$\emptyset$ Marks the Spot

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Describing the population characteristics in a large game with a non-atomic, purely finitely additive probability p means that ϵ-equilibria may not exist. This happens because a mass of agents and their characteristics seem to belong to $\emptyset$. This paper uses p to characterize the mislaid agents and their characteristics. Restoring them to the model yields equilibrium existence and a well-behaved equilibrium correspondence.

Fighting the Tide

From de Finetti's *Theory of Probability* (p. 555), wrote of Kingman's "Additive set functions and the theory of probability," *Proc. Cambridge Philosophical Society* (1967),

> *The basic idea is the possibility of stretching the interpretation in such a way as to be able to attribute the 'missing' probability in the partition to new fictitious entities in order that everything adds up properly. In some cases, in order to salvage countable additivity, it is even claimed that the new entities are not fictitious, but real.*

A Game

From Khan, Qiao, Rao, and Sun (KQRS).

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- Population/types: $(T = [10, \infty), \mathcal{B}, \mu)$, μ a non-atomic probability.
- Utility for t depends action $a \in \{0, 1\}$ and $\nu := \mu(\{t \in T : a(t) = 1\})$, given by $\mathcal{G}(t)(a, t) = a \cdot u(t, \nu)$ where

$$u(t,\nu) = \begin{cases} 1 & \text{if } \nu \in [0, \frac{1}{2}], \\ 1 - t(\nu - \frac{1}{2}) & \text{if } \nu \in (\frac{1}{2}, \frac{1}{2} + \frac{2}{t}], \text{ and} \\ -1 & \text{if } \nu \in (\frac{1}{2} + \frac{2}{t}, 1]. \end{cases} \tag{1}$$

Equilibria

A measurable $t \mapsto a(t)$ is an ϵ-**equilibrium** if

$$\mu(\{t : \mathcal{G}(t)(a(t), \nu_a) \geq \max_{b \in A} \mathcal{G}(t)(b, \nu_a) - \epsilon\}) \geq (1 - \epsilon), \tag{2}$$

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Do the obvious if $t \mapsto a(t) \in \Delta(\{0, 1\})$ is a mixed strategy.

Equilibrium Strategies: Countably Additive Case

Utility is $a \cdot u(t, \nu)$ where

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Equilibrium strategies: $a^*(t) = 1_{\{t \leq t^\circ\}}$ where cutoff t° satisfies $F_\mu(t^\circ) = \frac{1}{2} + \frac{1}{t^\circ}$ for the cdf $F_\mu(t) = \mu((-\infty, t])$.

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Equilibrium distribution of utility: $u(t) \equiv 0$ for $t > t^\circ$, for $t \leq t^\circ$, utility is $1 - t/t^\circ$. Letting $P(Y \in A) = \mu(A|(-\infty, t^\circ])$ and $X = 1 - Y/t^\circ$, $\mathcal{L}(X)$ is the conditional distribution of utilities for the approximately half of the population receiving positive utility.

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Details of μ/p matter.

NO Approximate Equilibria: PFA

$([10,\infty),\mathcal{B},\mu)$, assume $\mu([n,\infty))\equiv 1$ even though $[n,\infty)\downarrow\emptyset$.
Utility functions

$$u(t,\nu)=\begin{cases}1 & \text{if } \nu\in[0,\frac{1}{2}],\\ 1-t(\nu-\frac{1}{2}) & \text{if } \nu\in(\frac{1}{2},\frac{1}{2}+\frac{2}{t}], \text{ and}\\ -1 & \text{if } \nu\in(\frac{1}{2}+\frac{2}{t},1].\end{cases} \tag{5}$$

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- $[\nu_a \le \frac{1}{2}] \Rightarrow (\forall t)[Br(t) = \{1\}]$, utility loss of $a = 0$ is 1, hence ϵ-best responses must put mass at least $1 - \epsilon$ on $a = 1$.

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μ and the measurable $t \mapsto \mathcal{G}(t) \in C(A \times [0,1])$ induce a probability, p, on the unit ball, $U \subset C(A \times [0,1]))$.

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Implications: c-tight $\Rightarrow$ tight $\Rightarrow$ n-tight, none reverse.

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If not n-tight: approximate equilibria may not exist (KQRS example above); though they may; restoring mislaid agents and their characteristics to the game restores countable additivity. Equilibria depend on details of p.

