, $V^m(\cdot)$, for the myopic strategy, that is, to act as soon as $p_t \geq p^m$. This gives the increasing value function $V^m(p) > 0$ for $p < p^m$ and $V^m(p) = u(p)$ for $p \geq p^m$. This means that the value function for this policy jumps downwards at p^m. There are two ways to see why this must be a suboptimal strategy: acting at $p_t = p^m$ guarantees a payoff of 0, not acting means that there is a strictly positive probability of a strictly positive payoff tomorrow while the worst that will happen in the future is a payoff of 0; there exists an $\epsilon > 0$ with the property that $V^m(p^m - \epsilon) > V^m(p^m + \epsilon)$, that is, giving up some of your reward would help you. In other words, if p_t is just over p^m, it is worthwhile to wait, p_{t+1} might be higher, and this piece of information is worth waiting for, even worth the risk that p_{t+1} is lower, provided the "just over" is small enough.
For any $x > 0$, let τ_x be the random variable $\min\{t \geq 0 : p_t \geq x\}$. For any value of τ_x, define $\Pi(x) = \sum_{t=0}^{\tau_x - 1} 0 + \beta^{\tau_x} u(p_{\tau_x})$, define $f(x,p) = E\,(\Pi(x)|p_0 = p)$ and define the value function by $V(p) = \sup_{x>0} f(x,p)$. Under the assumptions given here, standard envelope theorem results tell us that $V(\cdot)$ is smooth which means that at the crucial p^*, we will have $V'(p^*) = u'(p^*)$.

C. Repeated Games and their Equilibria

We begin our analysis of repeated games with a classic example, the Prisoners' Dilemma. We then turn to a method of constructing strategies called Penal Codes, in which irreversibility and partial irreversibility both play central roles. After this analysis, we will give the formal notation for repeating games, both finitely often, corresponding to playing the game until a known end point in time to the repeated interaction, and "infinitely" often, corresponding to playing the game until a random ending point in time, T, with $P(T = t) > 0$ for an infinite number of t's. After this, we turn to the formalities of Abreu's Simple Penal Codes and Optimal Simple Penal Codes.

C.1. Repeated Prisoners' Dilemma. Recall the we had a joint investment problem with the payoffs given by

	Don't invest	Invest
Don't invest	$(2,2)$	$(12,0)$
Invest	$(0,12)$	$(9,9)$

One analysis of this game suggested that a contract, binding only if both sign it, specifying damages at least 3 in case that one player invests but the other does not, "solves" the problem of having a unique dominant strategy equilibrium that is so bad for both players. For example, if the contract specifies damages of, say, 4, it changes the payoffs to

	Don't invest	Invest
Don't invest	$(2,2)$	$(8,4)$
Invest	$(4,8)$	$(9,9)$

An alternate source of solutions is to note that we expect these firms to interact many times in the future, and, as primates, we are very good at linking future actions to present choices. Not to put too fine a point on it, we are good at rewarding nice behavior and punishing bad behavior. In parallel with the motivational structures sometimes used for donkeys, mules, and horses, we sometimes call this "carrots and sticks."

C.1.1. *Finite Repitition is Not Enough.* The following is ridiculously easy to prove, though it is still rather difficult to believe. It tells us that threats to retaliate for "bad" behavior and reward "good" behavior are not credible in the N-times repeated Prisoners' Dilemma when N is known.

LEMMA IV.1. *If Γ is the Prisoners' Dilemma just given, then for every finite N, the unique equilibrium for Γ^N is "Don't Invest" in each period.*

When N is not known, the situation is quite different, but one must specify reactions to an infinite number of situations.

C.1.2. *An Equilibrium Sequence of Carrots and Sticks.* Consider the following three histories, two of which are equal but are labelled differently. We let d and v represent "*d*on't invest" and "in*v*est."

$$\begin{aligned} q^0 &= ((v,v),(v,v),(v,v),\ldots), \\ q^1 &= ((d,d),(d,d),(d,d),\ldots), \text{ and} \\ q^2 &= ((d,d),(d,d),(d,d),\ldots). \end{aligned} \tag{C.1}$$

(C.2)

From these three histories, we construct a strategy $F(q^0;(q^1,q^2))$ as follows:

1. at $t=0$, the "pariah counter," c is set to 0, the "within history" time is set to $\tau=0$, and both agents pick the action for q^c at τ;
2. at each $t \geq 1$, the agents update the pariah and within history time counter, c' and τ', as specified below and then play the τ'th element of q^c with the updated value of τ and c. If the pariah counter last period was c and the within history time was τ,
   a. and if the τ' element of $q^{c'}$ was played at $t-1$, then the pariah counter stays the same and the within history counter is set to $\tau+1$;
   b. the τ' element of $q^{c'}$ was not played at $t-1$ and only agent i failed to play their part of the τ' element of $q^{c'}$, then the pariah counter is set to i and the within history time is set to $\tau=0$;
   c. and if the τ' element of $q^{c'}$ was not played at $t-1$ and both agents failed to play their part of the τ' element of $q^{c'}$, then the pariah counter is set to 1 and the within history time is set to $\tau=0$.

