Assignment #1 for **Mathematics for Economists**
Fall 2018

Due date: Wed. Sept. 19.

Readings: CSZ, Ch. 3, 4.3-5, 4.7, 4.10, and 4.11

A. **From Chapter 3.6**: 3.6.4 (p. 91)
B. **From Chapter 3.6**: 3.6.6 (p. 92).
C. **From Chapter 3.7**: 3.7.12 (p. 96), 3.7.16 and 17 (p. 98).
D. About the triangle inequality in $\mathbb{R}^\ell$, $\ell \geq 1$.
1. For $r, s \in \mathbb{R}$, show that $|r + s| = |r| + |s|$ iff $rs \geq 0$, that is, iff r and s have the same sign.
2. For $r, s, t \in \mathbb{R}$ with $r \leq t$, show that $|r - s| + |s - t| = |r - t|$ iff $r \leq s \leq t$.
3. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^\ell$, show that $\|\mathbf{x} - \mathbf{y}\|_2 + \|\mathbf{y} - \mathbf{z}\|_2 = \|\mathbf{x} - \mathbf{z}\|_2$ iff $(\exists \alpha \in [0, 1])[\mathbf{y} = \alpha\mathbf{x} + (1 - \alpha)\mathbf{z}]$.
4. For $r > 0$, $B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^\ell : \|\mathbf{x} - \mathbf{y}\|_2 < r\}$. Show that if $\mathbf{y}, \mathbf{z} \in B_r(\mathbf{x})$, then $(\forall \alpha \in [0, 1])[\alpha\mathbf{y} + (1 - \alpha)\mathbf{z} \in B_r(\mathbf{x})]$.

E. Show the following.
1. If $x_t = a_t\beta^t$, $|a_t| \leq B$ for some $B \in \mathbb{R}_+$, then the sequence $a_t\beta^t$ is summable provided $|\beta| < 1$.
2. [Infinite decimal expansions] For any sequence of integers, $n_t \in \{0, 1, 2, \ldots, 9\}$, the sequence $x_t = \sum_{i=1}^{t} \frac{n_t}{10^t}$ is Cauchy and converges to the number $0.n_1n_2n_3\cdots$.
3. If $x_t = 1/t^a$, $a > 0$, then the sequence $(x_t)_{t=1}^\infty$ is summable iff $a > 1$.
4. If $x_t = \frac{a}{bt+c}$ for $t \in \mathbb{N}$ and $a, b \neq 0$, then x_t is not summable, but the partial sums of $y_t = \frac{a}{(-1)^t bt+c}$ do converge.
5. If $E_N(r) := \sum_{n=0}^{N} \frac{r^n}{n!}$, then for all $r \in \mathbb{R}$, the sequence $E_N(r)$ converges. [The limit provides one definition of e^r.]

F. Problems about long-run averages and discounting. For this problem, sequences start at $t = 0$, that is, a sequence is $(x_0, x_1, x_2, \ldots)$ and all sequences are bounded.
1. Suppose that $x_t \to x$, show that $\frac{1}{T+1}\sum_{t \leq T} x_t \to x$.
2. Let x_t be the sequence $1, 7, 9, 3, 1, 7, 9, 3, 1, 7, 9, 3, \ldots$. Show that $x_t \not\to x$ for any x, give $\lim_T \frac{1}{T+1}\sum_{t \leq T} x_t$ and prove that your answer is correct.
3. Consider the sequence

$$(x_t)_{t=0}^\infty = (\underbrace{0, 0}_{2^1}, \underbrace{1, 1, 1, 1}_{2^2}, \underbrace{0, 0, 0, 0, 0, 0, 0, 0}_{2^3}, \ldots).$$

Show that $\liminf_T \frac{1}{T+1}\sum_{t \leq T} x_t = \frac{1}{3}$ and $\limsup_T \frac{1}{T+1}\sum_{t \leq T} x_t = \frac{2}{3}$.
4. For $0 < \beta < 1$, express the density on $\{0, 1, 2, \ldots\}$ given by $p_t = (1 - \beta)\beta^t$ as a convex combination of uniform densities on $\{0, 1, \ldots, t\}$, $t = 0, 1, 2, \ldots$.
5. Show that if $\lim_{T\uparrow\infty} \frac{1}{T+1}\sum_{t \leq T} x_t = r$, then $\lim_{\beta\uparrow 1}(1 - \beta)\sum_{t=0}^\infty x_t\beta^t = r$. [The reverse implication is also true. It's called the Hardy-Littlewood Tauberian theorem. It is a good deal more difficult to prove.]

