Assignment #1 for **Mathematics for Economists**
Fall 2017

Due date: Mon. Sept. 18.

Readings: CSZ, Ch. 2.8, Ch. 5.6, 5.8, 5.9, and 5.10.

Monotone Comparative Statics Problems

Definition 1. *For $X \subset \mathbb{R}$ and $T \subset \mathbb{R}$, a function $f : X \times T \to \mathbb{R}$ has* ***increasing differences*** *or is* ***supermodular*** *if for all $x' > x$ and all $t' > t$,*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t), \tag{1}$$

and it is ***strictly supermodular*** *if for all $x' > x$ and all $t' > t$,*

$$f(x', t') - f(x, t') > f(x', t) - f(x, t). \tag{2}$$

For ***submodularity/decreasing differences****, and* ***strictly submodular/decreasing differences*** *functions, reverse the inequalities.*

For $T \subset \mathbb{R}$, the correspondence $t \mapsto S(t) = [g(t), h(t)]$, $g(t) \leq h(t)$ for all $t \in T$, is **non-decreasing** if for all $t' > t \in T$, $g(t') \geq g(t)$ and $h(t') \geq h(t)$. Under study are problems of the form

$$V(t) := \max_{x \in S(t)} f(x, t), \;\; x^*(t) := \{x \in S(t) : f(x, t) = V(t)\}. \tag{3}$$

Many of the problems below will use the following result.

Theorem 1 (Topkis)**.** *If $f : X \times T \to \mathbb{R}$ is supermodular and $t \mapsto S(t)$ is non-decreasing, then for all $t' > t$,*

$$\max\{x \in x^*(t')\} \geq \max\{x \in x^*(t)\} \text{ and } \min\{x \in x^*(t')\} \geq \min\{x \in x^*(t)\}.$$

The first three problems give results that make the Topkis theorem easier to apply.

A. One can sometimes use the Topkis result is to compare the argmax for problems

$$\max_{x \in S(t)} g(x) \text{ and } \max_{x \in S(t)} h(x).$$

The device is to define $f(x, t) = g(x) + t[h(x) - g(x)]$ and set $T = \{0, 1\}$. Show that $f(x, t)$ is strictly supermodular if $[h(x) - g(x)]$ is a strictly increasing function, and supermodular if it is non-decreasing.

B. On cancellation: show that if $f(x, t) = g(x, t) + h(x) + m(t)$, then $f(\cdot, \cdot)$ is supermodular if and only if $g(\cdot, \cdot)$ is supermodular.

C. A biotech firm spends $x \geq 0$ looking for a cure for a rare disease (think of the Orphan Drug Act). Its expected benefits are $b_1(x)$, social benefits not captured by the firm are given by $b_2(x)$, an increasing function.
   1. Show that the socially optimal level of x is larger than the firm's profit maximizing level.
   2. Show that allowing the firm to capture some portion of $b_2(x)$ increases the firm's spending.

D. Firms that provide training to their employees often lose their employees, and all of their skills, to competitors. Show that this leads to suboptimal levels of training.

E. For $x, t \in [1200, 2000]$, let $f(x,t) = xt$. Show that $f(\cdot,\cdot)$ is strictly supermodular, that $g(x,t) := \log(f(x,t))$ is both supermodular and submodular, and that $h(x,t) := \log(g(x,t))$ is strictly submodular. Give $x^*(t)$ for the problems

$$\max_{x \in [1200,2000]} f(x,t), \quad \max_{x \in [1200,2000]} g(x,t), \text{ and } \max_{x \in [1200,2000]} h(x,t).$$

How are your results **not** a contradiction to the Topkis result given above?

F. This problem contains more information relevant to the previous problem. A function $f : X \times T \to \mathbb{R}$ is **quasi-supermodular** if for all $x' > x$ and all $t' > t$,

$$f(x',t) - f(x,t) > 0 \text{ implies } f(x',t') - f(x,t') > 0, \text{ and}$$
$$f(x',t) - f(x,t) \geq 0 \text{ implies } f(x',t') - f(x,t') \geq 0.$$

1. Show that a supermodular function is quasi-supermodular.
2. The previous problem showed that an increasing function of a supermodular function need not be supermodular. Show that an increasing function of a quasi-supermodular function is quasi-supermodular.
3. Give a function that is quasi-supermodular but not supermodular.
4. Prove the Topkis result given above replacing "supermodular" with "quasi-supermodular."