The idea is that when/if the players fail to go along with the strategy recommendations, they are named "pariah." Provided no-one else ever goes against the strategy, this is irreversible.

Because the histories in (C.1) have so simple a structure, $F(q^0;(q^1,q^2))$ can be equivalently expressed as "start by playing v and so long as only v has been played

in the past, continue to play v, otherwise play d." Sometimes this strategy is called the "grim trigger strategy" because there is a trigger, someone deviating from v, and a grim response, play d forever thereafter.
For some values of δ, the grim trigger strategy is a **subgame perfect equilibrium** for the utility functions $V(\cdot, \delta)$. Being subgame perfect means that the strategies are an equilibrium starting after any possible history. What makes this easy to check in this case is that, for these strategies, there are only two kinds of histories that we need check, depending on what the strategy calls for the players to do.

· First, if the strategy $F(q^0; (q^1, q^2))$ calls for both players to play d, then it will call for d forever thereafter. A best response to the other player playing d forever thereafter is d now and forever.
· Second, if the strategy $F(q^0; (q^1, q^2))$ calls for both players to play v, then it will call for v forever thereafter provided neither player deviates from the strategy. The best possible payoff to deviating at t is

$$(1-\delta) \cdot \left(\delta^t 12 + \delta^{t+1} 2 + \delta^{t+2} 2 + \cdots\right) = (1-\delta)\delta^t (12 + \tfrac{\delta}{(1-\delta)2}. \tag{C.3}$$

The payoff to not deviating at t is

$$(1-\delta) \cdot \left(\delta^t 9 + \delta^{t+1} 9 + \delta^{t+2} 9 + \cdots\right) = (1-\delta)\delta^t \tfrac{1}{(1-\delta)} 9. \tag{C.4}$$

Not deviating beats deviating if

$$9\tfrac{1}{(1-\delta)} > 12 + 2\frac{\delta}{(1-\delta)}, \text{ equivalently} \tag{C.5}$$

$$\frac{1}{(1-\delta)}(9 - 2\delta) > 12, \text{ or} \tag{C.6}$$

$$9 - 2\delta > 12 - 12\delta, \text{ that is } \delta > \frac{3}{10}. \tag{C.7}$$

In terms of the payoff box at t if no-one has played d in the past and we presume that the strategy $F(q^0; (q^1, q^2))$ will be played in the future, we have the payoffs being δ^t times the following,

	Don't invest	Invest
Don't invest	$(2, 2)$	$(12(1-\delta) + 2\delta, 0(1-\delta) + \delta 2)$
Invest	$(0(1-\delta) + \delta 2, 12(1-\delta) + 2\delta)$	$(9, 9)$

Again, $\delta > \frac{3}{10}$ is enough to make v a best response to the expectation of v by the other player. Taking $\delta = \frac{1}{2}$ yields the payoff matrix

	Don't invest	Invest
Don't invest	$(2, 2)$	$(7, 1)$
Invest	$(1, 7)$	$(9, 9)$

Note the difference from the contractual solution — v is not dominant, expecting the other to play d makes d a best response, while expecting the other to play v makes v a best response.

C.1.3. *Temporary Joint Penance.* Penance need not be everlasting. Consider $F(q^0; (q^1, q^2))$ constructed from the following strategies,

$$\begin{aligned} \text{(C.8)} \qquad q^0 &= ((v,v),(v,v),(v,v),\ldots), \\ q^1 &= (\underbrace{(d,d),(d,d),(d,d),\ldots,(d,d)}_{T \text{ times}},(v,v),(v,v),\ldots), \text{ and} \\ q^2 &= (\underbrace{(d,d),(d,d),(d,d),\ldots,(d,d)}_{T \text{ times}},(v,v),(v,v),\ldots). \end{aligned}$$

(C.9)

One description of the associated strategy is that it is a somewhat softer trigger, if anyone deviates, we both spend T periods doing pennance playing d, then go back to playing v provided everyone has gone along with T periods of penance.

