Assignment #3 for **Mathematics for Economists**
Fall 2018

Due date: November 12, 2018

Topics: envelope theorems; monotone comparative statics; increases in risk; informativeness of signals and risk; savings in the face of risk.

Readings: CSZ, 2.7-8, 4.8-9, 6.6

"Supermodularity and Complementarity in Economics: An Elementary Survey," Rabah Amir, *Southern Economic Journal*, 71(3), 636-660 (2005).

Handout on the KT theorem, envelopes, and absolutely continuous functions.

"Envelope Theorems for Arbitrary Choice Sets," Milgrom and Segal, *Econometrica*, 70(2), 583-601 (2002).

"Increasing Risk and Increasing Informativeness: Equivalence Theorems," Erin Baker, *Operations Research*, 54(1), 26-36 (2006).

A. [Single rotation versus Faustmann rotation in forestry] An area of the forest is planted at $t = 0$. If it is cut down at $t > 0$, the benefits are given by $Q(t) > 0$. Typically, $Q(\cdot)$ is "ess-shaped," that is, it is nearly flat for a while, then has a region of convex increase, then it becomes concave and flattens out. This can make first order conditions have two solutions, one of which will be a minimum.

With an interest rate r, the single rotation problem is

$$\max_{t \geq 0} Q(t)e^{-rt},$$

and t_1^* denotes the solution (set). The Faustmann rotation problem is

$$\max_{t \geq 0} Q(t) \left[e^{-rt} + e^{-r2t} + e^{-r3t} + \cdots\right],$$

and t_F^* denotes the solution (set).

1. How does t_1^* depend on r? Explain this mathematically and in economic terms (think opportunity cost of capital).
2. Repeat the previous problem for t_F^*.
3. Compare t_1^* and t_F^*, mathematically and economically.
4. Suppose now that the forest burns down at a random time t after planting having cdf $F(\cdot)$. If it burns down before you cut it, you receive nothing (but in the Faustmann case you can replant at that point in time). How does this change t_1^*? What about t_F^*? Explain the mathematics and the economics of this.
5. Suppose that as well as the profits, $Q(t)$, the forest also yields a flow of amenities $A(t) > 0$ so that cutting it down at t yields total benefit $Q(t) + \int_0^t e^{-\rho x} A(x)$. How does this change t_1^*? What about t_F^*? Explain the mathematics and the economics of this.

B. [Laws of unintended consequences] Suppose that there is a policy, to be set at a level $t \geq 0$. For examples, this could be a frequency of vehicle inspection, a tax level, a maximal amount of pollution that any given vehicle can emit per mile. Sometimes only parts of the benefits, $B(t)$, or parts of the costs, $C(t)$, are included.

We want to see what happens to the optimal t when they are all included. We suppose for the first parts of the problem that both the social benefits and social costs increasing in t.

1. Carefully compare the properties of the *sets* of optimal t's for the problems
$$\max_{t\geq 0}\,[B(t) - C(t)] \text{ and } \max_{t\geq 0}\,[(B(t) + B_2(t)) - C(t)]$$
where $B_2(\cdot)$ is another increasing benefit function.
2. Carefully compare the properties of the *sets* of optimal t's for the problems
$$\max_{t\geq 0}\,[B(t) - C(t)] \text{ and } \max_{t\geq 0}\,[B(t) - (C(t) + C_2(t))]$$
where $C_2(\cdot)$ is another increasing cost function.
3. Carefully compare the properties of the *sets* of optimal t's for the problems
$$\max_{t\geq 0}\,[B(t) - C(t)] \text{ and } \max_{t\geq 0}\,[(B(t) + B_2(t)) - (C(t) + C_2(t))]$$
when
a. the net benefits $B_2(\cdot) - C_2(\cdot)$, are increasing, and
b. the net benefits $B_2(\cdot) - C_2(\cdot)$, are decreasing.
4. Explain how one or more of these comparisons apply to vehicle inspection rates that are designed to decrease accidents to an optimal rate, but also benefit the environment.
5. Explain how one or more of these comparisons apply to limits on pollution per vehicle that are designed to increase air quality to some optimal level but have the effect of making the cheaper cars less available, where the cheaper cars are the older and more polluting ones.
6. Explain how one or more of these comparisons apply to royalties charged to private logging firms on public lands when illegal logging is an option.
7. Explain how one or more of these comparisons apply to white collar crime by executives when the Justice Department fails to prosecute executives and instead seeks monetary penalties paid by the shareholders.
8. Explain how one or more of these comparisons apply to the following slippery slope: a system of cameras installed to deter crime can be linked to facial recognition software allowing governments to track all citizens' movements.

