# THE FINITISTIC THEORY OF INFINITE GAMES

C. J. HARRIS, M. B. STINCHCOMBE, AND W. ZAME

Abstract. The additions to, or fixes for, game structures necessitated by use of the usual model of infinite sets can be unified in the concept of a game expansion. This paper identifies a class of expansions, the finitistic ones, that sharpens the previous fixes, delivers a well-behaved theory for infinite games, and clarifies the relation between classes of games and the requisite expansions. Finitistic equilibria are the minimal closed set of expansion equilibria consistent with the idea that continuous sets are the limits of finite approximations.

## 1. Introduction

The usual models of infinite sets are perfectly adequate for many mathematically intensive inquiries, but not for the study of strategic interactions. A series of examples show that a theory of games based on the usual models fails even minimal standards. The extant responses to, or fixes for, these failures varies with the class of games. For some classes, the addition of cheap talk suffices. For others, public signals or more general forms of correlation are required. For yet other classes, endogenous sharing rule equilibria have been invented. These fixes, as well as the information leakages and the as-yet-unnamed additions to game structures described below, are unified in the definition of a game expansion.

Game expansions organize the previous literature as a series of matchings between classes of games and corresponding small expansions that deliver a class-specific well-behaved theory. The relation between classes of games and the necessary fixes can seem rather complicated. This paper identifies one class of expansions, the finitistic ones. Finitistic expansions sharpen the previous fixes, make

*Date*: March 17, 2000. Many thanks to seminar participants in the U. T. Microeconomics Seminar Series for helpful advice.

transparent the relation between games and the necessary expansions, and deliver a well-behaved theory of infinite games, both extensive and normal form.

Finitistic sets are a class of limiting cases of large finite sets. While behaving logically as if they are finite, they contain each element of the infinite set they replace. The next section discusses this seeming paradox as an 'intersection of limit sets,' a more roundabout but perhaps more intuitive, equivalent formulation of finitistic sets. Finitistic expansions of games replace the usual model of infinite sets by finitistic models of those sets.

Examples of the failure to meet minimal standards are in Section 4, as are sketches of the finitistic expansions and the previous fixes. Section 5 contains three equivalent formulations of finitistic sets, proofs that the finitistic approach meets the minimal standards for a theory of games — non-emptiness, closure, and upper hemi-continuity, and a discussion of the minimality of the finitistic class of expansions. Appendix A is a primer on the nonstandard analysis used for the most convenient treatment of finitistic sets.

Section 6 defines general extensive form games and their expansions. Finitistic expansion equilibria are equal to the limit sets identified in Section 5. Section 7 shows that finitistic expansions give equilibrium sets that are subsets of the previous expansion equilibria. Models of infinite sets are fundamental to many mathematically intensive fields. Section 8 is a discussion of some of the broader issues involved in the change suggested here.

## 2. Finitistic Approximations

In order to take finitistic approximations, we proceed as follows.

## 3. An Intersection of Limits Intuition for Finitistic Sets

The following two person, normal form game with compact metric actions spaces and continuous payoffs explains how sequences of finite sets can behave as if they were exhaustive, that is, as if they contain every element in an infinite set.

**Example 3.1** (Simon and Stinchcombe)**.** *The action spaces of the two players are* $A_1 = A_2 = \{-1\} \cup [0,1]$*, and the symmetric utility functions are*

$$u_i(a_i, a_j) = \begin{cases} 0 & \textit{if } a_i = -1 \\ 2 & \textit{if } a_i, a_j \in [0,1] \\ -a_i & \textit{if } a_j = -1 \textit{ and } a_i \in [0,1] \end{cases}$$

The strategy $a_i = 0$ weakly dominates every other strategy, but $(a_1, a_2) = (0,0)$ is only a limit undominated equilibrium along sequences of finite approximations that contain the point $(0,0)$. In more detail, consider playing the game on a sequence $F_i^n \to_H A_i$ ("$\to_H$" is convergence in the usual *H*ausdorff metric on closed sets) of finite subsets of $A_i$ that do not contain $a_i = 0$. Let $\mathbb{E}(F^n)$ be the undominated equilibria when the strategy sets $A_i$ are replaced by $F^n$. The limit set (*limes superior*), $\lim_n \mathbb{E}(F^n)$, contains the point $(a_1, a_2) = (-1,-1)$, a weakly dominated strategy pair.

Omissions of single points can change game theoretic analyses. By taking the appropriate intersections, it is possible for sequence analyses to take account of every point. For any $F = F_1 \times F_2 \subset A_1 \times A_2$, $F$ finite, define

(1) $$\mathbb{L}_F(\mathbb{E}) = \{\lim_n \mathbb{E}(F^n) : F \subset F^n \text{ a.a.}\}, \text{ and}$$

(2) $$\mathbb{L}(\mathbb{E}) = \bigcap_F \mathbb{L}_F(\mathbb{E}),$$

where "a.a." stands for "almost always," i.e. for all $n$ sufficiently large. $\mathbb{L}(\mathbb{E})$ is called the set of **anchored $\mathbb{E}$-limits**, and, with $\mathbb{E}(\cdot)$ defined as above, contains only $(0,0)$. In this way, $\mathbb{L}(\mathbb{E})$ is the set of (limits of) undominated equilibria for large finite versions of the game that contains "all" of the points in $A_1 \times A_2$, an uncountably infinite set of strategy vectors.

A finitistic set, "$F$," is an idealization of the intersection of limits of sequences, an idealization taken from Robinson's (1970) nonstandard analysis. For a finitistic $F = \times_{i \in I} F_i$, $\mathbb{E}(F)$ is the set of equilibria when the game is played with the finitistic strategy sets $F_i$. The set $\mathbb{L}(\mathbb{E})$ will be shown to be equal to the union of the points in $\Delta(A)$ (the probabilities on $A$) closest to $\mathbb{E}(F)$, the union being taken over finitistic $F$. The above analysis becomes, "$a_i = 0$ belongs to any finitistic $F_i$, it weakly dominates all other strategies, therefore weakly dominates all strategies in $F_i$, so that $\mathbb{E}(F)$ contains only $(0,0)$."

Finitistic sets, idealizations of limits of sequences of finite sets, are closely tied to infinitesimals, which are idealizations of sequences of points in $[0,1]$ converging to 0. For example, the Hausdorff distance between a finitistic $F$ and $[0,1]$ is infinitesimal. Appendix A is a primer on the nonstandard analysis used for this, the most convenient treatment of these limit sets and limit quantities.

## 4. The Failure to Meet Minimal Standards

Games formulated with the usual models of the continuum often require expansion to be even moderately well-behaved. There is a wide and unpredictable variety of requisite expansions. Finitistic expansions can make transparent what these must expansions entail to deliver a well-behaved theory.

### 4.1. **Overview of the Examples.**

There are two types of examples, five extensive form games with continuous payoffs, and three normal form games with discontinuous payoffs. Extensive form games with discontinuous payoffs are contained in the finitistic theory of games, but such games are not needed to see how badly behaved the theory of games is with the usual model of the continuum.

#### 4.1.1. *Continuous Extensive Form Examples.*

The first extensive form game has compact action sets, continuous payoffs, one player, and no equilibrium. Any finitistic version of this game has a sensible equilibrium. To achieve this same outcome with the usual model of the continuum requires an expansion of the information available to the player, specifically, a leakage of information.

The second example is a continuous signaling game. If the receiver plays in her strict best response set, then, with the usual models of infinite set, it has no Bayesian Nash equilibrium. Manelli [28] shows that adding cheap talk to continuous signaling games delivers a well-behaved theory. This expansion is larger than necessary for a well-behaved theory — a non-empty, closed subset, sometimes a strict subset, of the cheap talk equilibria outcomes arise as the equilibrium outcomes of finitistic expansions of the game.

The third example is a continuous game of almost perfect information with no equilibrium. [16] and [17] show that the addition of public signals to this class of games gives a well-behaved theory. Again, this expansion, the addition of a public correlating device, is larger than necessary for a well-behaved theory — a non-empty closed subset, sometimes strict, of the public signal equilibrium outcomes arise as the equilibrium outcomes of finitistic expansions of the game.

Like the signaling example and unlike the previous example, the fourth example has differential information. The players observe different continuous signals then simultaneously pick their actions in compact sets. It is not known whether such games have equilibria. Known sufficient conditions include restrictions on the joint distribution of the signals [30], or that the game be 0-sum [27]. A complicated argument for the existence of correlated equilibria can be found in [5]. Again, this expansion is larger than necessary for a well-behaved theory — the finitistic equilibrium outcomes form a non-empty closed subset, sometimes strict, of the correlated equilibrium outcomes [42].

With the exception of information leakage, the expansions mentioned so far have been familiar. It might be hoped that a well-behaved theory of games can be constructed from pasting together information leakage and these familiar expansions. The fifth example is as continuous and regular an extensive form game as can be imagined, but it requires complex expansions never seen before. Different possible overlap patterns of the finitistic versions of the players' action sets explain the qualitative aspects of the requisite game expansions.

4.1.2. *Dis-Continuous Normal Form Examples.* A separate class of expansion issues arise when utilities are discontinuous. In the continuous extensive form games, the requisite expansions of the signal sets or the action sets had no direct effect on utilities. With discontinuous payoffs, this will not be true in general.

The first normal form game satisfies known sufficient conditions for equilibrium existence ([6], hence [38]). However, the equilibria that exist are mixed, while any finitistic version of the game has an obvious and more intuitive pure strategy equilibrium. To represent this equilibrium as an expansion requires that utilities depend directly on the added points. The pure strategy equilibrium of this game can also be formalized as a pure strategy endogenous sharing rule equilibrium (ESR). ESR's are known to exist [40], but as expansions, they are too large for two separate kinds of reasons.

The second normal form game has a unique Nash equilibrium which is also the unique finitistic equilibrium outcome. However, because ESR's can ignore single points, it has (too) many ESR's. The third normal form game does not even have approximate equilibria when the usual models of infinite sets are used. The finitistic approach to this game shows that ESR's take too liberal a view of convexification, allowing randomization that cannot arise as a limit of indepedent randomization.

4.1.3. *Taken Together.* Taken together, the examples and the results make several points. A well-behaved theory of games requires expansions if we use the usual mathematical models of infinite sets. The requisite expansions can vary widely and unpredictably across different classes of games. Finitistic expansions are always constructed the same way, and they deliver a well-behaved theory of games. As part of making transparent the properties of the requisite class of expansions, they show that the expansions in the literature are larger than necessary. Finally, because finitistic sets behave logically as if they were finite the entire theory of finite games can be brought to bear on the analyses.[1]

[1]Formally, this is by transfer of statements that are true about finite games.

4.2. **Extensive Form Examples.** The presentation of these five compact and continuous extensive form games may seem to have extraneous pieces. These pieces are included to clarify the general structure into which infinite games fit.

An **expansion of a set** $X$ is a pair $(\widehat{X}, \varphi_{\widehat{X}})$ where $\varphi_{\widehat{X}}$ is an onto mapping from $\widehat{X}$ to $X$. To construct a game expansion, $\widehat{\Gamma}$, from a game, $\Gamma$, one expands some or all of the spaces in the definition of $\Gamma$. Some care must be taken to extend the signals and utilities conformably with the expansion from $\Gamma$ to $\widehat{\Gamma}$, and the details will be laid out below. If $X$ is (say) a set of actions available in a game $\Gamma$, and $x$ is a point in $X$, then $\varphi_{\widehat{X}}^{-1}(x)$ is the larger set of (say) actions that replace $x$ in the expanded game $\widehat{\Gamma}$.

4.2.1. *An Informational Discontinuity.* In this single agent game, $\Gamma$, with continuous signals, continuous payoffs, and compact action sets, there is no equilibrium, at least with the usual model of infinite sets. The finitistic equilibria are intuitive, and can be represented by a game expansion with information leakage.

**Example 4.1.** *Nature picks $\omega \in \Omega = \{-1, +1\}$ with probability $\frac{1}{2}$ each at $t = 0$. The single agent does not observe Nature's pick. Equivalently, the single agent observes the uninformative signal $s_0(\omega) \equiv 0$. If Nature picked $\omega$ at $t = 0$ and the agent picks $a_1 \in A_1 = [-1, +1]$ at $t = 1$, then she sees the signal $s_1 = -(a_1 - \omega)^2$. After seeing $s_1$ (and remembering $s_0$), the situation is repeated at time $t = 2$: the agent picks $a_2 \in A_2 = [-1, +1]$, and the resultant signal is $s_2 = -(a_2 - \omega)^2$. The agent's utility at the end of the two periods is the sum of the last two signals she receives, $u = s_1 + s_2$.*

Knowing $\omega$ would make everything easy for the agent, $a_t^* = \omega$ at $t = 1, 2$ is optimal. Not knowing Nature's pick at time $t = 2$ would make the optimal choice $a_2^* = 0$. However, playing $a_1 = 0$ makes the first signal completely uninformative and gives a maximal utility of $-2$. By contrast, for every $\epsilon \neq 0$, picking $a_1 = \epsilon$ gives the completely informative signal $s_1 = -(\epsilon - \omega)^2$. With $\omega$ known at $t = 2$, $a_2^* = \omega$, giving a maximal utility of $-1 - \epsilon^2 + 0$. This is greater than $-2$, but the supremum is not achievable, meaning that there is no equilibrium.

With $S_t$ being the range of the signals, $t = 0, 1, 2$, the history space for this game is

$$H = (\Omega \times S_0) \times (A_1 \times S_1) \times (A_2 \times S_2). \tag{3}$$

Let $\Delta(H)$ denote the set of probabilities on $H$. Let $\mu^\epsilon \in \Delta(H)$ denote the $\epsilon$-equilibrium distribution given above, i.e.

$$\mu^\epsilon = \tfrac{1}{2}\delta_{h^\epsilon_-} + \tfrac{1}{2}\delta_{h^\epsilon_+}, \tag{4}$$

$h^\epsilon_- = ((-1,0),(\epsilon,-(\epsilon+1)^2),(-1,0))$, $h^\epsilon_+ = ((+1,0),(\epsilon,-(\epsilon-1)^2),(+1,0))$, and $\delta_x$ is point mass on $x$. The associated utility is $-1-\epsilon^2$. The limit outcome as $\epsilon \to 0$ is

$$\mu^0 = \tfrac{1}{2}\delta_{h^0_-} + \tfrac{1}{2}\delta_{h^0_+} \tag{5}$$

where $h^0_- = ((-1,0),(0,-1),(-1,0))$ and $h^0_+ = ((+1,0),(0,-1),(+1,0))$. The associated utility is $-1$. Neither $\mu^0$ nor the utility $-1$ is achievable when the game is modeled with the usual model of infinite sets unless we expand the game. The set of possible outcomes is not closed, and because this failure of closure interacts with equilibrium conditions, the equilibrium set is empty.