G. [Forestry/renewable resource problems] For this problem, we suppose that for $t > 0$ (interpreted as time since planting), $B(t) > 0$ (interpreted as a societal benefit to harvesting), that $S > 0$ is a "scrap value," that $A(t) > 0$ is a flow value (of other societal benefits from having the forest growing), and that $r > 0$ is the social interest rate.

1. Compare, if possible, the solutions to the problems

$$\max_{t \geq 0} e^{-rt} B(t) \text{ and}$$
$$\max_{t \geq 0} e^{-rt}(B(t) + S).$$

   If there is a difference, give the economic intuition for it.

2. Compare, if possible, the solutions to the problems

$$\max_{t \geq 0} e^{-rt} B(t) \text{ and}$$
$$\max_{t \geq 0} e^{-rt} B(t) + \int_0^t e^{-rx} A(x)\, dx.$$

   If there is a difference, give the economic intuition for it.

3. Compare, if possible, the solutions to the problems

$$\max_{t \geq 0} e^{-rt} B(t) \text{ and}$$
$$\max_{t \geq 0} e^{-rt} B(t) + e^{-r \cdot 2t} B(t) + e^{-r \cdot 3t} B(t) + \cdots.$$

   Here the idea is that after harvesting at t, one can replant the forest and start the cycle anew. If there is a difference, give the economic intuition for it.

H. For this problem, we again suppose that for $t > 0$, $B(t) > 0$, that $S > 0$ is a "scrap value," that $A(t) > 0$ is a flow value. Now we have two interest rates, $r' > r > 0$.

1. Compare, if possible, the solutions to the problems

$$\max_{t\geq 0}\ e^{-rt}B(t) \text{ and}$$
$$\max_{t\geq 0}\ e^{-r't}B(t).$$

If there is a difference, give the economic intuition for it.

2. Compare, if possible, the solutions to the problems

$$\max_{t\geq 0}\ e^{-rt}B(t) + \int_0^t e^{-rx}A(x)\,dx \text{ and}$$
$$\max_{t\geq 0}\ e^{-r't}B(t) + \int_0^t e^{-r'x}A(x)\,dx.$$

If there is a difference, give the economic intuition for it.

3. Compare, if possible, the solutions to the problems

$$\max_{t\geq 0}\ e^{-rt}B(t) + e^{-r\cdot 2t}B(t) + e^{-r\cdot 3t}B(t) + \cdots \text{ and}$$
$$\max_{t\geq 0}\ e^{-r't}B(t) + e^{-r'\cdot 2t}B(t) + e^{-r'\cdot 3t}B(t) + \cdots \text{ and}$$

If there is a difference, give the economic intuition for it.

I. Suppliers who have only one buyer, e.g. the workers in a company town or the makers of fighter jet engines, are at a disadvantage in the market because there is market power on the other side of the market. When there is unbalanced market power, there is almost always inefficiency. Suppose that the supply curve is the strictly positive, increasing function $p(q)$, that is, for a supply of q to be forthcoming, the buyer must offer $p(q)$ and for $q' > q$, $p(q') > p(q)$. The supplier surplus at q is $S(q) := \int_0^q [p(q) - p(x)]\,dx$. If the buyer orders q, their profits are $\Pi(q) = R(q) - q \cdot p(q)$ where $R(q)$ is their revenue as a function of q.

1. Give $S'(q)$ and show that it is positive.
2. Show that the single buyer buys less than the socially optimal quantity.