C. Suppose that a monopolist sells to N identical customers so their profit function is

(1) $$\pi(x, N) = Nxp(x) - c(Nx) = N \cdot \left[xp(x) - \tfrac{c(Nx)}{N}\right].$$

1. If $c(\cdot)$ is convex, how does $x^*(N)$ depend on N?
2. If $c(\cdot)$ is concave, how does $x^*(N)$ depend on N?

[Concave cost functions correspond to declining marginal costs, the case of a natural monopoly. Convex cost functions typically correspond to a monopoly that is maintained by some non-market institution.]

D. You start with an amount x, choose an amount, c, to consume in the first period, and have $f(x - c)$ to consume in the second period, and your utility is $u(c) + \beta u(f(x - c))$ where $u' > 0$ and $u'' < 0$. We suppose that $r \mapsto f(r)$ is increasing.

1. Consider the two-period consumption problem,

(2) $$P_c(x) = \max \quad [u(c) + \beta u(f(x - c))] \text{ subject to } c \in [0, x].$$

Prove that this problem is equivalent to the two-period savings problem,

(3) $$P_s(x) = \max \quad [u(x-s) + \beta u(f(s))] \text{ subject to } s \in [0, x].$$

2. Prove that savings, $s^*(x)$, are weakly increasing in x and β.
3. Now define $V(y) = \max_{c_0, c_1, \cdots} \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $x_0 = y$, $s_t = x_t - c_t$, $x_{t+1} = f(s_t)$, and $c_t \in [0, x_t]$, $t = 0, 1, \ldots$. Assuming that the maximization problem for V has a solution, show that $V(\cdot)$ is increasing. From this, prove that the solution to the following infinite horizon savings problem is weakly increasing in x and β,

(4) $$P(x) = \max \quad [u(x-s) + \beta V(f(s))] \text{ subject to } s \in [0, x].$$

E. For random variables X and Y taking values in an interval $[0, M]$, Y is **riskier** than X if $E\, f(Y) \geq E\, f(X)$ for all *convex* $f : [0, M] \to \mathbb{R}$.
1. If X is riskier than Y then $E\, X = E\, Y$.
2. Y is riskier than X if and only if $E\, f(Y) \geq E\, f(X)$ for all f of the form $f(r) = -r$ and $f(r) = \max\{r - a, 0\}$, $a \in \mathbb{R}$.
3. If $Y = X + \varepsilon$ and ε is a random variable satisfying $E\,(\varepsilon|X) = 0$, then $E\, Y = E\, X$ and Y is riskier than X.
4. Let $R \geq 0$ be a random variable, and let ε be a random variable such that $E\,(\varepsilon|R) = 0$ and $R + \varepsilon$ takes values in some bounded interval. For $t \in [0, 1]$, let $V(t) = \max_{x \in [0,1]} E\, u((1-x)W + x(R + t\varepsilon))$ for a strictly concave, increasing $u(\cdot)$. Show that $x^*(\cdot)$ and $V(\cdot)$ are both decreasing in t.
5. Consider the two-period problem $\max_{s \in [0, w_0]} \; u(w_0 - s) + E\, \beta u(W_1 + (1+r)s)$ where W_1 is a random variable not under the control of the decision maker. Give conditions on $u(\cdot)$ guaranteeing that if we replace W_1 by a riskier random variable W_1', then the corresponding optimal s is larger. Prove that your answer is correct. [Hint: prudence.]

We now turn to some background useful for choice under uncertainty problems. Consider the problems

(5) $$\max_{x \in X} \; U(x, \theta) := \int u(x, s) f(s; \theta)\, ds$$

where $f(\cdot, \theta)$ is a class of densities, $\theta \in \Theta \subset \mathbb{R}^\ell$. To have $U(\cdot, \cdot)$ supermodular, we need, for $x' > x$ and $\theta' > \theta$, to have

(6) $$\int [u(x', s) - u(x, s)]\; f(s; \theta')\, ds \geq \int [u(x', s) - u(x, s)]\; f(s, \theta)\, ds.$$

There is a well-known condition that is sufficient for this, see problem L below.