Expand $S_0$ to $(\widehat{S}_0, \varphi_{\widehat{S}_0})$ where $\widehat{S}_0 = S_0 \times S'_0$, $S'_0$ contains $\Omega$, $\varphi_{\widehat{S}_0}$ is projection onto $S_0$, make utility independent of $s'_0$, and set $\widehat{s}_0(\omega) = (s_0, s'_0(\omega)) = (0, \omega)$. This models information leakage, and $\mu^0$ is the unique equilibrium in this expanded game.

If each set in $H$ is replaced by a finite set and the signals are to make sense, it must be the case that each finite replacement for $S_t$ contains the range of the signal, $s_t$, restricted to the finite replacement of its domain. Such a finite replacement scheme is called **conformable**. For example, if $A_1$ is replaced by $F_{A_1}$, then a conformable $S_1$ must contain all points of the form $-(f_1 - \omega)^2$, $f_1 \in F_{A_1}$, $\omega \in \Omega$ (being finite, $\Omega$ is its own finitistic replacement).

If $\mathbb{E}(F)$ is the set of equilibrium histories when each set in $\Gamma$ is *conformably* replaced by a finitistic version of itself, then $\mathbb{E}(F)$ contains only the point $\mu^\epsilon$ where $\epsilon$ is a smallest non-zero element of $F_{A_1}$, the finitistic replacement of $A_1$. Either smallest non-zero element of $F_{A_1}$ is infinitesimal, and the unique closest point in $[-1, +1]$ is 0 itself. The finitistic definition of $\mathbb{L}(\mathbb{E})$ is the union of closest points in $\Delta(H)$ to $\mathbb{E}(F)$, the union being taken over finitistic $F$. Thus, $\mathbb{L}(\mathbb{E})$ contains only $\mu^0$. The statement, "When any non-zero $a_1$ is played, the signal $s_1$ is perfectly informative," is still true in the finitistic game. The means that associated with $\mu^0$ there must be a perfectly informative signal, in other words, information leakage happens in all finitistic equilibria.

This is an example of the general pattern, game models built with the usual models of infinite sets have outcome sets that are not closed. Containing the limits requires an expansion of the game. If the failure of closure interacts with the equilibrium conditions, the equilibrium set can be empty. In this case, non-empty equilibrium set also requires an expansion of the game. Finitistic expansions work, giving a closed set of possible outcomes and non-empty set of equilibria. They also identify the qualititative characteristics of the requisite expansion, in this case, that information leakage is required.

4.2.2. *Signaling Games.* This signaling game, $\Gamma$, has compact action and signal spaces, continuous payoffs, but has no Bayesian Nash equilibrium.

**Example 4.2** (van Damme, Manelli). *Nature picks $\omega \in \Omega = \{\text{Heads}, \text{Tails}\}$ with probability $\frac{1}{2}$ each at time $t = 0$. Player 1 sees the signal $s_0(\omega) \equiv \omega$, identified with player 1's type. After seeing $s_0$, player 1 picks $a_1 \in A_1 = [0, 2]$ at $t = 1$. This gives rise to the signal $s_1(\omega, a_1) \equiv a_1$. At $t = 2$, player 2 see $s_1$ and picks $a_2 \in A_2 = \{0, 2\}$ giving rise to an irrelevant signal, $s_2$. Player 2's payoffs are given by $u(a_1, a_2) = a_2(1 - a_1)$, independent of $\omega$. Player 1's payoffs are given by $u(H, a_1, a_2) = a_1 \cdot a_2$ when $\omega =$ Heads, and by $u(T, a_1, a_2) = (2 - a_1) \cdot (2 - a_2)$ when $\omega =$ Tails.*

Player 2's strict best response to $a_1 < 1$ is $a_2 = 2$, her strict best response to $a_1 > 1$ is $a_2 = 0$, and she is indifferent between either action if $a_1 = 1$. Since 2's payoffs do not depend on $\omega$, in any Bayesian Nash equilibrium, player 2 must be playing her strict best responses, but can respond to $a_1 = 1$ with any mixture, $\nu_2$, over $A_2$. For any $\nu_2$, player 1's optimum will not exist at least $\frac{1}{2}$ of the time — no Bayesian Nash equilibrium.

Finitistic versions of a game always have equilibria. The following analysis shows how a finitistic analysis brings out what kinds of expansions are needed to restore equilibrium existence.

The history space for this game is

$$H = (\Omega \times S_0) \times (A_1 \times S_1) \times (A_2 \times S_2). \tag{6}$$

Let $\mathbb{E}(F)$ be the set of Bayesian Nash equilibrium distributions when each set in $H$ is replaced by a finite set in a conformable[2] fashion and $F$ is the corresponding finitistic version of $H$. If 2 plays her strict best responses and player 1 plays $1-\epsilon$ $(1+\epsilon')$, the first $a_1 \in F_{A_1}$ below (above) $a_1 = 1$ when $\omega =$ Heads ($\omega =$ Tails), then 1's expected payoffs are

$$\tfrac{1}{2}(1-\epsilon)\cdot 2 + \tfrac{1}{2}(2-(1+\epsilon'))\cdot(2-0) = 2-(\epsilon+\epsilon'), \tag{7}$$

that is, 2 minus an infinitesimal. Suppose, for the instant, that $\nu_2$, player 2's response to $a_1 = 1$, is $(\frac{1}{2}, \frac{1}{2})$ randomization on $A_2$. In this case, $\mathbb{E}(F)$ contains only the point $\mu^{\epsilon,\epsilon'} = \frac{1}{2}\delta_{h(\epsilon)} + \frac{1}{2}\delta_{h(\epsilon')}$ in $\Delta(F)$ where

$$h(\epsilon) = ((\omega,\omega),(a_1,a_1),(a_2,s_2)) = ((\text{Heads},\text{Heads}),(1-\epsilon,1-\epsilon),(2,s_2)), \tag{8}$$

$$h(\epsilon') = ((\omega,\omega),(a_1,a_1),(a_2,s_2)) = ((\text{Tails},\text{Tails}),(1+\epsilon',1+\epsilon'),(0,s_2)). \tag{9}$$

In this equilibrium, player 1 signals Heads or Tails to player 2 by playing $\epsilon$ below or $\epsilon'$ above $a_1 = 1$, and the cost is infinitesimal.[3]

[2]Each finite replacement of a signal space, $S_t$, contains the range of the corresponding $s_t(\cdot)$ restricted to the finite replacement of its domain. See above for more detail.
[3]If $\nu_2$ is point mass on either 0 or 2, the equilibrium involves 1 playing only below or only above $a_1 = 1$, but is otherwise unchanged.

Because $\mathbb{L}(\mathbb{E})$ is the union, over finitistic $F$, of the points in $\Delta(H)$ closest to $\mathbb{E}(F)$, $\mathbb{L}(\mathbb{E})$ contains only $\mu^{0,0}$, the probability on $H$ given by

$$\mu^{0,0} = \tfrac{1}{2}\delta_{h(\text{Heads})} + \tfrac{1}{2}\delta_{h(\text{Tails})} \tag{10}$$

where $h(\text{Heads})$ is the history $((\text{Heads}, \text{Heads}), (1, 1), (2, s_2))$ and $h(\text{Tails})$ is the history $((\text{Tails}, \text{Tails}), (1, 1), (0, s_2))$. The distribution $\mu^{0,0}$ is not achievable, it is in the closure of the achievable when the usual models of infinite sets are used, and the failure of closure interacts with equilibrium conditions. If, with the usual models of infinite sets, player 1 can talk cheaply, she can play $a_1$ and costlessly signal $\omega$'s value to player 2, mimicking the finitistic equilibrium.

Let $\widehat{A}_1$ be $F_{A_1}$ with $\varphi_{\widehat{A}_1}(f_1)$ being the closest point to $f_1$ in $A_1$ (known as the standard part of $f_1$). By playing different points in $\varphi^{-1}_{\widehat{A}_1}(a_1)$, player 1 can, at the cost of at most an infinitesimal, signal to 2. Intuitively, this is why the limits of equilibrium outcomes taken along finite approximations are contained in the set of cheap talk equilibria as in [28].

4.2.3. *Continuous Games with Almost Perfect Information.* This game demonstrates that the lack of closure of the set of achievable outcomes using the usual models of infinite sets can prevent the existence of equilibria even in games of almost perfect information.

**Example 4.3** (Harris, Reny, and Robson)**.** *Nature makes an extraneous choice at $t = 0$. Players $A$ and $B$ move simultaneously at $t = 1$, players $C$ and $D$ observe the choices made at $t = 1$, and then move simultaneously at $t = 2$. The player action sets are $X_A = [-1, +1]$, $X_B = X_C = X_D = \{L, R\}$. With $a$ being the typicaly element of $X_A$, utilities are*

$$u_A = -|a| \cdot 1_{\{x_B = x_C\}} + |a| \cdot 1_{\{x_B \neq x_C\}} \quad u_B = \begin{cases} 2 \cdot 1_{\{x_C = L\}} - 1 & \text{if } x_B = L \\ 4 \cdot 1_{\{x_C = R\}} - 2 & \text{if } x_B = R \end{cases},$$
$$-10 \cdot 1_{\{x_C \neq x_D\}} - \tfrac{1}{2}a^2,$$

$$u_C = \begin{cases} -a & \text{if } x_C = L,\ x_A = a \\ +a & \text{if } x_C = R,\ x_A = a \end{cases}, \qquad u_D = \begin{cases} -a & \text{if } x_D = L,\ x_A = a \\ +a & \text{if } x_D = R,\ x_A = a \end{cases}.$$

Following [17], note that players $C$ and $D$ "will choose $L$ if $A$ plays strictly to the left of zero, and $R$ if $A$ plays to the right of zero. Player $B$ wishes to guess the choice of player $C$. Player $A$ has to trade off" preventing "player $B$ from guessing the action of player $C$," wanting "players $C$ and $D$ to coordinate," and the "cost $\frac{1}{2}a^2$ associated with any non-zero action."

[17] shows that this game has no subgame perfect equilibrium, though it and other continuous games of almost perfect information do have equilibria once they are expanded by the addition of a rich public signal at each stage. Specifically, players $C$ and $D$ observe the expanded signal is $\widehat{s}_1 = (s_1, s_1^e)$ where $s_1^e$ is e.g. uniformly distributed on $[0,1]$ and independent of $s_1$.

Only $X_A$ is infinite in this game. Therefore, in any finitistic version of this game, only player $A$ has more options. $A$ can randomize over points infinitely close to the left and right of $x_A = 0$ in their replacement set, $F_A$. Since all players at time $t = 2$ see the same signal, this, at no cost, can duplicate the public signal, $s_1^e$. Intuitively, the new information in the infinitesimals above and below $x_A = 0$ is seen equally by later players, that is, the new information is a public signal. The arguments in [17] also show that $\mathbb{L}(\mathbb{E})$ contains only public signal equilibria in this game.

Again, the public signal outcomes necessary for equilibrium existence cannot be realized using the strategies in the original game without expansion. The expansion is provided by finitistic sets.

4.2.4. *Differential Information and Simultaneous Moves.* The players in this game see different continuous signals about Nature's choice, then simultaneously pick actions in compact sets. Payoffs are continuous in Nature's choice and in the players' choices. It is not known whether such games have equilibria. In the example, the space of achievable histories is not closed.[4] This game also demonstrates that the finitistic versions of Nature's move need to be chosen carefully.

[4] Finding an existence counter-example requires matching up the failure of closure of the history space with the equilibrium conditions, simple to say and, apparently, difficult to implement.

**Example 4.4.** *At time $t = 0$, Nature picks a point in $\Omega = \Omega_1 \times \Omega_2$, $\Omega_i = [0,1]$ (with the usual Borel $\sigma$-field), according to $P$, the uniform distribution on the diagonal in $\Omega$. At $t = 1$, each player $i$ sees $\omega_i$, and picks $a_i \in A_i$, the two-point set $\{a_i, b_i\}$. Utilities $u_i(\omega, a)$ are independent of $\omega$ and given in the matrix*

| | $a_2$ | $b_2$ |
|---|---|---|
| $a_1$ | $(6,6)$ | $(3,0)$ |
| $b_1$ | $(0,3)$ | $(9,9)$ |

The matrix game has three equilibria, $\text{Na} = \{(a_1, a_2), (b_1, b_2), ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. Strategies, $\sigma = (\sigma_1, \sigma_2)$ are measurable mappings from $\Omega$ to $\Delta_1 \times \Delta_2$, equilibrium strategies have the property that $\sigma \in \text{Na}$ $P$-a.e. Consider the sequence of strategies $\sigma^n$ that play $(a-1, a_2)$ on the intervals $[2k \cdot 2^{-n}, (2k+1) \cdot 2^{-n})$, $k = 1, \ldots, 2^{n-1}$ and $(b_1, b_2)$ on the complementary intervals. With $\mu^n$ being the distribution over outcomes generated by play of $\sigma^n$, $\mu^0 = \lim_n \mu^n$ is not achievable by any strategy when the usual model of $\Omega$ is used. In this game, the set of limits of equilibrium strategies is the set of public signal equilibria.