Problem C.1. *Give conditions an T and δ making the strategy $F(q^0; (q^1, q^2))$ just given a subgame perfect equilibrium.*

C.1.4. *Temporary Individualized Penance.* Penance can be individualized. Consider $F(q^0; (q^1, q^2))$ constructed from the following strategies,

$$\begin{aligned} \text{(C.10)} \qquad q^0 &= ((v,v),(v,v),(v,v),\ldots), \\ q^1 &= (\underbrace{(v,d),(v,d),(v,d),\ldots,(v,d)}_{T \text{ times}},(v,v),(v,v),\ldots), \text{ and} \\ q^2 &= (\underbrace{(d,v),(d,v),(d,v),\ldots,(d,v)}_{T \text{ times}},(v,v),(v,v),\ldots). \end{aligned}$$

(C.11)

One description of the associated strategy is that the pariah must make amends for T periods before being forgiven.

Problem C.2. *Give conditions an T and δ making the strategy $F(q^0; (q^1, q^2))$ just given a subgame perfect equilibrium.*

C.2. Finitely and Infinitely Repeated Games. Here we take a game $\Gamma = (A_i, u_i)_{i \in I}$ and play it once at time $t = 0$, reveal to all players which $a_i \in A_i$ each player chose, then play it again at time $t = 1$, reveal, etc. until all the plays in $\mathbb{T}$ have happened, $\mathbb{T} = \{0, 1, \ldots, N-1\}$ or $\mathbb{T} = \{0, 1, \ldots\}$. The first case, $\mathbb{T} = \{0, 1, \ldots, N-1\}$, corresponds to repeating the game N times, the second case, $\mathbb{T} = \{0, 1, \ldots\}$, corresponds to repeating the game until some random time in the future where the unknown time may be arbitrarily large.
If $a^t = (a_i^t)_{i \in I}$, and $(a^t)_{t \in \mathbb{T}}$ is the sequence of plays, then the period utilities for the players are $u = (u_t)_{t \in \mathbb{T}} := (u_i(a^t))i \in I, t \in \mathbb{T}$, and the payoffs will either be

$$\text{(C.12)} \qquad A(u, N) = \frac{1}{N}\sum_{t=0}^{N-1} u_t \in \mathbb{R}^I \text{ or } V(u,\delta) = (1-\delta)\sum_{t=0}^{\infty} u_t \delta^t \in \mathbb{R}^I$$

in the finite or the infinite case respectively. Because of the normalizations, $A(u, N), V(u, \delta) \in$ $\mathbf{co}\,(u(A))$. We denote the game played N times by Γ^N, the game played infinitely often with discount factor δ by Γ^∞_δ.

When playing the game $N < \infty$ times, the possible history space for the game is H^N, the product space

$$H^N = \underbrace{A \times \ldots \times A}_{N \text{ times}}. \tag{C.13}$$

When playing the game "infinitely often," the possible history space is

$$H^\infty = (a^0, a^1, \ldots) \in A \times A \times \cdots . \tag{C.14}$$

For $h^N \in H^N$ or $h^\infty \in H^\infty$, we associate utilities as above.

A strategy for $i \in I$ specifies what i will do at $t = 0$, what they will do in response to each and every vector of choices $a^0 \in A$, what they will do in response to each and every vector of choices $(a^0, a^1) \in H^2$, and so on through all $t \in \mathbb{T}$. Without loss, we restrict attention to mixed strategies that specify a distribution in $\Delta(A_i)$ for $t = 0$, specify a distribution in $\Delta(A_i)$ in response to each and every vector of choices $a^0 \in A$, each and every $(a^0, a^1) \in H^2$, and so forth.

Since the strategy sets are very different in Γ, Γ^N, and Γ^∞_δ, the way that we will be comparing the equilibrium sets is to compare $u(Eq(\Gamma))$, $U^N(Eq(\Gamma^N))$, $U^N(SGP(\Gamma^N))$, $U^\infty_\delta(Eq(\Gamma^\infty_\delta))$ and $U^\infty_\delta(SGP(\Gamma^\infty_\delta))$. The starting point is

Lemma IV.2. *If $\sigma^* \in Eq(\Gamma)$, then $\sigma_i^t \equiv \sigma_i^* \in SGP(\Gamma^N)$, $i \in I$, $t = 1, \ldots, N$, and $\sigma_i^t \equiv \sigma_i^* \in SGP(\Gamma^\infty_\delta)$, $i \in I$, $t = 1, 2, \ldots$.*

Since every SGP is an equilibrium and $Eq(\Gamma) \neq \emptyset$, immediate corollaries are

$$\emptyset \neq u(Eq(\Gamma)) \subset U^N(SGP(\Gamma^N)) \subset U^N(Eq(\Gamma^N)), \text{ and}$$
$$\emptyset \neq u(Eq(\Gamma)) \subset U^\infty_\delta(SGP(\Gamma^\infty_\delta)) \subset U^\infty_\delta(Eq(\Gamma^\infty_\delta)).$$

In this sense, we've "rigged" the results, the only kinds of results are increases the set of equilibria when the game is repeated.

D. The Logic of Repeated Interactions

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