G. This problem concerns the macro growth/fishery model of §3.7 in CSZ. Suppose that the fish growth curve is given by $f(x) = 200\sqrt{x}$ and that the in-period utility of consuming fish is given by $u(c_t) = \log(c_t)$, and that the total utility is given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$ for some $0 < \beta < 1$.

1. Suppose that $x_0 > 0$ and $x_{t+1} = f(x_t)$ for all t. Characterize the time path and limit point(s) if any for all starting values x_0 assuming no fishing.
2. Give, as a function of β, the long-run steady state optimal consumption of fish, $c^*(\beta)$, and the corresponding fish stock left to reproduce each period.
3. Give the highest sustainable long-run fish consumption and fish stock. Show that it is the limit, as $\beta \uparrow 1$, of the answer you found in the previous part of this problem.
4. Give the Euler equation and the law of motion for the consumption path that maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$ given an initial stock of fish $x_0 > 0$.
5. Show that the consumption path you just found converges to $c^*(\beta)$.

H. Write down the infima and suprema of the following sets. In each case say whether the infima and suprema are minimum and maximum elements.
1. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.
2. $\{\frac{3n}{4n+1} : n \in \mathbb{N}\}$.
3. $\mathbb{Q} \cap (0, 1)$.
4. $\{q \in \mathbb{Q} : q^2 < 7\}$.

I. Verify carefully from the definition of supremum that
1. $\sup\{\frac{3n-1}{4n} : n \in \mathbb{N}\} = \frac{3}{4}$.
2. $\sup\{\cos(\frac{1}{n}) + (-1)^n : n \in \mathbb{N}\} = 2$.

J. Assume that the sequence x_n is nonnegative and converges to x, and that $y = \limsup_n y_n$ is finite and positive. Show that $\limsup_n x_n y_n = xy$.

K. Let E_n be a sequence of subsets of a set X and define $f_n(x) = 1_{E_n}(x)$. Show the following.
1. $\limsup_n 1_{E_n}(x) = 1$ iff x belongs to infinitely many of the E_n.
2. $\limsup_n 1_{E_n}(x) = 0$ iff x belongs to at most finitely many of the E_n.
3. $\liminf_n 1_{E_n}(x) = 1$ iff x belongs to all but finitely many of the E_n.
4. $\liminf_n 1_{E_n}(x) = 0$ iff x fails to belong to infinitely many of the E_n.

L. Let A_n and B_n be sequences of subsets of a non-empty set Ω. Let $A_- = \{\omega \in \Omega : \liminf_n 1_{A_n}(\omega) = 1\}$, $A^+ = \{\omega \in \Omega : \limsup_n 1_{A_n}(\omega) = 1\}$, with corresponding definitions for B_- and B^+. Give the relations between the sets A_-, A^+, B_-, B^+ and the sets

$$(AB)_- := \{\omega \in \Omega : \liminf_n 1_{A_n}(\omega) 1_{B_n}(\omega) = 1\} \text{ and}$$
$$(AB)^+ := \{\omega \in \Omega : \limsup_n 1_{A_n}(\omega) 1_{B_n}(\omega) = 1\}.$$

M. Define the

$$x(n, m) = \begin{cases} 0 & \text{if } m > n \\ -1 & \text{if } m = n \\ 2^{m-n} & \text{if } m < n \end{cases}.$$

1. For each n, calculate $S(n) = \sum_{m=1}^{\infty} x(n, m)$ and $\sum_n S(n)$.
2. For each m, calculate $T(m) = \sum_{n=1}^{\infty} x(n, m)$ and $\sum_m T(m)$.

N. Consider the function defined by $f(x,y) = x/y$ for $x, y > 0$ and $f(0,0) = 2$. Prove that your answers are correct in all of the following.