Finitistic versions of sets can be modeled as equivalence classes of (generalized) sequences of finite sets. When a finitistic version of $\Omega$ is produced from a sequence $\Omega^\alpha$, there should be a corresponding sequence $P^\alpha$ giving the finitistic version of $P$. The **equality a.a. condition** imposed is that for any (measurable) $E \subset \Omega$, $P^\alpha(E) = P(E)$ a.a. Letting $E$ be the diagonal in this game, finitistic approximations are constrained to have equal signals for the two players. Violating this condition even by an infinitesimal widens the set of limit distributions. Formulating the approximations and the strategies as sequences: model Nature's move as $F^\alpha \times F^\alpha$ where $F^\alpha = \{f_1^\alpha, f_2^\alpha, \ldots, f_{K^\alpha}^\alpha\}$ (where $K^\alpha \to \infty$), let 1 play the strategy "$a_1$ if the signal is $f_k^\alpha$ and $k$ is odd, $b_1$ if $k$ is even," let 2 play the strategy "$a_2$ if the signal is $f_k^\alpha$ and $k$ is odd, $b_2$ if $k$ is even," let $Q^\alpha$ assign equal measures to all points of the form $(f_{2k}^\alpha, f_{2k-1}^\alpha)$, $(f_{2k}^\alpha, f_{2k}^\alpha)$, and $(f_{2k+1}^\alpha, f_{2k}^\alpha)$, $k = 1, \ldots, K^\alpha/2 - 1$, and let $Q^\alpha$ conditional on the diagonal satisfy the equality a.a. condition; with this probability, the given strategies are an equilibrium. $Q^\alpha$ puts only $\frac{1}{3}$ of its mass on the diagonal, puts all of its mass within an infinitesimal of the diagonal.

Diagram here

The point in $\Delta(H)$ closest to the outcome $\lim_\alpha \mu^\alpha$ of the given equilibrium strategies puts mass $\frac{1}{3}$ on the three points $(a_1, a_2)$, $(a_1, b_2)$, and $(b_1, b_2)$ independent of $\omega$. This is a correlated equilibrium that is not a public signal equilibrium.

4.2.5. *New Additions.* In this extensive form game, there need not be an equilibrium with the usual model of infinite sets. What distinguishes this example from the previous one is that crucial strategic aspects of the game depend on fine details of the finitistic $F$ that is used. If one wishes to use the usual models of infinite sets in this game and to have a well-behaved theory, it will require inventing a new class additions to mimic the finitistic expansions.

**Example 4.5.** *At $t = 0$, Nature picks $\omega \in \Omega = \{-1, +1\}$ with probability $\alpha$, $0 < \alpha < 1$, giving rise to the (uninformative) signal $s_0(\omega) \equiv 0$. At $t = 1$, player 1, who observes $s_0$, picks $a_1 \in A_1 = [0, 1]$ which gives rise to the signal $s_1(\omega, s_0, a_1) \equiv a_1$. At $t = 2$, player 2 observes $s_1$ and picks $a_2 \in A_2 = [0, 1]$ giving rise to the (continuous) signal $s_2 = \omega \cdot |a_1 - a_2|$. At $t = 3$, player 3 observes $s_2$ and picks $a_3 \in A_3$ (giving rise to an irrelevant signal). Player 2's utility is $-(a_1 - a_2)^2$, player 3's utility is arranged so that her optimal action depends on whether or not $\omega$ can be inferred.*

The history space for this game is

$$H = (\Omega \times S_0) \times (A_1 \times S_1) \times (A_2 \times S_2) \times (A_3 \times S_3). \tag{11}$$

Again, conformability requires that the choice of the finitistic $F_{A_\tau}$, $\tau \leq t$, constrains the choice of the $F_{S_t}$ to contain the finitistic range of the signals $s_t$.

Four observations:

1. if the spaces in $H$ are replaced with finitistic $F_{A_t}$ and $F_{S_t}$, then $F_{A_1} \backslash F_{A_2} \neq \emptyset$ is possible, so that 1 can guarantee that 3 perfectly infers $\omega$,
2. for $A \subset [0,1]$, it is possible that $(F_{A_1} \cap A) \backslash (F_{A_2} \cap A) \neq \emptyset$, $(F_{A_1} \cap A^c) \subset (F_{A_2} \cap A^c)$ while so that 1 can only guarantee that 3 perfectly infers $\omega$ if she avoids $A$,
3. $F_{A_1} \subsetneq F_{A_2}$ is also possible, in which case player 2 can choose whether or not player 3 infers the correct value of $\omega$,
4. if it is not required that the $F_{A_t}$ contain each element of the $A_t$, then $F_{A_1} \cap F_{A_2} = \emptyset$ is possible, making it impossible for 3 to be ignorant of $\omega$ no matter what players 1 and 2 desire.

The finitistic equilibrium outcome set may contain very different types of outcomes depending on which of the first three observations holds. The exhaustive aspect of finitistic sets implies that $F_{A_1} \cap F_{A_2} \neq \emptyset$, so that the fourth strategic situation is not relevant.[5]

Again, the usual models of infinite sets do not capture many of the limit phenomena in game models. If we keep to the usual models by expanding the game, required, at a minimum, are expansions allowing the 1'st and 2'nd players, but not both simultaneously, to choose whether or not the 3'rd player receives complete information about $\omega$. Further, these informational expansions may be different in different parts of 1's action space.

4.3. **Normal Form Examples.** There are three normal form examples with compact action spaces but discontinuous payoffs. They are presented with some extraneous pieces.

4.3.1. *Omitting the Obvious.* The first game has mixed but no pure equilibria. The pure strategy finitistic equilibrium is quite intuitive, but omitted in analyses conducted with the usual models of infinite sets.

[5]Suppose one were to take the position that the usual models of infinite sets are not only inconvenient, but more generally irrelevant (see, for example, [33]). In this case, being exhaustive loses some of its appeal, making $F_{A_1} \cap F_{A_2} = \emptyset$ a reasonable proposition.

**Example 4.6.** *At $t = 0$, Nature makes an irrelevant pick, $\omega_0$, in the set $\Omega = \{\omega_0\}$, giving rise to the uninformative signals $s_{i,0}(\omega_0)$. At $t = 1$, two legislators offer bills $a_1$ and $a_2$ in the interval $[0, 1]$. The probability of a bill $a_i$ winning passage is given by the proportion of the interval closer to $a_i$ than to $a_j$, $i \neq j$, and ties are evenly split. Legislator 1 is in the minority party and believes that the only politically feasible bills are in the interval $A_1 = [0, \frac{1}{3}]$, while Legislator 2 believes that $A_2 = [\frac{1}{4}, 1]$ is available. The interval $[\frac{1}{4}, \frac{1}{3}]$ may be chosen by either.*

As the Legislators are interested in winning, we may assume that payoffs are linear in the probabilities, making this a constant sum game. It has no pure strategy equilibrium because

$$(\forall \mu_2)\left[\sup_{a_1} u_1(a_1, \mu_2) \geq \frac{1}{3}\right] \quad \text{and} \quad (\forall \mu_1)\left[\sup_{a_2} u_2(\mu_1, a_2) \geq \frac{2}{3}\right], \tag{12}$$

and no vector of pure strategies delivering the payoffs $(\frac{1}{3}, \frac{2}{3})$ is an equilibrium.

This game satisfies [6]'s hence [38]'s sufficient conditions for the existence of an equilibrium. Indeed, equation (12) and a bit of further analysis implies that there is a mixed strategy equilibrium in which both agents randomize over $(\frac{1}{4}, \frac{1}{3})$. While characterizing the mixed strategy equilibrium for this game is not particularly difficult,[6] one can surely understand the first impulse to search for the equilibrium in which Legislator 1 plays as far to the right as possible, $a_1 = \frac{1}{3}$, while Legislator 2 plays just to the right of Legislator 1, giving payoffs of approximately $(\frac{1}{3}, \frac{2}{3})$.

In any finitistic version of the game, play of $a_1 = \frac{1}{3} \in F_{A_1}$ and $a_2$ the least element of $F_{A_2}$ greater than $\frac{1}{3}$ is an equilibrium with payoffs infinitesimally close to $(\frac{1}{3}, \frac{2}{3})$. Thus, with $\mathbb{E}(F)$ being the equilibrium strategies, and for any finitistic $F = F_{A_1} \times F_{A_2}$, $\mathbb{L}_F(\mathbb{E})$ contains the point $(\frac{1}{3}, \frac{1}{3})$.

A direct game expansion that captures this equilibrium sets to $\widehat{A}_i = A_i \times A_i^e$. Here $A_i^e$ contains two points that can be understood as "+" and "−." The equilibrium can then be represented as $\widehat{a} = (\widehat{a}_1, \widehat{a}_2) = ((\frac{1}{3}, -), (\frac{1}{3}, +))$ with $u(\widehat{a}) = (\frac{1}{3}, \frac{2}{3})$. There is a crucial difference in this expansion — the expanded strategies

[6]Legislator 2's density $\varphi_2$ must satisfy $\int_y^{1/3} \varphi_2(a)da - \int_{1/4}^{y} \varphi_2(a)da = 2y\varphi_2(y)$ for Lebesgue almost every $y$ in $(\frac{1}{4}, \frac{1}{3})$, and a similar formula holds for $\varphi_1$.

directly affect utility (e.g. play of $((\frac{1}{3},+),(\frac{1}{3},-))$ would reverse the payoffs, giving $(\frac{2}{3},\frac{1}{3})$).

A less direct but more generally applicable class of game expansions are the endogenous sharing rule equilibria (ESR's) due to [40].

4.3.2. *Endogenous Sharing Rules as Expansions of Normal Form Games.* To construct an ESR in a normal form game, close the graph of $u$ in $\times_{i\in I} A_{i,1} \times [-1,+1]^I$. Denote the correspondence with the resulting graph by $\psi(\cdot)$, and take the point-wise convex hull of $\psi$. A sharing rule is a (measurable) selection $v = (v_i)_{i\in I}$ from the resulting upper hemicontinuous, convex-valued correspondence. An ESR, $(v, \sigma^*)$, is a sharing rule $v$ and a vector of mixed strategies, $\sigma^*$, that form an equilibrium of the game with strategy spaces $A_i$ and payoffs $(v_i)_{i\in I}$. ESR's exist provided the $A_i$ are dense in compact metric spaces and the payoffs are bounded.[7]

Any ESR, $(v, \sigma^*)$, can be represented as a game expansion: Expand $\Omega$ to $\widehat{\Omega} = \{\omega_0\} \times [0,1]$ with $\varphi_{\widehat{\Omega}}(\omega_0, r) \equiv \omega_0$. Have $\widehat{P}$ be point mass on $\omega_0$ times Lebesgue measure, $\lambda$. For each $A_i$, let $\widehat{A}_i = A_i$ with $\varphi_{\widehat{A}_i}$ equal to the identity map (so there is no distinction between the $A_i$ and the $\widehat{A}_i$). Take the extended signals, $\widehat{s}_{i,0}$, to be everywhere equal to the uninformative $s_{i,0}$. For each $a \in A$, take $\widehat{u}((\omega_0, r), a)$ to be a point in $\psi(a)$ such that $\int_{[0,1]} \widehat{u}((\omega_0, r), a)\, d\lambda(r) = v(a)$. Standard selection theorems (e.g. [20, Theorem ?.??, p. ??]) show that it is possible to pick $\widehat{u}$ to have this property and to be a jointly measurable function. For each $i \in I$, and each $a_i \in A_i$, playing $a_i$ against $\sigma^*$ gives $i$ expected utility $v_i(\sigma^* \backslash a_i)$. Thus, $\sigma^*$ is an equilibrium for the given expansion. Further, it is easy to see that an equilibrium $\sigma'$ for a given $\widehat{u}$ in this expansion is an ESR.

4.3.3. *ESR's Include too Much.* ESR's involve a referee, either in the form of a convexifier, $v$, or in the form of a choice of $\widehat{u}$, picking how to assign utilities at discontinuities. The next two examples show that way payoffs are picked in ESR's

[7]The intuitive pure strategy equilibrium in Example 4.6 is represented by a selection satisfying $v(\frac{1}{3},\frac{1}{3}) = (\frac{1}{3},\frac{2}{3})$ in addition to the play of the pure strategy $(\frac{1}{3},\frac{1}{3})$.

is disconnected from the players' actions in at least two unpalatable ways. The first lack of connection arises because ESR's have no exhaustiveness properties.

**Example 4.7.** *The game starts, at $t = 0$, with an irrelevant move of Nature and an uninformative signal. At $t = 1$, the two players simultaneously pick in their action spaces, $A_i = [0, 1]$, and the utility functions are*

$$u_1(a_1, a_2) = \begin{cases} 2 & \text{if} \quad a_1 = 0 \\ a_1 & \text{if} \quad a_1 > 0 \end{cases}, \qquad u_2(a_1, a_2) = \begin{cases} 1 - a_2 & \text{if} \quad a_1 = 0 \\ a_2 & \text{if} \quad a_1 > 0 \end{cases}.$$

It is easy to check that $v_1(a_1, a_2) \equiv a_1$ and $v_2(a_1, a_2) \equiv a_2$ is a selection from $\psi(\cdot)$, indeed, it is the unique continous selection. For this selection, the unique equilibrium is $(a_1, a_2) = (1, 1)$. What is missing from this ESR is the crucial strategic aspect of $a_1 = 0$, player 1's ability to guarantee her/himself a payoff of $u_1 = 2$. This is not missing from any finitistic expansion of the game, and the unique finitistic expansion equilibrium outcome is $(0, 0)$.

The second lack of connection arises because taking the convex hull of $\psi(a)$ casts too wide a net, only the limits of the products of independent randomizations should be included.