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Suppes and Zanotti (1989), "Conditions on upper and lower probabilities to imply probabilities," *Erkenntnis* fixed de Finetti's treatment of upper/lower probabilities, Stinchcombe (2016), "Objective and Subjective Foundations for Multiple Priors," *Journal of Economic Theory* showed that $\Pi(p)$, as a set of priors, models a small but sometimes interesting class of problems.

Failing N-Tightness

N-tightness (mass $p(K^\delta) > (1-\epsilon)$ for all $\delta > 0$) fails,

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Arzelà-Ascoli: compact subsets of U are equi-continuous. For $t \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{t})$, |slope| is t, and $\mu(\{t : t \geq n\}) \equiv 1$, the pfa p fails n-tightness (does not put any mass on the ball with radius $1/4$ around any compact set of continuous functions).

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Splits difference between Kingman and de Finetti. Pragmatically, it's useful.

Reminder 1

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Lemma. For a probability p, the following are equivalent.

(a) p is pfa.

(b) There is a countable partition $\{F_n : n \in \mathbb{N}\} \subset \mathcal{X}$ with $p(F_n) \equiv 0$.

(c) There exists a strictly positive $g \in M_b(X)$ with $\int g\, dp = 0$.

Reminder $1'$

From Yosida and Hewitt (1952), "Finitely additive measures," *Transaction of the A.M.S.*, a pfa probability p on $(X, \mathcal{X})$ has a unique extension $\widehat{p}$ to the compact Stone space $(\widehat{X}, \widehat{\mathcal{X}})$, and $p \leftrightarrow \widehat{p}$ is one-to-one and onto.

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From Anderson (1982), "Star-finite representations of measure spaces,' *Transactions of the A.M.S.* and Anderson and Rashid (1978), "A nonstandard characterization of weak convergence," *Proceedings of the A.M.S.*, it's easier to use ${}^*p$ on $({}^*X, {}^*\mathcal{X})$, and ${}^*p$ will put mass 1 on points in ${}^*X$ that are not nearstandard.

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$\mathrm{proj} : ({}^*X \setminus X) \to (\widehat{X} \setminus X)$ is onto and many-to-one.

Reminder 2

For a set X, the definition of ${}^*X$ is a two-step process.

- Define $x \sim_\mu y$ in $X^{\mathbb{N}}$ if $\mu(\{n \in \mathbb{N} : x_n = y_n\}) = 1$ where μ is a purely finitely additive probability on the integers with $\mu(E) = 0$ or $\mu(E) = 1$ for all $E \subset \mathbb{N}$.

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- Define ${}^*X = X^{\mathbb{N}} / \sim_\mu$.

Reminder 2′

$X = \mathbb{R}_+$, $\langle x_n \rangle$ denote the $\sim_\mu$ equivalence class of the sequence $n \mapsto x_n$, $r > 0$,

$$r = \langle r, r, r, r, \ldots \rangle, \tag{8}$$

$$dt = \langle 1, 1/2, 1/3, 1/4, \ldots \rangle, \tag{9}$$

$$(dt)^2 = \langle 1, 1/4, 1/9, 1/16, \ldots \rangle, \tag{10}$$

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For $x \in {}^*\mathbb{R}$: $x \simeq 0$ if $|x| < r$ for all $r \in \mathbb{R}_{++}$, and $y = \mathrm{st}(x)$ if $|y - x| = \langle |y - x_1|, |y - x_2|, |y - x_3|, |y - x_4|, \ldots \rangle \simeq 0$.

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The mislaid agent characteristics are in ${}^*U \setminus U$; analyze the game using p_L (or μ_L); interpret the elements of ${}^*U \setminus U$. Can also compactify U and work with $\widehat{U} \setminus U$.

Implementing the Plan

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- for $t \in {}^*[10, \infty)$, $F_{\mu_L}(t) = \mu_L((-\infty, t])$.

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Different choice of pfa μ/p leads to conditional distribution of utility being any element of $\Delta^{ca}([0,1])$.

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- This seems not to be the game we started with.
- All of the pieces came from the game we started with.

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Reality and represenation,

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Pragmatism, to optimally attain "clearness of apprehension,"

> *Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object. (Charles Sanders Pierce)*

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3. Best use is to capture "limit" phenomena in a pretty wide sense. Roughly, ${}^*X$ contains the compactification of X with respect to all the properties we can think of.

I'm Done

Questions?

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Thank you for listening.