U.T. Economics Summer 2011 Math Camp

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Date: First week-end review problems

Below is a collection of what seem to us to be the most important of the problems assigned this week, at least, the most important if you have absorbed and are beginning to be comfortable with the notation and definitions we have used. Please try to write out complete solutions to whichever of these problems you have not already handed in — the process of writing out solutions gives you direct feedback on whether or not you understand the material. If you choose not to hand in solutions, then be very sure that you *could* write them out.

A. A **sequence in** $\mathbb{R}$ is a list $x = (x_1, x_2, x_3, \ldots)$ such that each $x_n \in \mathbb{R}$. The set of all sequences is denoted $\mathbb{R}^{\mathbb{N}}$. Within the set of all sequences we have the following distinguished sets:

- $\ell_\infty = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists B \in \mathbb{R})(\forall n \in \mathbb{N})[\,|x_n| \leq B\,]\}$.
- $c = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists r \in \mathbb{R})(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[\,|x_n - r| \leq \epsilon\,]\}$.
- $c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : (\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[\,|x_n| \leq \epsilon\,]\}$.
- $\ell_2 = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists B \in \mathbb{R})(\forall N \in \mathbb{N})[\,\sum_{n=1}^{N} |x_n|^2 \leq B\,]\}$.
- $\ell_1 = \{x \in \mathbb{R}^{\mathbb{N}} : (\exists B \in \mathbb{R})(\forall N \in \mathbb{N})[\,\sum_{n=1}^{N} |x_n| \leq B\,]\}$.

You are now going to show that $\ell_1 \subsetneq \ell_2 \subsetneq c_0 \subsetneq c \subsetneq \ell_\infty$.

1. Show that $(\forall x \in \ell_1)[x \in \ell_2]$ and that $(\exists x \in \ell_2)[x \notin \ell_1]$.
2. Show that $(\forall x \in \ell_2)[x \in c_0]$ and that $(\exists x \in c_0)[x \notin \ell_2]$.
3. Show that $(\forall x \in c_0)[x \in c]$ and that $(\exists x \in c)[x \notin c_0]$.
4. Show that $(\forall x \in \ell_c)[x \in \ell_\infty]$ and that $(\exists x \in \ell_\infty)[x \notin c]$.

B. [Exchange efficiency via prices] Find the Pareto efficient allocations and the equilibrium for the following exchange economies.

1. Suppose that $I = \{1, 2\}$, $\ell = 2$, $\mathbf{y}_1 = (5, 2)'$, $\mathbf{y}_2 = (2, 5)'$, $\succ_1$ is given by the utility function $u_1(x_{1,1}, x_{1,2}) = x_{1,1} + x_{1,2}$, and $\succ_2$ is given by the utility function $u_2(x_{2,1}, x_{2,2}) = x_{2,1} \cdot x_{2,2}$.
2. Suppose that $I = \{1, 2\}$, $\ell = 2$, $\mathbf{y}_1 = (9, 1)'$, $\mathbf{y}_2 = (3, 8)'$, $\succ_1$ is given by the utility function $u_1(x_{1,1}, x_{1,2}) = \min\{x_{1,1}, x_{1,2}\}$, and $\succ_2$ is given by the utility function $u_2(x_{2,1}, x_{2,2}) = x_{2,1} + x_{2,2}$.
3. Suppose that $I = \{1, 2\}$, $\ell = 2$, $\mathbf{y}_1 = (2, 1)'$, $\mathbf{y}_2 = (6, 8)'$, $\succ_1$ is given by the utility function $u_1(x_{1,1}, x_{1,2}) = 2\log(x_{1,1}) + \log(x_{1,2})$, and $\succ_2$ is given by the utility function $u_2(x_{2,1}, x_{2,2}) = \log(x_{2,1}) + 3\log(x_{2,2})$.

C. Let X be the set containing the cities Austin, Tokyo, and Mumbai. Represent each of the following relations by filling out the following boxes.

1. xRy if city x strictly precedes city y alphabetically.

Mumbai			
Tokyo			
Austin			
	Austin	Tokyo	Mumbai

2. xSy if the second letter of city x's name is strictly earlier than the second letter of city y's name.

Mumbai			
Tokyo			
Austin			
	Austin	Tokyo	Mumbai

3. xEy if city x and city y are spelled with the same number of letters.

Mumbai			
Tokyo			
Austin			
	Austin	Tokyo	Mumbai

D. The observation about majority voting that appears in this problem is originally due to Nicolas de Caritat, the marquis de Condorcet (1743-1794). Suppose that X is a three point set, specifically $X = \{a, b, c\}$, and that persons $i = 1, 2, 3$ have complete transitive preferences orderings $\succeq_i$ over X that satisfy $a \succ_1 b \succ_1 c$ for person 1, $c \succ_2 a \succ_2 b$ for person 2, and $b \succ_3 c \succ_3 a$ for person 3.