**Example 4.8.** *The game starts, at $t = 0$, with an irrelevant move of Nature and an uninformative signal. At $t = 1$, the two players simultaneously pick in $A_i = \mathbb{N}$, and the utility functions are symmetric,*

$$u_i(a_i, a_j) = \begin{cases} (10, -10) - (\frac{1}{a_i}, \frac{1}{a_j}) & \text{if} \quad a_i \geq a_j + 2 \\ (8, 4) - (\frac{1}{a_i}, \frac{1}{a_j}) & \text{if} \quad a_i = a_j + 1 \\ (-2, -2) - (\frac{1}{a_i}, \frac{1}{a_j}) & \text{if} \quad a_i = a_j \\ (4, 8) - (\frac{1}{a_i}, \frac{1}{a_j}) & \text{if} \quad a_i = a_j - 1 \\ (-10, 10) - (\frac{1}{a_i}, \frac{1}{a_j}) & \text{if} \quad a_i \leq a_j - 2 \end{cases}$$

The $A_i$ are dense in $\bar{A}_i := \mathbb{N} \cup \{\infty\}$, their metrizable one point compactifications. Further, the only discontinuity point for the $u_i$ happens at $(\infty, \infty)$. The convex hull of the limits of the possible at payoffs at $(\infty, \infty)$ is $V = \mathbf{co}\,\{(10, -10), (8, 4), (-2, -2), (4, 8), (-10, 10)\}$. Any selection from $V$ combined with play of $(\infty, \infty)$ is an endogenous sharing rule equilibrium. In particular, the utility levels $(5,7)$ can occur because $(5,7) = \frac{3}{4}(4,8) + \frac{1}{4}(8,4)$. This is the unique expression of $(5,7)$ as a convex combination of points in $V$. However, no utilities for product probabilities are in the neighborhood of $(5,7)$.

The equilibrium in the finitistic version of the game gives: utility $(10,-10)$ if the largest point in $F_{A_1}$ is 2 or more larger than the largest point in $F_{A_2}$; utility $(8,4)$ if the largest point in $F_{A_1}$ is 1 larger than the largest point in $F_{A_2}$; and so forth, with symmetry.

4.4. **Summary of Lessons from the Examples.** With the usual model of infinite sets, a minimally well-behaved game theory, a non-empty, closed set of predictions immune to the inclusion of particular points, requires that games be expanded. Even from the small set of examples given here, the requisite expansions must include forms of information leakage, public randomization, cheap talk, complex correlating devices, choices of conditions under which early agents can choose whether or not later agents are informed, and endogenous sharing rules. Further, the exact shape of the requisite expansions depends critically on the game in question. By contrast, finitistic versions of these games, needing no further expansion, are relatively simple to analyze.

As a preview of why finitistic expansions work so well, note that when $F_X$ is a finitistic version of $X$, for any topology $\tau$ on $X$, no matter how large, any $\tau$-compactification of $X$ is contained in $F_X$ as a collection of equivalence classes. In other words, $F_X$ contains a huge number of idealizations of the behavior of generalized sequences in $X$. In discussing related results, Lindstrøm [21, p. 58] notes "that a very general limit construction is built, once and for all, into the existence of sufficiently saturated, nonstandard models."

## 5. Finitistic Models of Infinite Sets

There are three characterizations of the class of large finite sets we use: (i) they are the anchored limits of sequences of finite approximations, (ii) they are the limit sets when the class of finite sets is partially ordered by reverse inclusion, and (iii) they are exhaustive, star-finite sets. The three results in this section are (a) the three approaches are equivalent, (b) if $\mathbb{E}(B)$ is non-empty for finite $B$, then the limit sets are non-empty, and (c) sufficient conditions for the upper-hemicontinuity of the the correspondence from utility functions to limit sets. The proof of the equivalence contains a minimality result for finitistic game expansions. These results also hold when there are restrictions on the allowable class of finite subsets.

The equivalences are expressed in terms of $\mathbb{E}$-sets, approximate $\mathbb{E}$-sets, and $\mathbb{L}(\mathbb{E})$-sets, "$\mathbb{E}$" being a mnemonic for "equilibrium," "$\mathbb{E}^\epsilon$" for "approximate equilibrium," and $\mathbb{L}$ for "limit." Throughout, $X$ is a compact Hausdorff space, the set of Borel probabilities on $X$, $\Delta(X)$, has the weak$^*$ topology, $\mathcal{P}_X$ denotes the finite subsets of $X$, and when $X$ is a product space, $\mathcal{P}_X$ denotes the set of products of finite subsets, $\mathcal{P}_X = \times_{i \in I} \mathcal{P}_{X_i}$.

Under study is the limit behavior of a correspondence $\mathbb{E}(\cdot)$ (respectively, $\mathbb{E}^\epsilon(\cdot)$) that maps each $F$ in $\mathcal{P}_X^C$ (respectively, and each $\epsilon > 0$) to a *non-empty* subset of probabilities in $\Delta(X)$ that are concentrated on $F$. Limits are taken along a subset, $\mathcal{P}_X^C$ of $\mathcal{P}_X$ that is **rich**, that is, for each $x \in X$, $(\exists F \in \mathcal{P}_X^C)[x \in F]$, and has a **lattice structure**, that is, if $F, F' \in \mathcal{P}_X^C$, then $(\exists F'' \in \mathcal{P}_X)[F'' \supset F \cup F']$. Clearly $\mathcal{P}_X$ is rich and has a lattice structure. In the next section, we will see that conformable finitistic $F$ replacements of $H$ is rich and has a lattice structure in extensive form games. It is assumed throughout that $\mathcal{P}_X^C$ is rich and has a lattice structure, the theory is only interesting when $\mathcal{P}_X^C$ is infinite.

The 'typical' interpretations of $\mathbb{E}$ and $\mathbb{E}^\epsilon$ are irrelevant to results in this section, though they can help the intuition. If $X = \times_{i \in I} X_i$ is a metric space of strategy vectors and $F \in \mathcal{P}_X^C$, then $\mathbb{E}(F)$ (respectively $\mathbb{E}^\epsilon(F)$) will typically be equilibria (respectively $\epsilon$-equilibria) of the game in which the players, $i \in I$, are restricted to the actions in $F_i$. If $X$ is the set of possible histories for a dynamic game with infinite action sets, then $\mathbb{E}(F)$ (respectively $\mathbb{E}^\epsilon(F)$) will typically denote a set of equilibrium outcomes (respectively $\epsilon$-equilibrium outcomes) when $P$ is replaced by a generalized sequence of distributions of Nature's move, $P^n$, that satisfy the equality a.a. condition of Section 4.2.4.

5.1. **Definitions.** A sequence or generalized sequence $F^n$ in $\mathcal{P}_X^C$ converges to $X$, $F^n \to X$, if for each $x \in X$ and each neighborhood $G_x$ of $x$, $F^n \cap G_x \neq \emptyset$ a.a.[8] Because $\mathcal{P}_X^C$ is rich and has a lattice structure, there exist generalized sequences in $\mathcal{P}_X^C$ that converge to $X$.

For any $F \in \mathcal{P}_X^C$, $S(F)$ denotes the set of $F' \in \mathcal{P}_X^C$ such that $F \subset F'$.

**Definition 1.** *The set of* **anchored $\mathbb{E}$-limits** *is* $\mathbb{L}^{an}(\mathbb{E}) = \bigcap_{F \in \mathcal{P}_X^C} \mathbb{L}_F^{an}(\mathbb{E})$ *where*

$$\mathbb{L}_F^{an}(\mathbb{E}) = \{\lim_n \mu^n : \mu^n \in \mathbb{E}(F^n),\ F^n \to X,\ F^n \in S(F)\ a.a.]\}. \tag{13}$$

*The set of* **anchored approximate $\mathbb{E}$-limits** *is* $\mathbb{L}^{an}(\mathbb{E}^+) = \bigcap_{\epsilon > 0,\, F \in \mathcal{P}_X^C} \mathbb{L}_F^{an}(\mathbb{E}^\epsilon)$ *where* $\mathbb{E}^\epsilon$ *replaces* $\mathbb{E}$ *in eqn.* (13).

While $\mathbb{L}_F^{an}(\mathbb{E})$ and $\mathbb{L}_F^{an}(\mathbb{E}^\epsilon)$ depend on the choice of $F$ and $\epsilon$, the sets $\mathbb{L}^{an}(\mathbb{E})$ and $\mathbb{L}^{an}(\mathbb{E}^+)$ do not.

Partially order $\mathcal{P}_X^C$ by $F_1 \succ F_2$ if $F_1 \supset F_2$. Limits as $F \uparrow \infty$ are defined in $\mathcal{P}_X^C$ relative to this partial ordering.

**Definition 2.** *The set of* **partial order $\mathbb{E}$-limits** *is*

$$\mathbb{L}^{po}(\mathbb{E}) = \{\lim_{F \uparrow \infty} \mu^F : \mu_F \in \mathbb{E}(F'),\ F' \succ F\}, \tag{14}$$

*and the set of* **partial order approximate $\mathbb{E}$-limits** *is*

$$\mathbb{L}^{po}(\mathbb{E}^+) = \{\lim_{\epsilon \downarrow 0,\, F \uparrow \infty} \mu_F : \mu^F \in \mathbb{E}^\epsilon(F'),\ F' \succ F\}. \tag{15}$$

Nonstandard objects[9] belong to an $\aleph$-saturated extension, ${}^*V(Z)$, of a superstructure $V(Z)$ where the base set, $Z$, contains $X$, $\mathbb{R}$, and $\aleph$ is a cardinal greater

[8] This is the usual Hausdorff convergence of $F^n$ to $X$ when $X$ is a compact metric space.
[9] See Lindstrøm (1988) or Hurd and Loeb (1985) for good expositions, Appendix A is a primer that tries to make it clear that the crucial properties of nonstandard objects are at least reasonable.

than the cardinality of $V(X)$. The class of finite subsets of any bounded $Y$ in the superstructure $V(Z)$ is denoted $\mathcal{P}_Y$, and is itself bounded in $V(Z)$. The $^*$-finite (read star-finite or hyperfinite) subsets of $Y$ are $^*\mathcal{P}_Y$.

**Definition 3.** *An $F \in {}^*\mathcal{P}_X^C$ is* **exhaustive** *if for all $x \in X$, $x \in F$. An element of ${}^*\mathcal{P}_X^C$ is a* **finitistic version of** $X$*, or simply,* **finitistic***, if it is exhaustive. The set of finitistic subsets of $X$ is denoted $\mathbb{F}$.*

Because $\mathcal{P}_X^C$ is rich and has a lattice structure, $\aleph$-saturation implies $\mathbb{F} \neq \emptyset$. When $X$ is infinite, any finitistic version of $X$ is a strict expansion of $X$ — it contains all of the points in $X$ and others as well (e.g. [21, Lemma 2.?]). When $(Z, \tau)$ is a topological space and $z \in {}^*Z$ is nearstandard, the standard part of $z$ is denoted by $^\circ z$ or $\mathrm{st}\,(z)$. The standard part of $z$ the point $z'$ in $Z$ closest to $z$. Recall that $\Delta(X)$ has the weak$^*$ topology.

**Definition 4.** *The set of* **finitistic $\mathbb{E}$-limits** *and* **finitistic approximate $\mathbb{E}$ limits** *are*

$$\text{(16)} \qquad \mathbb{L}^{fi}(\mathbb{E}) = {}^\circ\{{}^*\mathbb{E}(F) : F \in \mathbb{F}\}, \ \textit{and } \mathbb{L}^{fi}(\mathbb{E}^+) = {}^\circ\{{}^*\mathbb{E}^\epsilon(F) : F \in \mathbb{F},\ \epsilon \simeq 0\}.$$

There is a topological interpretation of finitistic sets directly tied to the partial order on $\mathcal{P}_X^C$. Let $\mathcal{C}$ denote the set of bounded, real-valued functions, $c$, on $\mathcal{P}_X^C$ with the property that $\lim_{F\uparrow\infty} c(F)$ exists. Because $\mathcal{C}$ is a sup norm closed algebra of functions separating points in $\mathcal{P}_X^C$, there is a compactification, $Z_X$, of $\mathcal{P}_X^C$ defined by the property that each $c \in \mathcal{C}$ has a unique continuous extension from $\mathcal{P}_X^C$ to $Z_X$, and that all continuous functions on $Z_X$ are extensions of elements of $\mathcal{C}$ (see e.g. [19, Thm. III.7.3, p. 157]). The filterbase $\mathcal{S} = \{S(F) : F \in \mathcal{P}_X^C\}$ converges to a unique point, call it $\infty$, in $Z_X$. The finitistic sets are the infinitesimal neighborhood, i.e. the monad, of $\infty$ in $Z_X$.

**Definition 5.** *When $(Z, \tau)$ is a topological space, and $z$ is a point in $Z$, the* **monad of** $z$ **in** ${}^*Z$ *is the set $m(z) = \bigcap\{{}^*U : z \in U,\ U \in \tau\}$.*

**Lemma 1.** *$F \in \mathbb{F}$ if and only if $F \in m(\infty)$ and $F \neq \infty$.*

**Proof**: If $F \in \mathbb{F}$, then for all $c \in \mathcal{C}$, ${}^*c(F) \simeq {}^*c(\infty)$, which is equivalent to $F \in m(\infty)$ because the $c$'s define the topology on $Z_X$. If $F \notin \mathbb{F}$, then there exists $B \in \mathcal{P}_X^C$ such that $F$ is not a superset of $B$. Define $c$ to be equal to 1 for all supersets of $B$ and equal to 0 otherwise so that $c \in \mathcal{C}$. Because $c(F) = 1$ and $c(\infty) = 0$, $F$ is not in $m(\infty)$. ∎

5.2. **Equivalence, Existence, and Minimality.** All three of the approaches, anchored, partial order limits, and finitistic, give the same closed, non-empty limiting sets, and there is a strong sense that this set is minimal.

**Theorem 1** (Equivalence)**.** *With $D$ being $\mathbb{E}$ or $\mathbb{E}^+$, $\mathbb{L}^{an}(D)$, $\mathbb{L}^{po}(D)$, and $\mathbb{L}^{fi}(D)$ are equal, closed, and non-empty.*

**Proof**: The proof for $\mathbb{E}^+$ is a minor modification of the one for $\mathbb{E}$, and is not given here. For the purposes of this proof, introduce the set $\mathbb{L}'(\mathbb{E}) = \bigcap_{F \in \mathcal{P}_X^C} \mathrm{cl}\, \mathbb{E}(S(F))$. The set $\mathbb{L}'(\mathbb{E})$ is closed and non-empty because it is the intersection of the class of sets $\{\mathrm{cl}\, \mathbb{E}(S(F)) : F \in \mathcal{P}_X^C\}$, a collection of closed subsets of the compact space $\Delta(X)$ having the finite intersection property. The steps $\mathbb{L}' \subset \mathbb{L}^{po}$, $\mathbb{L}^{po} \subset \mathbb{L}^{an}$, $\mathbb{L}^{an} \subset \mathbb{L}'$, and $\mathbb{L}' = \mathbb{L}^{fi}$ will complete the proof.