Definition 1. *A family of densities on $\mathbb{R}$, $\{f(s; \theta) : \theta \in \Theta\}$, $\Theta \subset \mathbb{R}$, has the* ***monotone likelihood ratio property (MLRP)*** *if there exists a $s \mapsto T(s)$ such that for any $\theta' > \theta$, $f(s; \theta')$ and $f(s; \theta)$ are the densities of different distributions, and $\frac{f(s;\theta')}{f(s;\theta)}$ is a nondecreasing function of $T(s)$.*

Comment: if $s \mapsto T(s)$ is monotonic, as it often is, we can simplify the assumption to $\frac{f(s;\theta')}{f(s;\theta)}$ is a nondecreasing function of s.

F. An exponential distribution with parameter β has density $f(s;\beta) = \frac{1}{\beta}e^{-s/\beta}$ for $s \geq 0$, $\beta \in \Theta = (0,\infty)$.
  1. Does this class have the MLRP?
  2. For $\gamma > 0$, let $f(s;\gamma,\beta)$ be the density of X^γ where X has an exponential β.
     a. For fixed γ, does the class parametrized by β have the MLRP?
     b. For fixed β, does the class parametrized by γ have the MLRP?
  3. Let $Y = \sum_{i=1}^N X_i$ where the X_i are iid exponentials with parameter β. Does Y have the MLRP?

G. Consider the class of uniform distributions, $U[a,b]$.
  1. Does the class of uniform distributions $U[0,\theta]$ have the MLRP?
  2. What about the class of uniform distributions $U[-\theta,\theta]$?
  3. What about the class of uniform distributions $U[\theta - r, \theta + r]$?

H. Consider the class of distributions $f(s;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-\mu)^2/2\sigma^2}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $s \in \mathbb{R}$.
  1. For fixed σ, does the class parametrized by μ have the MLRP?
  2. For fixed μ, does the class parametrized by σ have the MLRP?
  3. Show that for fixed σ, if $\mu' > \mu$, then the distribution $f(s;\mu',\sigma)$ first order stochastically dominates the distribution $f(s;\mu,\sigma)$.
  4. Show that for fixed μ, if $\sigma' > \sigma$, then the distribution $f(s;\mu,\sigma')$ is riskier than the distribution $f(s;\mu,\sigma)$.

I. Consider one-parameter families of pdfs on $\mathbb{R}$, $f(s;\theta) = C(\theta)e^{Q(\theta)T(s)}h(s)$ with $\theta \mapsto Q(\theta)$ a strictly increasing function. Verify that they have the MLRP. For this class of distributions, verify the existence of a uniformly most powerful test for $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$ by considering test of the form that they reject if $x > C$, accept if $x < C$ (and accept with probability γ if $x = C$).

MLRP classes are a special type of Pólya distribution, and more general results than the following are in Karlin's "Pólya Type Distributions, II," *Annals of Mathematical Statistics*, 28(2), 281-308 (1957).

Theorem 1. *Let $f(s;\theta)$ be a family of densities on $\mathbb{R}$ with the MLRP.*

(a) *If $s \mapsto g(s)$ is nondecreasing, then $\theta \mapsto \int g(s)\,df(s;\theta)\,ds$ is nondecreasing.*

(b) *If $X_1,\ldots,X_n$ are iid $f(s;\theta)$ and $(s_1,\ldots,s_n) \mapsto g(s_1,\ldots,s_n)$ is nondecreasing in each argument, then $\theta \mapsto \int g(s_1,\ldots,s_n)\,f(s_1;\theta)\cdots f(s_n;\theta)ds_1\cdots ds_n$ is nondecreasing.*

(c) *$\theta' > \theta$, then $f(\cdot;\theta')$ first order dominates $f(\cdot,\theta)$.*

(d) *If $s \mapsto g(s)$ crosses 0 from below at most once, i.e. for some s_0, $g(s) \leq 0$ for $s < s_0$ and $g(s) \geq 0$ for $s \geq s_0$, then either $\psi(\theta) := \int g(s)f(s;\theta)\,ds$ is everywhere positive or everywhere negative, or there exists θ_0 with $\psi(\theta) \leq 0$ for $\theta < \theta_0$ and $\psi(\theta) \geq 0$ for $\theta \geq \theta_0$.*

Proof. For part (a), define $A = \{s : f(s;\theta') < f(s;\theta)\}$, $B = \{s : f(s;\theta') > f(s;\theta)\}$, set $a = \sup_{s\in A} g(s)$, $b = \inf_{s\in A} g(s)$, and note that $a \leq b$. We must show that