1. Fill in the tables below that describe $\succeq_i$, $i = 1, 2, 3$.

c			
b			
a			
	a	b	c

$\succeq_1$

c			
b			
a			
	a	b	c

$\succeq_2$

c			
b			
a			
	a	b	c

$\succeq_3$

2. For non-empty $B \in \mathcal{P}(X)$, define

$$C_V(B) = \{x \in B : (\forall y \neq x \in B)[\#\{i : x \succ_i y\} \geq 2]\},$$

i.e. $C_V(B)$ is the set of options in the set B that win a head-to-head vote against everything else in B. [Mnemonically, $C_V(B)$ is the "chosen by pairwise voting on options in B" set.] Give $C_V(B)$ for each subset of X containing either 2 or 3 points.

3. Define $\succeq_V^* \subset X \times X$ by $x \succeq_V^* y$ if $(\exists B \in \mathcal{P}(X))[[x, y \in B] \wedge [x \in C_V(B)]$. Fill in the following table for $\succeq_V^*$.

c			
b			
a			
	a	b	c

4. Prove that $\succeq_V^*$ cannot be represented by any utility function.

E. Represent each of the following relations on the given sets X as subsets of $X \times X$, and answer the questions: is the relation complete? transitive? reflexive? a linear ordering?
   1. X is a set of people having the last names of Adams, Avery, Banach, Briarsmith, Bucemi, Zame and Zappa. In centimeters, the heights of these 7 people are 181, 202, 194, 156, 179.3, 185, and 162. For $x, y \in X$, the relation $(\alpha\beta)$ is defined by $x(\alpha\beta)y$ if x's last name is alphabetically before y's last name.
   2. With the same 7 people as given in the previous problem, define the relation E by $(x, y) \in E$ if the first letter of x's last name and the first letter of y's last name are the same.
   3. With the same 7 people as given in the previous problem, define the relation T by xTy if x is taller than y.
   4. With the same 7 people as given in the previous problem, define the relation (ET) by $(x, y) \in (ET)$ if the first letter of x and y's last name are the same and x is taller than y.

F. [Partially non-excludable goods] One class of goods that tend to be underprovided are the non-excludable ones, that is, the goods for which it is not possible to prevent people who have not paid for it from having access to it. Examples include clean air, clean water, the sight of a beautiful building, the use of roads that do not have toll booths, fish in the public lake, lighthouses, national defense, widespread vaccination. An excludable good is one for which it is possible to prevent people who have not paid for it from having access to it. Examples include food, private beach front property (with effective gates/fences). There are degrees of non-excludability between 0 and 1: only some of the hikers in the national parks have permits; only some of the fishers have licenses; private beach front property may be legally accesible from the water if not from land.

   Suppose that producing $x \geq 0$ of a good costs $C(x)$. Let $W(x)$ denote the total societal willingness to pay for x, and suppose that $W(\cdot)$ is a non-decreasing function. Let $t \in [0, 1]$ denote the proportion of the total willingness to pay that can be collected. The collectible surplus maximization problem is

$$P(t): \quad \max_{x \geq 0}[t \cdot W(x) - C(x)].$$

   1. Let $x^*(t)$ denote the solution to $P(t)$. How does $x^*(\cdot)$ depend on t? Explain your answer both with mathematics and intuitively.

2. Show that $x^*(t)$ is always at least weakly lower than the level that would maximize social surplus.

G. [Private amenities from a 'forest'] A private woodlot owner receives an amenity flow of $A(t)$ while the trees are growing on her lot and the lot was replanted at $t = 0$. We assume that $A(t) > 0$ for $t > 0$, and that she is interested in both the amenity flow and the present value of net revenue from a single rotation. That is, we assume that she cuts down her lot of trees at T_A defined by

$$T_A = \text{argmax}_{t \geq 0} Q(t)e^{-rt} + \int_0^t A(s)e^{-rs}\, ds.$$

1. Show that the function $g(t) = \int_0^t A(s)e^{-rs}\, ds$ is increasing by giving its derivative for $t > 0$.
2. Show that T_A is larger than, or at least as large as, T_S, the solution to the problem $\max_{t \geq 0} Q(t)e^{-rt}$. Explain this both mathematically and intuitively.
3. Formulate and solve the same problem for the case of the optimal rotational forest in perpetuity (the Faustmann rotation) and show that adding the amenity flows into the optimization problem makes the optimal rotation longer.