*Step* 1 — $\mathbb{L}' \subset \mathbb{L}^{po}$: $\mu \in \mathbb{L}'$ if and only if for all $F \in \mathcal{P}_X^C$, there is a sequence $\mu_F^\alpha$ in $\mathbb{E}(S(F))$ such that $\lim_\alpha \mu_F^\alpha = \mu$. This implies that there exists $F \mapsto \alpha(F)$ such that $\lim_{F \uparrow \infty} \mu_F^{\alpha(F)} = \mu$ and $\mu_F^{\alpha(F)}$ belongs to $\mathbb{E}(F')$ for some $F' \succ F$ so that $\mu \in \mathbb{L}^{po}$.

*Step* 2 — $\mathbb{L}^{po} \subset \mathbb{L}^{an}$: If $\mu \in \mathbb{L}^{po}$, then $\mu \in \mathbb{L}^{an}$ because $F^n \uparrow \infty$ implies both that $F^n \to X$ and for every $F$, $F^n \in S(F)$ a.a.

*Step* 3 — $\mathbb{L}^{an} \subset \mathbb{L}'$: Suppose that $\mu \notin \mathbb{L}'$, i.e., for some $F \in \mathcal{P}_X^C$, $\mu \notin \mathrm{cl}\, \mathbb{E}(S(F))$. This implies that $\mu \notin \mathbb{L}_F^{an}$, which in turn implies that $\mu \notin \mathbb{L}^{an}$.

*Step* 4 — $\mathbb{L}' = \mathbb{L}^{fi}$: For any subset $S$ of a topological space, the standard part of ${}^*S$ is the closure of $S$ (e.g. [21, Thm. ??.?]), so that ${}^\circ\,{}^*\mathbb{E}(S(F)) = \mathrm{cl}\, \mathbb{E}(S(F))$. Thus, a point $\mu$ satisfies $\mu \notin \mathbb{L}'$ if and only if $\mu \notin \bigcap_{F \in \mathcal{P}_X^C} {}^\circ\,{}^*\mathbb{E}(S(F))$ if and only if there exists an $F \in \mathcal{P}_X^C$ such that $\mu \notin {}^\circ\,{}^*\mathbb{E}(S(F))$. Because $F' \in \mathbb{F}$ if and only if $F' \succ F$ for all $F \in \mathcal{P}_X^C$, this is in turn equivalent to $(\forall F' \in \mathbb{F})[\mu \notin {}^{\circ *}\mathbb{E}(F')]$. Combining, $\mu \notin \mathbb{L}'$ if and only if $\mu \notin \mathbb{L}^{fi}$. ∎

The intersection formulation of $\mathbb{L}'$ shows that $\mathbb{L}^{an}$, $\mathbb{L}^{po}$, and $\mathbb{L}^{fi}$ identify the smallest closed set containing the limits of $\mathbb{E}(\cdot)$ applied to finite sets. The limits sets are the minimal closed sets consistent with the idea that continuous quantities are the limits of finite approximations.

In light of Theorem 1, the superscripts on $\mathbb{L}^{an}$, $\mathbb{L}^{po}$, and $\mathbb{L}^{fi}$ will be omitted.

5.3. **Upper-hemicontinuity.** Let $\mathbb{E}(B,u)$ denote the set $\mathbb{E}(B)$ when $u$ is the utility function, and $\mathbb{L}(\mathbb{E},u)$ the corresponding limit set. For many of the interesting $\mathbb{E}$'s, $u \mapsto \mathbb{E}(B,u)$ is not upper-hemicontinuous even for fixed $B \in \mathcal{P}_X^C$. Re-consider Example 3.1 with the new but still symmetric utility functions,

$$u_i^\epsilon(a_i,a_j) = \begin{cases} 0 & \text{if } a_i = -1 \\ 2 & \text{if } a_i, a_j \in [0,1] \\ -a_i - \epsilon & \text{if } a_j = -1 \text{ and } a_i \in [0,1] \end{cases}$$

The sup norm distance between $u$ and $u^\epsilon$ is $\epsilon$, but for all $\epsilon > 0$, there are no weakly dominated strategies. If $\mathbb{E}(B,u^\epsilon)$ is defined as, for example, the perfect equilibria, then for all $\epsilon > 0$, $\mathbb{L}(\mathbb{E},u^\epsilon)$ contains both play of $(0,0)$ and play of $(-1,-1)$. By contrast, $\mathbb{L}(\mathbb{E},u)$ contains only play of $(0,0)$.

Metrize the possible utility functions, $\mathcal{U}$, with the sup norm, $\|\cdot\|_\infty$.

**Definition 6.** *The* **closure** *of* $u \mapsto \mathbb{L}(\mathbb{E},u)$ *from* $\mathcal{U}$ *to* $\Delta(\widehat{X}) \times [-1,+1]^I$ *is*

$$\overline{\mathbb{L}}(\mathbb{E},u) = {}^\circ\{{}^*\mathbb{E}(F,u^\epsilon) : F \in \mathbb{F},\ \|u-u^\epsilon\|_\infty \simeq 0\}. \tag{17}$$

If $u \mapsto \mathbb{E}(B,u)$ is uhc for fixed $B \in \mathcal{P}_X^C$, then $d(\mathbb{E}(B,u), {}^*\mathbb{E}(B,u^\epsilon)) \simeq 0$ whenever $\|u-u^\epsilon\|_\infty \simeq 0$ for each $u \in \mathcal{U}$.

**Theorem 2** (Upper hemi-continuity). *The correspondence* $u \mapsto \overline{\mathbb{L}}(\mathbb{E},u)$ *is upper-hemicontinuous.*

**Proof**: Let $\Psi$ be the correspondence on $Z_X \times \mathcal{U}$ having as graph the closure of the graph of the correspondence $(B,u) \mapsto \mathbb{E}(B,u)$ on $\mathcal{P}_X^C \times \mathcal{U}$. The correspondence $u \mapsto \overline{\mathbb{L}}(\mathbb{E},u)$ is the correspondence $u \mapsto \Psi(\infty,u)$, the graph of which is closed because it is the intersection of two closed sets. In the presence of the compactness of the range space, this is sufficient. ∎

Replacing $\mathbb{E}$ by $\mathbb{E}^+$ in the definition of closure leads to a version of Theorem 2 for $\mathbb{E}^+$ that is true for the same reasons.

## 6. Games, Equilibria, and Expansions

This section specifies general extensive form games and their expansions. A general extensive form game is one in which Nature's actions, the players' actions, and the signals on which they base their actions take their values in measure spaces. An expansion of an extensive form game expands each measure space in the game, and extends the signals and utilities in a conformable fashion (to be carefully specified below). Replacing the spaces that define an extensive form game by finitistic versions of the same space gives rise to an particularly interesting class of expansions, especially for compact and continuous extensive form games. The following are in force throughout.

**Assumption 6.1** (Blanket Technical Regularity)**.**

1. *The sets below are non-empty and come equipped with a $\sigma$-field of measurable subsets for which point masses are well-defined;*
2. *probabilities are countably additive;*
3. *the set of probabilities on any $\sigma$-field is given the smallest $\sigma$-field containing sets of the form $\{\mu : \mu(E) > r\}$ where $E$ is measurable and $r \in \mathbb{R}$;*
4. *product spaces have the product $\sigma$-field; and*
5. *the functions and sets below are universally measurable.*

6.1. **Games.** Specifying a game requires: the finite set of agents, $I = \{1, \ldots, I\}$; the moves of Nature; a specification of the possible outcomes; of when and under what conditions which agents can move; what the agents know when they make their moves; and the utilities. Formally, a **game** is 4-tuple

$$\Gamma = \big(\underbrace{(\Omega, P)}_{1},\ \underbrace{(A_{i,t})_{i\in I, t\in\mathbb{T}}}_{2},\ \underbrace{(S_{i,t}, s_{i,t}(\cdot))_{i\in I, t\in\{0\}\cup\mathbb{T}}}_{3},\ \underbrace{(u_i(\cdot))_{i\in I}}_{4}\big). \tag{18}$$

The rest of this subsection gives the definitions of the parts of a game, the assumptions they satisfy, as well as the definitions of outcomes and equilibria.

6.1.1. *The pieces.* The agents pick their actions at any time $t$ in $\mathbb{T} = \{1, \dots, T\}$, $T$ an integer, while Nature's move is a point $\omega$ in $\Omega$ picked at time $t = 0$ according to $P$. Signals at $t \in \{0\} \cup \mathbb{T}$ take their values in $S_t = \times_{i\in I} S_{i,t}$. Player $i$'s actions at $t \in \mathbb{T}$, points in $A_{i,t}$, are based on their signals at $t-1$, points in $S_{i,t-1}$.

The random initial history, $h_0$, is of the form $(\omega, (s_{i,0})_{i\in I})$ where $s_0 = (s_{i,0})_{i\in I}$ is a function of $\omega$. Thus, $h_0$ is a point in

$$H_0 := \Omega \times S_0. \tag{19}$$

Each player $i$'s actions at $t = 1$ are points $a_{i,1}$ in $A_{i,1}$. The actions are picked as functions of the signals $s_{i,0}$. Signals at time $t \in \mathbb{T}$, $s_t$, on which actions at $t+1$ will be based, are a function of $h_{t-1}$ and $a_t$, i.e. the domain of $s_t$ is $H_{t-1} \times A_t$. The set of $t$-partial histories is thus a subset of

$$H_t := (H_{t-1} \times A_t) \times S_t. \tag{20}$$

A behavioral strategy for $i$ is a function, $\sigma_{i,t}(s_{i,t-1})$, of their signal. Each $\sigma_{i,t}$ takes values in the set of probabilities on $A_{i,t}$. If $A_{i,t}$ has only one point, the intended interpretation is that player $i$ has no choice to make at $t$.

Utilities are given by $u_i : H_T \to [-1, +1]$, and $u = (u_i)_{i\in I}$.

6.1.2. *Perfect recall.* By assumption, the game is one of perfect recall. Define $\pi_{i,t}(h_{t-1})$ to be the subvector of $h_{t-1}$ containing all of $i$'s signals and actions previous to $t$, $\pi_{i,t}(h_{t-1}) = (s_{i,\tau}, a_{i,\tau})_{0\le\tau\le t-1}$.

**Assumption 6.2** (Perfect Recall)**.** *For all $i$, $t$, and for all partial histories $h_{t-1}$, the vector $s_{i,t}(h_{t-1})$ contains $\pi_{i,t}(h_{t-1})$ as a subvector.*

6.1.3. *Outcomes.* Fix a vector $\sigma = (\sigma_{i,t})_{i\in I,\ t\in\mathbb{T}}$ of strategies, and a partial history $h_{t-1}$. Define $\sigma_{\cdot,t} = (\sigma_{i,t})_{i\in I}$, and after each $h_{t-1}$, identify the vector $\sigma_{\cdot,t}(h_{t-1})$ of probabilities with the product probability on $\times_{i\in I} A_{i,t}$ with the given marginals. Also, define $\sigma_{i,\cdot} = (\sigma_{i,t})_{t\in\mathbb{T}}$. The one-step-ahead distribution over $A_t \times S_t$ that follows $h_{t-1}$ when $\sigma$ is being played is defined by its value on rectangles, $E \times F$,

$$Q_t(E \times F\big|h_{t-1}) := \sigma_{\cdot,t}(h_{t-1})(E \cap (s(h_{t-1},\cdot)^{-1}(F))). \tag{21}$$

Given a distribution, $\mu_{t-1}$ on $H_{t-1}$ and $D \subset H_{t-1}$ define $\mu_t$ on $H_t$ by

$$\mu_t(D \times (E \times F)) = \int_D Q_t(E \times F\big|h_{t-1})\ d\mu_{t-1}(h_{t-1}). \tag{22}$$

As an example, if $\mu_{t-1}$ is point mass on $h_{t-1}$, then $\mu_t(\cdot)$ is the product of point mass on $h_{t-1}$ and the one-step-ahead probability $Q_t(\cdot\big|h_{t-1})$.

Inductively applying this construction to $\mu_0$, the initial distribution on $H_0$ determined by Nature, gives the **outcome, $\mathbb{O}(\sigma)$, associated with play of** $\sigma$. Applying this construction to a point mass on a point $h_{t-1}$ gives the **outcome, $\mathbb{O}(\sigma\big|h_{t-1})$, associated with play of $\sigma$ following the partial history** $h_{t-1}$.

6.1.4. *Equilibria.* For $i \in I$ and $t \in \mathbb{T}$, $\mathcal{S}_{i,t-1}$ is the minimal $\sigma$-field making $s_{i,t-1}$ measurable, $u(\sigma) := u(\mathbb{O}(\sigma))$, and $u(\sigma\big|h_{t-1}) := u(\mathbb{O}(\sigma\big|h_{t-1}))$.

**Definition 7.** *$\sigma^*$ is an* **equilibrium** *and $\mathbb{O}(\sigma^*)$ an* **equilibrium outcome** *if for all $i \in I$ and for all strategies $\sigma_i$, $u_i(\sigma^*) \geq u_i(\sigma^*\backslash\sigma_i)$.*

**Definition 8.** *$\sigma^*$ is a* **Bayesian Nash equilibrium** *and $\mathbb{O}(\sigma^*)$ a* **Bayesian Nash equilibrium outcome** *if for all $i \in I$ and all $t \in \mathbb{T}$, $E\left(u_i(\sigma^*)\big|\mathcal{S}_{i,t-1}\right) \geq E\left(u_i(\sigma^*\backslash\sigma_i)\big|\mathcal{S}_{i,t-1}\right)$ with $\mathbb{O}_{t-1}(\sigma)$ probability 1 where $\mathbb{O}_{t-1}(\sigma)$ is the marginal of $\mathbb{O}(\sigma)$ on $H_{t-1}$, and conditional expectations are taken with respect to $\mathbb{O}_{t-1}(\sigma^*)$.*

6.2. **Expansions.** An expansion of $\Gamma$ is a game, $\widehat{\Gamma}$, in which there may be extra moves by Nature, extra signals, and extra actions. The extras arise from the expansion of the measure spaces in $\Gamma$.

