Assignment #2 for **Mathematics for Economists**
Fall 2018

Due date: Monday, Oct 15, 2018

Topics: Compactness and continuity; convexity and concavity; FOCs for concave functions; maximization of concave functions over convex sets; the separating hyperplane theorem and the Kuhn-Tucker theorem; differentiable comparative statics.

Readings: CSZ, Ch. 4.4-9, 4.11, Ch. 5.1-8, Ch. 6.1-2.

Handout with Ben-Porath's proof of the Kuhn-Tucker theorem.

A. CSZ, Exercise 4.8.4.
B. CSZ, Exercise 4.8.17.
C. The boundary of a set E in a metric space (M, d) is defined by $\partial(E) = \text{cl}(E) \cap \text{cl}(E^c)$.
   1. Give $\partial B_r(x)$.
   2. Give $\partial(E)$ if both E and E^c are dense in M.
   3. Show that $E \cup \partial(E) = \text{cl}(E)$.
   4. Show that $x \in \partial(E)$ if and only if it is an accumulation point of both E and E^c.
   5. Show that if $E \subset \mathbb{R}^n$ is convex, then so is $\text{cl}(E)$.
   6. If $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, give $\partial(E)$.
   7. If $E = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1\}$, give $\partial(E)$. [The answer is different than the previous one.]
   8. CSZ, Exercise 5.5.7.
D. Let $d(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ be two metrics on M. The metrics are equivalent if $[d(x_n, x) \to 0] \Leftrightarrow [\rho(x_n, x) \to 0]$.
   1. Let $d(\cdot, \cdot)$ be a metric on M and define $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Show that $\rho(\cdot, \cdot)$ is a metric and that it is equivalent to $d(\cdot, \cdot)$.
   2. Generalize the previous to show that if $h : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and concave, then $\rho(x, y) := h(d(x, y))$ is a metric equivalent to $d(\cdot, \cdot)$. To what extent can you remove the adjective "strictly" and still have this result be true?
   3. For $M = \mathbb{R}$, define $e(x, y) = |\Phi(x) - \Phi(y)|$ where $\Phi(r) = \frac{e^r}{1+e^r}$. Show that $e(\cdot, \cdot)$ is a metric and that it is equivalent to $d(x, y) = |x - y|$.
   4. In the previous problem, characterize the e-Cauchy sequences and prove that your characterization is correct.
E. Compactness is a very thorough form of completeness: show that (K, d) is compact if and only if (K, ρ) is a complete metric space for every metric $\rho(\cdot, \cdot)$ that is equivalent to $d(\cdot, \cdot)$.
F. Compactness in $\mathbb{R}$ and $\mathbb{R}^k$.
   1. If $r_n \to r$ is a sequence in $\mathbb{R}$, then there exists a monotone subsequence r_{n_k}.
   2. Using the previous, show that if r_n is a bounded sequence in $\mathbb{R}$, then there exists a convergent subsequence.
   3. Using the previous, show that if $\boldsymbol{r}_n$ is a bounded sequence in $\mathbb{R}^k$, then there exists a convergent subsequence.

4. Using the previous, show that $K \subset \mathbb{R}^k$ is compact if and only if it is closed and bounded.
5. Using the previous, show that $K \subset \mathbb{R}^k$ is compact if and only if every continuous $f : K \to \mathbb{R}$ achieves its maximum on K.
6. The previous statement is true for all metric spaces. Find its proof in CSZ and figure out what makes it more difficult to prove. [There is nothing to hand in for this problem.]

G. Suppose that K is a compact set of possible decisions and allocations that a society consisting of individuals $i = 1, \ldots, I$ could make, and that each i has preferences that can be represented by a continuous $u_i : K \to \mathbb{R}$. A point $x^* \in K$ is **weakly Pareto optimal** if there is no $y \in K$ such that $u_i(y) > u_i(x^*)$ for each i. Let WP denote the set of weakly Pareto optimal x^* in K.
1. Let $x^*(\Lambda) = \text{argmax}_{x\in K} \sum_i \lambda_i u_i(x)$ where $\Lambda = (\lambda_i)_{i=1}^I > 0$ (i.e. is weakly positive in each component and is not equal to 0). Show that each $x^*(\Lambda)$ is a non-empty subset of WP.
2. Give an example in which WP contains elements that are not of the form $x^*(\Lambda)$ for any Λ. [It is sufficient to give the set of possible utility levels for this.]
3. For each i, let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing function and define $u_i^\circ(x) = \varphi_i(u_i(x))$. Show that WP does not change with these new utility functions. Give an example in which the union of the set of $x^*(\Lambda)$ changes after this kind of monotonic transformation of the utility functions.
4. For a vector $v \in \mathbb{R}^I$ and $\Lambda > 0$, define $U(x; v, \Lambda) = \min_i \lambda_i(u_i(x) - v_i)$ and $x^*(v, \lambda) = \text{argmax}_{x\in K} U(x; v, \Lambda)$. Show that WP is the union of the $x^*(v, \Lambda)$.
5. Show that replacing the $u_i(\cdot)$ by the monotonic transformations $u_i^\circ = \varphi_i(u_i(\cdot))$ does not change the union of the $x^*(v, \Lambda)$. [There is a hard way to do this, and an easy way.]