J. The value to starting next period, $t = 1$, with capital k_1 is $V(k_1)$ where $V(\cdot)$ is an increasing function. We discount this to the present, $t = 0$, as having value $\beta V(k_1)$. You have capital k_0 now, will receive w_1 at the beginning of next period, and can save at a rate of return r. Thus, saving s from present capital yields consumption $c_0 = (k_0 - s)$ now and $c_1 = (w_1 + (1 + r)s)$ in the next period. The two period problem is

$$\max_{s\in[0,k_0]}\ [u(k_0 - s) + \beta V(w_1 + (1 + r)s)].$$

At various points in the following, you may want to invoke the assumption that $u(\cdot)$ and $V(\cdot)$ are twice continuously differentiable with strictly positive first derivatives and strictly negative second derivatives. Be explicit about when you do *not* need to invoke these assumptions.

1. What happens to optimal savings if r increases to $r' > r$?
2. What happens to optimal savings if k_0 increases to $k_0' > k_0$?
3. What happens to optimal savings if w_1 increases to $w_1' > w_1$?
4. What happens to optimal savings if β increases to $\beta' > \beta$?

Constrained Optimization Problems

We begin with some calculus, geometry and linear algebra, and then turn to maximization problems that use Lagrangean functions and the Kuhn-Tucker theorem.

K. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and for $t \in \mathbb{R}$, define

$$h(t) = f(\mathbf{x}^\circ + t(\mathbf{x} - \mathbf{x}^\circ)).$$

1. In terms of the gradient $Df(\mathbf{x}^\circ)$, give $h'(0)$. Geometrically, what does $h'(0) > 0$ mean about $f(\cdot)$ in a neighborhood of $\mathbf{x}^\circ$? What about $h'(0) = 0$? What about $h'(0) < 0$?
2. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, show that $\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle > 0$ if and only if the angle between $\mathbf{x}$ and $\mathbf{y}$ is acute. What about $\langle \mathbf{x}, \mathbf{y} \rangle = 0$? What about $\langle \mathbf{x}, \mathbf{y} \rangle < 0$? Relate this to the previous.
3. What is the direction of fastest increase of $f(\cdot)$ at $\mathbf{x}^\circ$?

L. Read up on and absorb the geometry of convex sets.

M. For an open, convex $C \subset \mathbb{R}^n$, a continuously differentiable $f : C \to \mathbb{R}$ is **concave** if it lies everywhere below its tangent planes, that is, if for all $\mathbf{x}^\circ \in C$ and all $\mathbf{x} \in C$,

$$f(\mathbf{x}^\circ) + Df(\mathbf{x}^\circ) \cdot (\mathbf{x} - \mathbf{x}^\circ) \geq f(\mathbf{x}).$$

1. Show that $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}$ is concave.
2. When M is a symmetric $n \times n$ matrix, the function
$$f(\mathbf{x}) = c_0 + \mathbf{x} \cdot \mathbf{y} + \frac{1}{2}(\mathbf{x} - \mathbf{v})' M (\mathbf{x} - \mathbf{v})$$
is called a **quadratic**. Give the conditions under which the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ is concave.
3. Let $C = \mathbb{R}^2_{++}$. Show that the function $f(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$, $\alpha, \beta > 0$, is concave.
4. For Σ a positive definite, $n \times n$ matrix and $\mu \in \mathbb{R}^n$, show that the function
$$f(\mathbf{x}; \mu, \Sigma) = \kappa e^{-\frac{1}{2}(\mathbf{x} - \mu)' \Sigma (\mathbf{x} - \mu)}$$
is quasi-concave in $\mathbf{x}$, $\kappa > 0$. What is the shape of its upper contour sets?

N. Suppose that $C \subset \mathbb{R}^n$ is a closed, convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable concave function.

1. Show that if $x^* \in C$ solves $\max_{x \in C} f(x)$, then for all $x \in C$, $Df(x^*)(x - x^*) \leq 0$.
2. Show that if for all $x \in C$, $Df(x^*)(x - x^*) \leq 0$, then x^* solves $\max_{x \in C} f(x)$.

O. CSZ (Corbae et al.) problems 5.8.1, 2, 3, and 4.

P. CSZ problem 5.8.8.

Q. CSZ problem 5.8.9.

R. CSZ problem 5.8.13.

S. CSZ problem 5.8.16.