**Definition 9.** *An* **expansion of a measure space** $(X, \mathcal{X})$ *is a measure space* $(\widehat{X}, \widehat{\mathcal{X}})$ *and a mapping* $\varphi_{\widehat{X}} : \widehat{X} \to X$ *that is both onto and measurable.*

If $x$ is a point in a set of (say) actions $X$, then $\varphi_{\widehat{X}}^{-1}(x)$ is a corresponding larger set of (say) actions available in an expanded game. Also, $(X, \mathcal{X})$ can be understood as the set of $\varphi_{\widehat{X}}$-equivalence classes in $\widehat{X}$, $X = \widehat{X}/\varphi_{\widehat{X}}$.

In increasing order of complexity, three expansions of $(X, \mathcal{X})$ are: $\widehat{X} = X$ and $\varphi_{\widehat{X}}$ is the identity mapping; $\widehat{X} = X \times [0,1]$ and $\varphi_{\widehat{X}}$ is projection onto $X$; and $\widehat{X}$ is a subset of $X \times [0,1]$ such that, when $\varphi_{\widehat{X}}$ is projection to $X$, it is onto. A fourth expansion, much used below, is available when $(X, \mathcal{X})$ is a compact Hausdorff space and its Borel $\sigma$-field, $(\widehat{X}, \widehat{\mathcal{X}})$ is a finitistic version of $X$ and a Loeb $\sigma$-field, and $\varphi_{\widehat{X}}$ is the standard part mapping.

6.2.1. *Examples of expansions in games.* Extra signals can give rise to information leakages that remove informational discontinuities. In Example 4.1, the action taken at time 2 depends on the signal $s_1 = -(a_1 - \omega)^2$. One informational leakage expansion takes $\widehat{S}_1$ to be $S_1 \times \Omega$ and have $\varphi_{\widehat{S}_1}^{-1}(s_1) = \{s_1\} \times \Omega$ and has the expanded signal $\widehat{s}_1 = (s_1, \omega)$. With this expanded signal, the player knows $\omega$ when choosing at $t = 2$ even if $s_1$ is uninformative.

Extra actions with no utility consequences, if observed by subsequent players, add cheap talk to signaling games such as Example 4.2. One formulation of the cheap talk expansion sets $\widehat{A}_1 = A_1 \times \Omega$ with $\varphi_{\widehat{A}_1}^{-1}(a_1) = \{a_1\} \times \Omega$. If player 1 plays the expanded strategy $\widehat{\sigma}_1 = (1, \omega)$ and $\widehat{s}_2 = \widehat{a}_1$, then at $t = 2$, player 2 can coordinate her action with $\omega$.

In a directly parallel fashion, extra moves with no utility consequences, if commonly observed by later players, can add a public signal to games of nearly perfect information such as Example 4.3. Different players seeing different aspects of an extra move by Nature adds a correlating device. In Forges ([11]) and Myerson ([31]), the correlating device depends on the entire previous history. Extra actions that determine the information content of the extra signals can make substantially alter the strategic aspects of a game such as Example 4.5. Section 4.3.2 discusses the game expansion that deliver ESR's, the addition of a referee determining payoffs associated with given actions.

6.2.2. *Formalities of expansions.* Fix a game

$$\Gamma = ((\Omega, P),\ (A_{i,t})_{i\in I, t\in\mathbb{T}},\ (S_{i,t}, s_{i,t}(\cdot))_{i\in I, t\in\{0\}\cup\mathbb{T}},\ (u_i(\cdot))_{i\in I}).$$

An expansion of $\Gamma$, $\widehat{\Gamma}$, is a game

$$\widehat{\Gamma} = ((\widehat{\Omega}, \widehat{P}),\ (\widehat{A}_{i,t})_{i\in I, t\in\mathbb{T}},\ (\widehat{S}_{i,t}, \widehat{s}_{i,t}(\cdot))_{i\in I, t\in\{0\}\cup\mathbb{T}},\ (\widehat{u}_i(\cdot))_{i\in I})$$

in which some or all of the spaces $\Omega$, $A_{i,t}$, and $S_{i,t}$ are replaced by expansions.

Because $P$ is fixed in $\Gamma$, the first requirement of $\widehat{\Gamma}$ that $P$ be the image law of $\widehat{P}$ under $\varphi_{\widehat{\Omega}}$. The additional requirements involve extending the signals $s_{i,t}$ and the utilities $u$ from their domains in $\Gamma$ to their larger domains in $\widehat{\Gamma}$.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measure spaces, $((\widehat{X}, \widehat{\mathcal{X}}), \varphi_{\widehat{X}})$ and $((\widehat{Y}, \widehat{\mathcal{Y}}), \varphi_{\widehat{Y}})$ two expansions, and $f : X \to Y$ a function from $X$ to $Y$. An extension, $\widehat{f}$, is a function from $\widehat{X}$ to $\widehat{Y}$ that "corresponds" to $f$. For each $x \in X$, an extension, $\widehat{f}$, of $f$, has $A_x = \varphi_{\widehat{X}}^{-1}(x)$ as part of its domain, and $B_x = \widehat{f}(A_x)$ is the image, in $\widehat{Y}$, of this part of the domain. Thus, the set $C_x = \varphi_{\widehat{Y}}(B_x)$ should bear some relationship to the set $\{f(x)\}$ if $\widehat{f}$ is to be an extension of $f$.

**Definition 10.** *$\widehat{f} : \widehat{X} \to \widehat{Y}$ is an* **exact extension** *of $f$ if*

$$(\forall x \in X)[\,\{f(x)\} = C_x\,].$$

*It is a* **selection extension** *if*

$$(\forall x \in X)[\,\{f(x)\} \subset C_x\,].$$

In finitistic expansions, exact extensions appear when extending continuous functions, selection extensions appear when extending measurable functions.

**Definition 11.** *Let* $(X, \mathcal{X})$ *be a compact Hausdorff space with its Borel* $\sigma$*-field.* $((\widehat{X}, \widehat{\mathcal{X}}), \varphi_{\widehat{X}})$ *is a* **finitistic expansion** *of* $(X, \mathcal{X})$ *if*

1. $\widehat{X}$ *is a finitistic version of* $X$*,*
2. $\mathcal{N}$ *is a finitistic version of* $\mathcal{X}$ *that meets every element of* $\widehat{X}$*,*
3. $\widehat{\mathcal{X}} = \sigma(\mathcal{N})$*, i.e.* $\widehat{\mathcal{X}}$ *is the* $\sigma$*-field generated by* $\mathcal{N}$*, and*
4. $\varphi_{\widehat{X}}$ *is the standard part mapping.*

**Lemma 2.** *Suppose that* $(X, \mathcal{X})$ *and* $(Y, \mathcal{Y})$ *are two compact Hausdorff measure spaces with finitistic expansions, that* $\widehat{Y}$ *contains* ${}^*f(\widehat{X})$ *where* $f$ *is a function mapping* $X$ *to* $Y$*, and set* $\widehat{f} = {}^*f_{|\widehat{X}}$*. The for each* $x \in X$*,* $C_x = \varphi_{\widehat{Y}}(\widehat{f}(\varphi_{\widehat{X}}^{-1}(x))$ *is equal to the closed set,* $\bigcap_{U \in \tau, x \in U} \operatorname{cl} f(U)$ *where* $\tau$ *is the topology on* $X$*.*

**Proof**: Note that $x' \in \varphi_{\widehat{X}}^{-1}(x)$ if and only if $x'$ is in the intersection of the monad of $x$ and $\widehat{X}$. The rest follows from the Overspill principle (e.g. Lindstrøm, Thm. ??.??) and the exhaustiveness of $\widehat{X}$. ∎

In particular, if $f$ is continuous (respectively measurable), then $\widehat{f}$ is an exact (respectively a selection) expansion of $f$.

**Definition 12.** $\widehat{\Gamma}$ *is an* **(exact) expansion of** $\Gamma$ *if each measure space in* $\Gamma$ *is replaced by an expansion, and the signals and utilities in* $\widehat{\Gamma}$ *are selection (exact) expansions of the corresponding signals and utilities in* $\Gamma$*.*

The vector of $\varphi$'s associated with an expansion is denoted $\varphi_{\widehat{\Gamma}}$ or $\varphi_{\widehat{H}}$.

**Definition 13.** $\widehat{\sigma}^*$ *is an* **(exact) expansion equilibrium of** $\Gamma$ *if it is a equilibrium of an (exact) expansion* $\widehat{\Gamma}$*. In this case, the image law of* $\mathbb{O}(\widehat{\sigma}^*)$ *on* $H_T$ *under* $\varphi_{\widehat{\Gamma}}$ *is called an* **(exact) expanded equilibrium outcome**.

6.3. **Finitistic Expansions of Compact Extensive Form Games.** Fix a game

$$\Gamma = ((\Omega, P),\ (A_{i,t})_{i \in I, t \in \mathbb{T}},\ (S_{i,t}, s_{i,t}(\cdot))_{i \in I, t \in \{0\} \cup \mathbb{T}},\ (u_i(\cdot))_{i \in I}).$$

**Definition 14.** *The game* $\Gamma$ *is* **compact** *if* $H_T$ *is compact,* $\Gamma$ *is* **compact and continuous** *if it is compact and the signals and utilities are continuous.*

For the rest of this paper, *all games are assumed to be compact.* This directly contains much of the previous literature: [28] studies cheap talk expansions of compact and continuous signaling games; [16] and [17] study public signal expansions of compact and continuous $\Gamma$ with almost perfect information; [11] and [31] study extensive form correlation expansions of more general compact and continuous $\Gamma$. Following Appendix 2, it can be made to contain the rest of the previous literature.

**Definition 15.** *A finite set,* $F \in \mathcal{P}_{H_T}$*, is* **conformable** *if the finite approximation to each* $S_{i,t}$ *contains* $s_{i,t}(F')$ *where* $F'$ *is the projection of points in* $F$ *into the domain of* $s_{i,t}$*.*

Induction on $t$ shows that $\mathcal{P}^C_{H_T}$, the set of conformable finite approximations is rich and has a lattice structure. The $\mathbb{L}(\mathbb{E})$ sets are defined using limits along $\mathcal{P}^C_{H_T}$ rather than allowing non-conformable versions of the game.

**Definition 16.** *When* $\Gamma$ *is compact,* $\widehat{\Gamma}$ *is a* **finitistic expansion of** $\Gamma$ *if each measure space in* $\Gamma$ *is replaced by a conformable finitistic expansion, and for all* $E \subset \Omega$*,* $\widehat{P}({}^*E \cap F_\Omega) = P(E)$*.*

The last condition is the *equality a.a. condition* discussed in Section 4.2.4. As shown there, it is strictly stronger than the condition that $P$ is the image law of $\widehat{P}$ under $\varphi_{\widehat{\Omega}}$.

Lemma 2 delivers the following.

**Lemma 3.** *Finitistic expansions of* $\Gamma$ *are expansions, and if* $\Gamma$ *is compact and continuous, finitistic expansions are exact.*

The following Theorem says that the standard parts of $\epsilon$ equilibrium outcomes, $\epsilon \simeq 0$, in finitistic conformable versions of a game are the finitistic expanded equilibria. It is a direct consequence of the lifting and pushing down theorems of [2, Theorems 5.2 and 5.3].

**Theorem 3.** *If $\mathbb{E}^{\epsilon}(\cdot)$ is the set of $\epsilon$ Bayesian Nash equilibrium outcome correspondence, then $\mathbb{L}(\mathbb{E}^{+})$ is the set of finitistic expanded Bayesian Nash equilibrium outcomes. If $\Gamma$ is compact and continuous, then $\mathbb{L}(\mathbb{E}^{+})$ is the set of finitistic exact expanded Bayesian Nash equilibrium outcomes.*

Taking $\mathbb{E}$ as a strict subset of $\mathbb{E}^{\epsilon}$ picks out further closed non-empty subsets of the expanded equilibrium outcome sets. For example, if $\mathbb{E}(B)$, $B \in \mathcal{P}^{C}_{H_T}$, is the set of sequential equilibrium outcomes, then $\mathbb{L}(\mathbb{E})$ provides the finitistic sequential equilibria of $\Gamma$. Replacing "sequential" by "Bayesian Nash" or "divine" or "stable" provides the corresponding set.

## 7. Finitistic Expansions & Previous Work

The definition of expansions unifies the previous work on compact games, while finitistic expansions sharpen and simplify the previous expansion results in the literature. We take the different classes of compact games in increasing order of complexity.

### 7.1. **Normal Form Games.**

In summary, compact and continuous normal form games need no expansion, the class of compact normal form games with discontinuous payoffs does need expansion. Endogenous sharing rule equilibria (ESR's) are expansions. ESR's form a superset, strict for at least two kinds of reasons, of the set of finitistic expansions.

$\Gamma$ is a normal form game if $\Omega$ is a singleton set, $\Omega = \{\omega_0\}$, and the time set is $\mathbb{T} = \{1\}$. A small extension of [12] (to account for exhaustiveness) shows that compact and continuous normal form games need no expansion.

**Theorem 4** (Fudenberg and Levine)**.** *If $\Gamma$ is a compact and continuous normal form game, then $\sigma^*$ is an equilibrium if and only if $\sigma^*$ belongs to $\mathbb{L}(\mathbb{E}^{+})$ where $\mathbb{E}^{\epsilon}(B)$ is the set of $\epsilon$-equilibria for the finite version of $\Gamma$ played on $B$.*

Compactness by itself is easy to arrange for general measure spaces (e.g. [2, §3]), and Appendix 2 discusses why this means that there is no loss in assuming that $\Omega$ is compact. However, functions on a measure space that is itself a product of measure spaces are often difficult to extend to the product of separate compactifications. Algebraic, measure theoretic, and compactification characterizations of abstract normal form games in which measurable extensions exist are available in [18]. The set-valued theory of integration needed to integrate non-measurable functions, including those that arise in games, is available in [43].