$\int g(s)\,[f(s;\theta') - f(s;\theta)]\;ds \geq 0$. Now,

$$\int g(s)\,[f(s;\theta') - f(s;\theta)]\;ds = \tag{7}$$

$$\int_A g(s)\,[f(s;\theta') - f(s;\theta)]\;ds + \int_B g(s)\,[f(s;\theta') - f(s;\theta)]\;ds \geq \tag{8}$$

$$a\int_A [f(s;\theta') - f(s;\theta)]\;ds + b\int_B [f(s;\theta') - f(s;\theta)]\;ds = \tag{9}$$

$$\left(a\int_A [f(s;\theta') - f(s;\theta)]\;ds + a\int_B [f(s;\theta') - f(s;\theta)]\;ds\right) + \tag{10}$$

$$\left(b\int_B [f(s;\theta') - f(s;\theta)]\;ds - a\int_B [f(s;\theta') - f(s;\theta)]\;ds\right) = \tag{11}$$

$$0 + (b-a)\int_B [f(s;\theta') - f(s;\theta)]\;ds \geq 0. \tag{12}$$

Part (b) follows by conditioning and induction, (c) by considering the functions $g(s) = 1_{(r,\infty)}(s)$.

For part (d), we shall show that the θ_0 we need is $\theta_0 := \inf\{\theta : \int g(s)f(s;\theta)\,ds > 0\}$. For this, it is sufficient to show that for any $\theta < \theta'$, $[\psi(\theta) > 0] \Rightarrow [\psi(\theta') \geq 0]$. There are two cases: (1) $\frac{f(s_0;\theta')}{f(s_0;\theta)} = \infty$, which requires $f(s_0;\theta) = 0$; and (2) $\frac{f(s_0;\theta')}{f(s_0;\theta)} = r$ for some $r \in \mathbb{R}_+$.
(1) Given the MLRP, $\frac{f(s_0;\theta')}{f(s_0;\theta)} = \infty$ and $f(s_0;\theta) = 0$ imply that $\psi(\theta) \leq 0$.
(2) Given that $\frac{f(s_0;\theta')}{f(s_0;\theta)} = r$, $g(s) \geq 0$ for all s in the set $C = \{s : f(s;\theta) = 0, f(s;\theta') > 0\}$. Integrating over the complement of C gives the first of the following inequalities (where we have avoided dividing by 0),

$$\int g(s)f(s;\theta')\,ds \geq \int_{C^c} g(s)\frac{f(s;\theta')}{f(s;\theta)}f(s;\theta)\,ds \tag{13}$$

$$\geq \int_{(-\infty,s_0)} rg(s)f(s;\theta)\,ds + \int_{[s_0,+\infty)} rg(s)f(s;\theta)\,ds, \tag{14}$$

and this last sum is equal to $r\int g(s)f(s;\theta)\,ds$. Since $r \geq 0$ and $\int g(s)f(s;\theta)\,ds > 0$, we conclude that $\int g(s)f(s;\theta')\,ds \geq 0$. □

J. If X has a distribution with density $s \mapsto f(s)$, then the class of densities $\{f(s-\theta) : \theta \in \mathbb{R}\}$ is a **location class**.
   1. For any location class, if $\theta' > \theta$, then $f(s-\theta')$ first order dominates $f(s-\theta)$.
   2. Having the MLRP is sufficient for first order dominance. By considering the location class of Cauchy distributions, show that the reverse is not true.

Definition 2. *A non-negative function $h(x,\theta)$ is **log supermodular** if $\log h(x,\theta)$ is supermodular, that is, if for all $x' > x$ and $\theta' > \theta$), $h(x',\theta')h(x,\theta) \geq h(x',\theta)h(x,\theta')$.*

K. At various points, you will be using log supermodularity of density functions and of marginal utilities.

1. Show that if the class $\{f(s;\theta) : \theta \in \Theta\}$ has the MLRP, then it is log supermodular in s and θ.
2. Show that if the support set for each density in the class $\{f(s;\theta) : \theta \in \Theta\}$ is the same interval, then being log supermodular in s and θ implies that the class has the MLRP.
3. Suppose that $u(\cdot)$ is a concave, increasing, twice continuously differentiable function. Show that $f(w, s) := u'(w + s)$ is log supermodular iff u has decreasing absolute risk aversion.