1. Show that $f(\cdot,\cdot)$ is continuous at every $(x^\circ, y^\circ) \in \mathbb{R}^2_{++}$.
2. Give a strictly decreasing sequence $(x_n, y_n) \to (0,0)$ with $\lim_n f(x_n, y_n) = f(0,0)$.
3. Give a strictly decreasing sequence $(x_n, y_n) \to (0,0)$ with $\limsup_n f(x_n, y_n) > f(0,0) > \liminf_n f(x_n, y_n)$.
4. Give a geometric charaterization of the set of strictly decreasing sequences $(x_n, y_n) \to (0,0)$ with the property that $\limsup_n f(x_n, y_n) \leq \liminf_n f(x_n, y_n)$.
5. Show that the previous yields a geometric characterization of the set of strictly decreasing sequences converging to $(0,0)$ for which $\lim_n f(x_n, y_n) = f(0,0)$.

O. [Tails of sequences] Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (M, d). For each $N \in \mathbb{N}$, define F_N as the closure of the set $\{x_{N+m} : m \in \mathbb{N}\}$.

1. Show that $x_n \to x$ if and only if $\cap_N F_N = \{x\}$ if and only if $d_H(F_N, \{x\}) \to 0$.
2. Show that $x_n \to x$ if and only if for every subsequence, x_{n_k}, we have $x_{n_k} \to x$.

P. [Some one-dimensional dynamical systems] For each of the following inductively defined sequences: check if the $f(\cdot)$ that gives $x_{n+1} = f(x_n)$ is a contraction mapping on a domain including the tail of the sequence; check for boundedness and for monotonicity of the sequence x_n; if the sequence x_n is convergent, prove it and give the limit; if the sequence x_n is not convergent, prove it.

1. $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$.
2. $x_1 > 1$ and $x_{n+1} := 2 - \frac{1}{x_n}$.
3. $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$.
4. $x_1 = 1$ and $x_{n+1} := \sqrt{2 + x_n}$.
5. $x_1 = \sqrt{p}$ for $p > 0$ and $x_{n+1} := \sqrt{p + x_n}$.
6. For $a > 0$, $x_1 > 0$, and $x_{n+1} := \sqrt{a + x_n}$.
7. For $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$.

Q. This problem concerns the dynamical system $x_{t+1} = f(x_t)$ in the case $M = [0,1]$ and $f(x) = 2\min\{x, 1-x\}$. For any x_0, define $K^*(x_0)$ to be the closure of the set $\{x_t : t = 0, 1, \ldots\}$.

The dyadic expansion of a number $r \in [0,1]$ as a sequence $(k_n)_{n=1}^\infty$ where each k_n is equal to 0 or is equal to 1. We think of this as the binary expansion $r = 0.k_1k_2k_3k_4\ldots$, equivalently as $r = \sum_{n=1}^\infty \frac{k_n}{2^n}$. There is ambiguity and sequences that end with an infinite string of 1's, for example $0.0111111\ldots$ specifies the same number as 0.100000000. The convention is that we rule out sequences ending in all 1's and put in $r = 1.000000\ldots$ to take care of the upper end of the interval.

Let $j_n = 1$ if $k_n = 0$ and $j_n = 0$ if $k_n = 1$. Note that for $r = 0.k_1k_2k_3\ldots > \frac{1}{2}$, we have $(1-r) = \sum_{n=1}^\infty \frac{y_n}{2^n}$. Further if $r < \frac{1}{2}$, then $2 \cdot r = \sum_{n=1}^\infty \frac{x_{n+1}}{2^n}$. Combining,

$$f(0.x_1x_2x_3x_4x_5\ldots) = \begin{cases} 0.x_2x_3x_4x_5\ldots & \text{if } x_1 = 0 \\ 0.y_2y_3y_4y_5\ldots & \text{if } x_1 = 1 \end{cases}$$

1. Note that $\frac{4}{7} = 0.100100100100\ldots$ and find $K^*(\frac{4}{7})$.
2. Show that the binary expansion of $r \in [0,1]$ repeats if and only if r is rational.
3. Show that if the binary expansion of r repeats, then $K^*(r)$ is finite.

4. Show that $\#K^*(r)$ is infinite if $r \notin \mathbb{Q}$.
5. Suppose that we pick $r = 0.x_1x_2x_3x_4 \in (0,1)$ at random by picking the x_t to be iid Bernoulli random variables with parameter $p \in (0,1)$. Show that, for a probability 1 set of r's, $\sup K^*(r) = 1$ and $\inf K^*(r) = 0$.
6. If you're feeling really ambitious, show that for a probability 1 set of r's, $K^*(r) = [0,1]$.