When $\Gamma$ is a compact normal form game that fails to be continuous, equilibria may not exist. [40] study existence questions through a superset of the class of expansions called endogenous sharing rule equilibria (ESR's). Section 4.3.2 defines ESR's, and, after modifying the definition of $\widehat{u}$ on a null set, it can be shown that ESR's are selection expansions. Intuitively, the proof adds an unobserved, random device that determines the players payoffs in $\mathbf{co}\,(\psi(a))$ when $a$ is played ($\psi(a)$ is the set of limits of payoff vectors taken as $a^n \to a$). Because the random device is not observed by the players, the players choices in ESR's are not correlated.

Limits of products of independent randomizations form a subset of the convex hull. Example 4.8 exploited this to show that ESR's need not be interpretable as limits of finite approximations — taking the convex hull of $\psi(a)$ casts too wide a net. Example 4.7 shows that by failing to satisfy exhaustiveness, ESR's cast too wide a net in a different direction.

**Theorem 5.** *If $\mathbb{E}^{\epsilon}(B)$ is the set of $\epsilon$-equilibria for the finite version of the normal form game $\Gamma$ played on $B$, then $\mathbb{L}(\mathbb{E}^{+})$ is a closed, non-empty subset, strict in some games, of the endogenous sharing rule equilibria.*

**Proof**: The closure and non-emptiness of $\mathbb{L}(\mathbb{E}^{+})$ are established in Theorem 1. Pick an arbitrary finitistic version of $A = \times_{i\in I} A_i$, $F = \times_{i\in I} F_i$. Let $\mu^*$ be an $\epsilon$-equilibrium for the game played on $F$, $\epsilon \simeq 0$. It suffices to show that there exists a selection $v$ from the correspondence $a \mapsto \mathbf{co}\,(\psi(a))$ such that play of $\mu = \text{st}\,(\mu^*)$ is an equilibrium when the action spaces are the $A_i$ and the payoffs $v$. In outline:

1) $\mu^*$ will be modified to $\mu'$ by shifting an infinitesimal amount of weight so as to put weight only on points $f_i \in F_i$ that deliver payoffs infinitesimally close to the maximum achievable against $\mu^*$; 2) $v$ will be a carefully constructed version of the conditional expectation $E(u|\mathcal{S})$ where $\mathcal{S}$ is the minimal $\sigma$-field making the standard part mapping measurable; 3) by the exhaustiveness of the $F_i$, playing any $a_i$ against $\mu$ cannot deliver a payoff higher than playing $\mu_i$.

1) Let $C_i \subset F_i$ be the set of $f_i \in F_i$ such that $\mu_i^*(f_i) > 0$, and let $\overline{u}_i$ be the $^*$max of $\{u_i(\mu^* \backslash f_i : f_i \in F_x\}$. Define the internal set $D_i \subset {}^*\mathbb{R}_+$ as the set of $\delta_i$ such that there exists an $S_i \subset C_i$ such that

$$u_i(\mu^* \backslash f_i) \geq \overline{u}_i - \delta_i, \text{ and } \sum_{f_i \in S_i} \mu_i^*(f_i) > 1 - \delta_i. \tag{23}$$

Because $\mu^*$ is an $\epsilon$-equilibrium, $\epsilon \simeq 0$, $D_i$ contains arbitrarily small non-infinitesimal numbers. Because $D_i$ is internal, overspill implies that it contains an infinitesimal. Pick one such infinitesimal, call it $\delta_i'$, and let $S_i'$ be the corresponding $S_i$. Let $\mu_i'(\cdot)$ be $\mu^*(\cdot|S_i')$ and note that $\mu'$ is in the weak$^*$ monad of $\mu$ because $\int g \, d\mu' \simeq \int g \, d\mu^*$ for any continous $g$ on $A$ (only an infinitesimal amount of mass was moved and $g$ is bounded). Redefine $\overline{u}_i$ as the $^*$max of $\{u_i(\mu' \backslash f_i) : f_i \in F_x\}$, changing its value by at most an infinitesimal.

2) Let $\mathcal{S}$ denote the minimal sub-$\sigma$-field making the vector of standard part mappings from $F_i$ to $A_i$ measurable and let $T_i$ be the standard part of $S_i'$. The function $v = E(^\circ u|\mathcal{S})$ is $\mathcal{S}$-measurable, so, by a Theorem of Doob's (e.g. [8, Ch. 1, No. 18]), $v$ can be taken to be a function of $a$, $a \in \times_{i \in I} T_i$. For each $i \in I$, take $\mathcal{S}_i^n$ to be a sequence of $\sigma$-fields generated by a nested sequence of disjointifications of finite $\epsilon_i^n$ covers of $T_i$, $\epsilon_i^n \to 0$. By the martingale convergence theorem, $v^n = E(^\circ u| \otimes_{i \in I} \mathcal{S}_i^n)$ converges a.s. to $v$. By construction, the distance from $v^n(a)$ to $\mathbf{co}\,(\psi(a))$ converges to 0 a.s. We can measurably modify $v$ on a set of measure 0 if necessary (e.g. [20, Theorem ?.??, p. ??]) and conclude that $v$ is a measurable selection from $a \mapsto \mathbf{co}\,(\psi(a))$. Define $v$ to be an arbitrary selection from $\mathbf{co}\,(\psi(a))$ on the complement $\times_{i \in I} T_i$.

3) Pick arbitrary $i \in I$ and $a_i \in A_i$. Because integration against Loeb measures is infinitely close to $^*$finite summation and pushing down gives a Loeb measurable function,

(24)

$$E\, v_i = \int_A v_i(a)\, d\mu(a) = \int_F {}^\circ u_i(f)\, dL(\mu')(f) \simeq \sum_{(f_j)_{j \in I} \in \times_{j \in I} S_j'} u_i((f_j)_{j \in I}) \Pi_{j \in I} \mu_j'(f_j)$$

where $L(\mu')$ is the Loeb measure generated by $\mu'$. Because $\mu'$ is an $\epsilon'$-equilibrium, $\epsilon' \simeq 0$, the last term in (24) is infinitesimally close to $\overline{u}_i$, that is, playing $\mu_i$ against $(\mu_j)_{j\neq i}$ when the payoffs are given by $v$, $i$ gets utility $^\circ\overline{u}_i$. Consider deviating from $\mu_i$ to $a_i$. This gives payoff

(25)

$$\int_A v_i(a)\, d(\mu\backslash a_i)(a) = \int_F {}^\circ u_i(f)\, dL(\mu'\backslash a_i)(f) \simeq \sum_{(f_j)_{j\neq i}\in \times_{j\neq i} S'_j} u_i(a_i, (f_j)_{j\neq i})\Pi_{j\neq i}\mu'_j(f_j).$$

Because $F_i$ is exhaustive, $a_i \in F_i$. This implies that the last term is less than or equal to $\overline{u}_i$, implying $\int_A v_i(a)\, d(\mu\backslash a_i)(a) \leq {}^\circ\overline{u}_i$, so that $\mu$ is an equilibrium when the payoffs are given by $v$. ∎

Finitistic equilibria also deliver an upper hemicontinuous theory for normal form games. Suppose that $\Gamma(u)$ is a compact normal form game with possibly discontinuous payoffs $u$, and that $u' \in {}^*\mathcal{U}$ is a utility function satisfying $\|u - u'\|_\infty \simeq 0$. Then $\mu^*$ is an $\epsilon$-equilibrium for $\Gamma(u)$ and for some $\epsilon \simeq 0$ if and only if $\mu^*$ is an $\epsilon'$-equilibrium for $\Gamma(u')$ and for some $\epsilon' \simeq 0$. In terms of Theorem 2, this means that $\overline{\mathbb{L}}(\mathbb{E}^+, u) = \mathbb{L}(\mathbb{E}^+)$, so that the mapping $u \mapsto \mathbb{L}(\mathbb{E}^+)$ from $\mathcal{U}$ to $\times_{i\in I}\Delta(A_i)$ is upper hemicontinuous.

7.2. **The Simplest Extensive Form Games.** The simplest extensive form games have non-trivial $\Omega$ and signals while sharing the time set $\mathbb{T} = \{1\}$ with normal form games. The study of this class of compact and continuous $\Gamma$ has a long history, though it is still not known if these games have equilibria in general. The example in Section 4.2.4 shows that the set of outcome distributions is not generally closed. [30] gives conditions on $P$ under which the set of outcome distributions is closed and the set of equilibria is non-empty and closed. Without the conditions on $P$, [5] gives a complicated existence proof when this class is expanded by the addition of a correlating device.

Passing through a finitistic expansion of $\Gamma$, [42] proves that $\mathbb{L}(\mathbb{E}^+)$ is a closed, non-empty subset, sometimes strict, of the correlated equilibria. In more detail, finitistic signals are informationally innocuous in this class of games — conditioning on the finitistic signal tells the players no more about any $E \subset \Omega$ than conditioning on the original signal. It is also true that any finitistic expansion is equivalent to expanding the game by the addition of a standard correlating device — each finitistic expanded equilibrium, that is, each element of $\mathbb{L}(\mathbb{E}^+)$, is an expanded equilibrium for an expansion in which: $\widehat{\Omega}$ is $\Omega \times [0,1]$; $\varphi_{\widehat{\Omega}}$ is projection; the probability $\widehat{P}$ is $P \times \lambda$; the expanded signals, $\widehat{s}_{i,0}$ are of the form $(s_{i,0}, s'_{i,0})$.

7.3. **Signaling Games.** If the expansion of a signaling game is again a signaling game, then each expansion outcome is the outcome of some cheap talk expansion. If it also true that $\widehat{u}$, the expanded utility function, is an exact extension of $u$, then each expansion equilibrium outcome is the equilibrium outcome of some cheap talk expansion. Finitistic expansions of signaling games are also signaling games, and they are exact expansions if the payoffs are continuous. This means that finitistic equilibrium outcomes form a closed and non-empty subset of the cheap talk equilibrium outcomes. The construction of a game in which they form a strict subset is below. The notation for signaling games was set in §4.2.2.

**Theorem 6.** *If $\widehat{\Gamma}$ is an expansion of a compact metric signaling game $\Gamma$ in which the Receiver sees the Sender's expanded action and nothing more, $\widehat{s}_2(\widehat{h}) \equiv \widehat{a}_1$, then every expanded outcome for $\Gamma$ can be realized as the outcome of a cheap talk expansion.*

Expansions in which $\widehat{s}_2$ can be directly informative about $\widehat{\omega}$ are not signaling games.

**Proof**: Fix expanded strategies $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ for players 1 and 2. The proof consists of showing that there is a vector of cheap talk strategies that induce the outcome $\varrho = \varphi_{\widehat{\Gamma}}(\mathbb{O}(\widehat{\sigma}_1, widehat\sigma_2)$.

Let $\widehat{Q}$ be the marginal of $\varrho$ on on $\Omega \times \widehat{A}_1$. Taking $\sigma_1'(\omega)$ to be a regular conditional probability (rcp) for $Q$ with respect to $\omega$ gives a strategy in $\Gamma$ that induces $Q$. Because $\widehat{s}_2(\widehat{h}) \equiv \widehat{a}_1$, all that is left is to show that the marginal of $\varrho$ on $A_1 \times A_2$ can be induced by cheap talk strategies.

Let $\widehat{R}$ be the marginal of $\varrho$ on $A_1 \times A_2$. Taking an rcp for $R$ with respect to $a_1$ gives a measurable mapping $a_1 \mapsto R_{a_1}$ such that for $R$-a.a. $a_1$, $R_{a_1}$ is concentrated on $\{a_1\} \times A_2$. Let $p(a_1)$ denote the marginal of $R_{a_1}$ on $A_2$. Play of the cheap talk strategy vector

$$(\sigma_1^{ct}, \sigma_2^{ct}) = ((\sigma_1'(\omega), p(\sigma_1'(\omega)), p(\sigma_1')) \tag{26}$$

induces $\varrho$. Because $[0,1]$ is Borel isomorphic to $\Delta(A_2)$ (all uncountable Borel subsets of compact metric spaces are Borel isomorphic), this strategy can be recast in the cheap talk form described above. ∎

**Theorem 7.** *If $\widehat{\Gamma}$ is an expansion of a compact metric signaling game $\Gamma$ in which $\widehat{s}_2(\widehat{h}) \equiv \widehat{a}_1$ and for every outcome $h$, $\{u(h)\} = \{\widehat{u}(\varphi_{\widehat{\Gamma}}^{-1}(h)\}$, then every expanded equilibrium outcome for $\Gamma$ is also cheap talk equilibrium outcome.*

**Proof**: Proof goes here. ∎

Theorems 6 and 7 directly imply

**Theorem 8.** *If $\mathbb{E}^{\epsilon}(B)$ is the set of $\epsilon$-equilibria for the compact and continuous signaling game $\Gamma$ played on the conformable, finite set $B$, then $\mathbb{L}^{+}(\mathbb{E})$ is a closed, non-empty subset of the cheap talk equilibrium outcomes for $\Gamma$.*

A crucial aspect of the strict subset result is the fact that $\mathrm{st}^{-1}(a)$ is a singleton set if $a$ is an isolated point.

7.4. **Games of Almost Perfect Information.** The set of finitistic expansion equilibria is a closed and non-empty subset, strict in some games, of the public randomization equilibria.

Again, a crucial aspect of the strict subset result is the fact that $\mathrm{st}^{-1}(a)$ is a singleton set if $a$ is an isolated point.

7.5. **A General Result on Exact Expansions.**

**Theorem 9.** *Forges-Myerson equilibria are exact expanded equilibria.*

## 8. Conclusions

We are arguing that a fundamental piece of our mathematical models in game theory should be expanded, not an argument lightly made. That the theory of games needs expansion is clear: Not having equilibria in simple games is not acceptable for a general theory of games; Precluding obvious equilibrium phenomena is equally unacceptable; Failures of closure of an equilibrium set mean that limits of equilibria may not be equilibria, and this is, at best, very hard to interpret. A potential counter argument to the need for expansion is the existence of many games in which the usual model of infinite sets do not cause problems. For such games, there is no need to switch to finitistic models. However, even in these games, finitistic expansions are often the easiest method of discovering whether or not such problems arise.