L. Suppose that $\{f(s;\theta) : \theta \in \Theta\}$ is a class of densities with the MLRP. Show that if $u(\cdot,\cdot)$ is supermodular, then the inequality in equation (6) holds.

M. Consider the following classes of portfolio choice problems,

$$\max_{x\in[0,w]} \int u(w - x + xs) f(s;\theta)\, ds \tag{15}$$

where $\{f(s;\theta) : \theta \in \Theta\}$, $\Theta \subset \mathbb{R}$, is a class of distributions on $[0,\infty)$ with the MLRP. Answer the following assuming uniqueness of solutions.

1. If $u(r) = \log(r)$, is $x^*(\theta)$ an increasing or decreasing function?
2. If $u(r) = r^\gamma$, $0 < \gamma < 1$, is $x^*(\theta)$ an increasing or decreasing function?
3. If $u(r) = r^\gamma$, $\gamma \geq 1$, is $x^*(\theta)$ an increasing or decreasing function?
4. In the previous three problems, characterize, if possible, the set of θ for which x^* increases with w. [This is where the last part of Theorem 1 may come in handy.]

N. [A function that is not absolutely continuous even though it has bounded variation.] Let p be the uniform distribution on the product space $\mathfrak{X} := \{0,2\}^{\mathbb{N}}$, define $\psi : \mathfrak{X} \to [0,1]$ by $\psi(x) = \sum_n x_n/3^n$, let $q = \psi(p)$ and for $t \in [0,1]$, define $F_q(t) = q([0,t]) = p(\{x \in \mathfrak{X} : \psi(x) \leq t\})$. Show that $F_q(\cdot)$ is continuous, that it is of bounded variation, but that it is not absolutely continuous. [The function $F_q(\cdot)$ sometimes goes by the name of the Cantor function.]

O. Let $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ be continuous and increasing, i.e. $[\mathbf{x} \gg \mathbf{y}] \Rightarrow [u(\mathbf{x}) > u(\mathbf{y})]$. Consider the problems

$$V(\mathbf{p}, w) = \max\ u(\mathbf{x}) \text{ subject to } \mathbf{x} \geq 0,\ \mathbf{p}\mathbf{x} \leq w. \tag{16}$$

1. Find a function g such that $V(\mathbf{p}, w) = \int_0^w g(s)\, ds$.
2. Find a function h such that $\partial V(\mathbf{p}, w)/\partial p_i = \int_0^{p_i} h(s)\, ds$.

P. Before choosing an action a in the compact metric space A, the decision maker observes a measurable random signal $S : \Omega \to \mathbb{R}$. The decision maker chooses the measurable function $a^*(\cdot)$ to solve

$$V_u(S) = \max_{a^*(\cdot)}\ E\, u(a^*(S), X) = \int_\Omega u(a^*(S(\omega), X(\omega))\, dP(\omega). \tag{17}$$

The signal $T : \Omega \to \mathbb{R}$ is a **Markov scramble** of S if $P(T \in A) = \int_A m(A; S(\omega))\, dP(\omega)$ where for all measurable A, $m(A,\cdot)$ is measurable, and for all s, $m(\cdot; s)$ is a probability distribution on $\mathbb{R}$.

The blanket regularity conditions: for all x, $u(\cdot, x) : A \to \mathbb{R}$ is continuous; for all a, $u(a,\cdot) : \mathbb{R} \to \mathbb{R}$ is measurable; and $\int_\Omega \max_{a\in A} |u(a, X(\omega)|\, dP(\omega) < \infty$.

The ultra-strong blanket regularity conditions: A is finite; S is a simple random variable; and X is a simple random variable. Under these conditions, the first four problems below are trivial, but the last one is still less than completely immediate.

1. Show that $u : A \times \mathbb{R} \to \mathbb{R}$ is jointly measurable.
2. Let S_n^L be the sequence of Lebesgue approximations to S. Give the solution to the problem $V_u(S_n^L)$.
3. Show that $V_u(S)$ has a solution. [This is a shockingly subtle problem, and if you don't see how to do it after some small amount of time, assume that it's true for the rest of the problem.]
4. Show that $V_u(S_n^L) \to V_u(S)$.
5. Show that for all u satisfying the blanket regularity conditions, $V_u(S) \geq V_u(T)$. Show that this is equivalent to S being more informative that T in the sense given in the Baker reading.