H. Suppose that (K, d) is a compact metric space and that $f : K \to K$ is **strictly non-expansive**, that is, suppose that f satisfies $d(f(x), f(y)) < d(x, y)$ for all $x, y \in K$.
1. Show that the function $(x, y) \mapsto d(f(x), f(x))$ from $K \times K$ to $\mathbb{R}_+$ is continuous (i.e. show that if $x_n \to x$ and $y_n \to y$, then $d(f(x_n), f(y_n)) \to d(f(x), f(y))$).
2. Show that f has a unique fixed point in K.
3. Let M be the non-compact metric space $\mathbb{R}_+$ with the usual metric and define $f : M \to M$ by $f(x) = x + 1/e^{x^2}$.
   a. Show that f is strictly non-expansive.
   b. Show that f has no fixed point.
   c. Define x° to be a nummerical fixed point if $|x^\circ - f^t(x^\circ)| < 1/1,000,000$ for all $t \in \{1, \ldots T\}$. If $T = 10$, how many steps will the numerical procedure with $x_0 = 1$ and $x_{t+1} = f(x_t)$ take to reach a numerical fixed point?

I. For a metric space (M, d), $C_b(M)$ denotes the set of continuous and bounded functions $f : M \to \mathbb{R}$. The distance between functions $f, g \in C_b(M)$ is given by $d(f, g) = \sup_{x\in M} |f(x) - g(x)|$. This problem asks you to show that $C_b(M)$ is a complete metric space, i.e. that every Cauchy sequence of functions in $C_b(M)$ has a limit that also belongs to $C_b(M)$.

1. Show that $d(\cdot, \cdot)$ is a metric.
2. Show that if f_n is a Cauchy sequence in $C_b(M)$, then for each $x \in M$, $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$. Let $f(x)$ denote $\lim_n f_n(x)$.
3. Show that $f \in C_b(M)$, that is, show that f is both bounded and continuous.
4. Show that $d(f_n, f) \to 0$.

J. Problems related to the Theorem of the Maximum.
   1. CSZ, Exercise 4.10.4.
   2. CSZ, Exercise 4.10.5.
   3. CSZ, Exercise 4.10.25.

K. More problems related to the Theorem of the Maximum.
   1. CSZ, Exercise 6.1.19.
   2. CSZ, Exercise 6.1.20

L. CSZ, Exercises 5.1.18 and 5.1.19.

M. CSZ, Exercise 5.1.40.

N. CSZ, Exercise 5.4.9.

O. CSZ, Exercise 5.4.24.

P. [A primitive version of the Kuhn-Tucker theorem] Suppose that: K is a compact convex subset of an open $G \subset \mathbb{R}^\ell$; K has a non-empty interior; $f : G \to \mathbb{R}$ is concave and has continuous first derivatives. Let $\mathbf{x}^*$ solve the problem $\max_{\mathbf{x} \in K} f(\mathbf{x})$. Show the following.
   1. If $\mathbf{x}^*$ is in the interior of K, then $Df(\mathbf{x}^*) = 0$.
   2. If $\mathbf{x}'$ is in the interior of K and $Df(\mathbf{x}') = 0$, then $\mathbf{x}'$ solves $\max_{\mathbf{x} \in K} f(\mathbf{x})$.
   3. If $\mathbf{x}^* \in \partial(K)$, then for all $\mathbf{x} \in K$, $(\mathbf{x} - \mathbf{x}^*) \cdot Df(\mathbf{x}^*) \leq 0$.
   4. Suppose now that $K = \{\mathbf{x} : g_m(\mathbf{x}) \leq b_m, m = 1, \dots, M\}$ where each $g_m(\cdot)$ is a continuously differentiable, convex function. Show that $Df(\mathbf{x}^*) = \sum_m \lambda_m Dg_m(\mathbf{x}^*)$ for a non-negative set of numbers λ_m, $m = 1, \dots, M$.