A second class of potential objections arise from the observation that finitistic sets are non-unique in two very different fashions. First, there are many different finitistic versions of any given set. Second, as noted in [24], within a given model of Zermelo-Frankel set theory, there is a unique (up to isomorphism) model of the continuum, but there are many nonstandard models. The first non-uniqueness is no more than the observation that there are many sequences/nets of finite sets converging to a given infinite set. Rather than picking one particular sequence/net and deciding it is the best one, we allow all sequences. Correspondingly, the nonstandard definition of $\mathbb{L}(\mathbb{E})$ is the union, taken over finitistic $F$, of the sets $\mathbb{E}(F)$. The second non-uniqueness is irrelevant for a simpler reason — independent of the choice of model of nonstandard objects, the nonstandard definitions used here are equivalent to standard limit definitions.

At a more general level, the choice between different models of quantities depends on the stories one wishes to tell, not on some deeper underlying truth.[10] There is no deep reason to prefer $\mathbb{R}$ over $\mathbb{Q}_2$ (the dyadic rationals) or $\mathbb{Q}_{10}$ (the decimal rationals) or some finitistic version of $\mathbb{R}$ as a model of continuous quantities. Distinguishing between any pair of these four models of quantities requires

[10]Finitistic sets and nonstandard probabilities have been used to provide a strictly coherent model of beliefs in infinite contexts ([41, §3]). Conditioning by Bayes' Law in finitistic probability spaces is equivalent to Popper systems of beliefs in infinite contexts ([26, Theorem 1]). Continuous time martingales and stochastic integration have transparent finitistic formulations ([1], [33]).

being able to distinguish "lengths" infinitely smaller than $1,000^{-1,000}$ times "diameter" of the smallest known subatomic particle, and it is not clear that this is meaningful.

The marvelous edifice of real analysis, supported by the usual model of the continuum, is internally consistent and powerful. Having all "lengths" represented, including the length of the diagonal of a unit square, is crucial to the reasoning in some fields — imagine not having a way to refer to the Gaussian distribution in statistics or the length of a diagonal in geometry. However, more is needed for game theory, and the "more" can only be represented using the usual models once one has figured out what the "more" is. With the proposed finitistic additions, the "more" is automatically represented, and, by minimality, only what is needed is added.

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## Appendix A. Primer on Nonstandard Analysis

The nonstandard constructions presented here are "all the same." We start with an object or class of objects, $G$. The nonstandard version of $G$, denoted by ${}^*G$ (read "star $G$"), is defined as the set of equivalence classes of $G$-valued sequences, where the equivalence relation is chosen carefully. There is a strong parallel with the construction of the real numbers as equivalence classes of Cauchy sequences rational numbers.

A.1. **A Cauchy detour.** A sequence in $G$ is a mapping from an infinite directed set $(\mathbb{M}, \geq)$ to $G$. A sequence $x^m$ in the rationals, $\mathbb{Q}$, is a **Cauchy sequence** if

$$(\forall q > 0,\ q \in \mathbb{Q})(\exists M \in \mathbb{M})(\forall m, m' \geq M)[|x^m - x^{m'}| < q].$$

Two Cauchy sequences, $x^m, y^m$, in $\mathbb{Q}$ are **Cauchy equivalent** if

$$(\forall q > 0,\ q \in \mathbb{Q})(\exists M \in \mathbb{M})(\forall m \geq M)[|x^m - y^m| < q].$$

The set of real numbers, $\mathbb{R}$, is, by definition, the set of equivalence classes of Cauchy sequences in $\mathbb{Q}$. Letting $[x^m]$ denote the equivalence class of a sequence

$\{x^m\}$, the operations of addition and multiplication are defined pointwise: $[x^m] + [y^m] = [x^m + y^m]$ and $[x^m] \cdot [y^m] = [x^m \cdot y^m]$. The mapping $q \mapsto [q, q, q, \ldots]$ embeds $\mathbb{Q}$ in $\mathbb{R}$, so that $\mathbb{Q}$ is regarded as a subset of $\mathbb{R}$. Any sequence approaching an irrational number is not equivalent to any $[q, q, q, \ldots]$, implying that $\mathbb{R}$ is a strict expansion of $\mathbb{Q}$.

A.2. **The equivalence relation for nonstandard expansions.** The equivalence relation, $\sim$, for constructing nonstandard versions of $G$ from the set of all sequences in $G$ is based on a finitely additive, $\{0, 1\}$-valued measure $\mu$ on the set of all subsets of $\mathbb{M}$, such that $\mu(\mathbb{M}) = 1$, and $\mu$ is equal to 0 on all finite sets.[11] Two sequences, $\{a^m\}$ and $\{b^m\}$ in $G$ are $\mu$-equivalent, written $\{a^m\} \sim_\mu \{b^m\}$, if $\mu\{n \in \mathbb{M} : a^m = b^m\} = 1$. ${}^*G$ is, by definition, the set of $\sim$ equivalence classes. The set ${}^*G$ is, for most but not all purposes, independent of $\mathbb{M}$ and $\mu$.

The following salient properties of $\mu$ are easily derived from its definition and imply that $\sim$ is an equivalence relation.

1. If $\{A_k : 1 \leq k \leq K\}$ is a finite partition of $\mathbb{M}$, then $\mu(A_k) = 1$ for one and only one of the $A_k$;
2. If $A = \cap\{A_k : 1 \leq k \leq K\}$ is a finite intersection of subsets of $\mathbb{M}$ such that $\mu(A_k) = 1$ for $1 \leq k \leq K$, then $\mu(A) = 1$.

Let $\langle a^m \rangle$ denote the equivalence class of a sequence $\{a^m\}$ in $G$. The mapping $a \mapsto \langle a, a, a, \ldots \rangle$ embeds $G$ in ${}^*G$, and $G$ is regarded as a subset of ${}^*G$. Equivalence classes of constant sequences are called standard points, others points in ${}^*G$ are called nonstandard. Standard points are denoted by the same symbol whether we are thinking of them as elements of $G$ or of ${}^*G$, e.g. $0 = \langle 0, 0, 0, \ldots \rangle$.

If $G$ is finite, then ${}^*G$ is equal to $G$ — any sequence $\{a^m\}$ in $G$ can be partitioned into finitely many sets $A_g = \{m \in M : a^m = g\}$, $g \in G$. By the first property of $\mu$, exactly one $A_g$, call it $g'$, has $\mu$-mass 1. Thus, $\{a^m\}$ is in the equivalence class of the constant sequence $\{g', g', \ldots\}$, written $\langle a^m \rangle = g'$. If $G$ is infinite, then ${}^*G$ is strictly larger than $G$. For example, the nonstandard point $\langle x^m \rangle$ in ${}^*\mathbb{R}$ is **infinitesimal** if $x^m$ is a sequence converging to 0 with each $x^m \neq 0$. This is written $\langle x^m \rangle \simeq 0$. If $\langle x^m \rangle \simeq 0$ and $\langle y^m \rangle \simeq 0$, then $\langle x^m \rangle + \langle y^m \rangle \simeq 0$ by the continuity of addition. For another type of nonstandard real, consider the point $\langle z^m \rangle$ in ${}^*\mathbb{R}$ when $z^m$ is a sequence increasing without bound. In that case $\langle z^m \rangle$ is called **infinite**.

A.3. **Extending relations and functions to ${}^*G$.** Given a relation $R$ on $G$, it is extended to ${}^*G$ by defining $\langle x^m \rangle R \langle y^m \rangle$ if $\mu\{m : x^m R y^m\} = 1$. Several examples:

[11] Zorn's Lemma implies the existence of a free ultrafilter, $\mathcal{U}$, on $\mathbb{M}$. Setting $\mu(A) = 1$ if $A \in \mathcal{U}$, and $\mu(A) = 0$ otherwise gives a measure with these properties. Conversely, any such measure determines a free ultrafilter.

1. $\langle x^m \rangle < \langle y^m \rangle$ if $\mu\{m : x^m < y^m\} = 1$, so that
2. if $\langle x^m \rangle$ is infinitesimal, then for every $\epsilon \in \mathbb{R}_{++}$, $0 < \langle x^m \rangle < \epsilon$, and
3. for every $n \in \mathbb{N}$, $n < \langle 1/x^m \rangle$.
4. Addition and multiplication are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, functions can be identified with their graphs and their graphs are relations. Following this logic, $\langle x^m \rangle + \langle y^m \rangle$ is defined as $\langle x^m + y^m \rangle$ and $\langle x^m \rangle \cdot \langle y^m \rangle$ as $\langle x^m \cdot y^m \rangle$.
5. If $S \subset \mathbb{R}$, then ${}^*S$ is the set of elements, $\langle x^m \rangle$, of ${}^*\mathbb{R}$ such that $\mu\{m : x^m \in S\} = 1$, written ${}^*S \subset {}^*\mathbb{R}$.
6. If $f$ is a function on $S \subset \mathbb{R}$, then $f(\langle x^m \rangle) = \langle f(x^m) \rangle$ so that if $\langle x^m \rangle \neq 0$, then $1/\langle x^m \rangle = \langle 1/x^m \rangle$.

Metric spaces play a crucial role in this paper. For any set $X$, the definition of ordered pairs gives ${}^*(X \times X) = {}^*X \times {}^*X$. A metric $d$ on $X$ is a mapping from $X \times X$ to $\mathbb{R}_+$, and its extension to ${}^*X$ is, following the logic above, defined by $d(\langle x^m \rangle, \langle y^m \rangle) = \langle d(x^m, y^m) \rangle \in {}^*\mathbb{R}_+$.

A.4. **Standard parts, transfer, and monads.** As one's familiarity with nonstandard objects grows, the need to keep track of the equivalence class construction shrinks. Here is the definition of standard parts and nearstandard points in a metric space $(X, d)$ without [and with] the equivalence classes notation.

**Definition 17.** *If there exists an $x \in X$ such that $d(x', x) \simeq 0$, then $x$ is called the* **standard part of** $x' \in {}^*X$. *A point $x' \in {}^*X$ is* **nearstandard** *if it has a standard part. [If there exists an $x \in X$ such that $\langle d(x, x^m) \rangle$ is the equivalence class of some sequence converging to 0, then $x$ is called the* **standard part of the equivalence class** $\langle x^m \rangle \in {}^*X$. *A point $\langle x^m \rangle \in {}^*X$ is* **nearstandard** *if it has a standard part.]*

Taking the standard part of a point in ${}^*X$ is much like taking the limit of a sequence in $X$. In particular, the limit may not exist, but if it does, it is unique. The uniqueness of limits is an application of the triangle law, for all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$. Using the equivalence class construction, a uniqueness proof can easily be transmuted into a proof of the statement that there is at most one standard part of a point in ${}^*X$. An alternative strategy of proof is to **transfer** the triangle law directly to ${}^*X$, i.e. to note that $d(x^m, y^m) + d(y^m, z^m) \geq d(x^m, z^m)$ for all $m$ implies that for all $x, y, z \in {}^*X$, $d(x, y) + d(y, z) \geq d(x, z)$. With this in hand, suppose that $x \neq z$ are two points in $X$ so that $d(x, z)$ is a strictly positive number in $\mathbb{R}$, while $d(x, y) \simeq 0$ and $d(y, z) \simeq 0$ both hold, i.e. $x$ and $z$ are both the standard part of $y$. The triangle law implies that $d(x, y) + d(y, z) \geq d(x, z)$, a contradiction that implies that the standard point of $y \in {}^*X$, if it exists, is unique.

The transfer principle is a powerful tool, a statement $P$ about objects in $G$ is true if and only if the corresponding statement, ${}^*P$, is about objects in $G$.

A statement of the transfer principle requires a bit of formal logic and will not be given here, though illustrative examples of its use are given below. At the simplest level, they expand on the notion used above — if it's true for every $m$ in the sequence, then it's true for the nonstandard equivalence class.

The monad of a point $x \in X$ is defined by $m(x) = \text{st}^{-1}(x)$ where $\text{st}(\cdot)$ maps the nearstandard points in ${}^*X$ to their standard parts in $X$. Thinking of the sequence construction of ${}^*X$, $m(x)$ consists of the set of equivalence classes of sequences converging to $x$. Since (generalized) sequences characterize topologies, so do monads. Here are three examples involving metric spaces $(X, d)$ and $(Y, \rho)$ with proofs leaning on transfer rather than the sequence construction.

**Lemma 4.** *$x$ is isolated if and only if $m(x) = \{x\}$.*

**Proof**: $x$ is isolated if and only if there exists a $\delta > 0$ in $\mathbb{R}$ such that

$$(\dagger) \quad (\forall y \in X)[d(x, y) < \delta \Leftrightarrow y = x].$$

If $x$ is isolated and $y \in m(x)$, then $y$ must satisfy $d(x, y) < \delta$. Transfer of the statement $(\dagger)$ is

$$({}^*\dagger) \quad (\forall y \in {}^*X)[d(x, y) < \delta \Leftrightarrow y = x].$$

If $x$ is not isolated, then

$$(\ddagger) \quad (\forall \epsilon \in \mathbb{R}_{++})(\exists y \in X)[y \neq x \ \& \ d(x, y) < \epsilon].$$

Transfer of the statement $(\ddagger)$ yields

$$({}^*\ddagger) \quad (\forall \epsilon \in {}^*\mathbb{R}_{++})(\exists y \in {}^*X)[y \neq x \ \& \ d(x, y) < \epsilon].$$

Any $y$ guaranteed by $({}^*\ddagger)$ when $\epsilon \simeq 0$ is another point in $m(x)$. ∎

**Lemma 5.** *A function $f : X \to Y$ is continuous at $x$ if and only if $d(x, x') \simeq 0$ implies $\rho(f(x), f(x')) \simeq 0$, that is, $f(m(x)) \subset m(f(x))$.*

**Proof**: Thinking of $m(x)$ as the sequences in $X$ converging to $x$ and $m(f(x))$ as the sequences in $Y$ converging to $f(x)$, the result is intuitively clear.

$f$ is continuous at $x$ iff $(\forall \delta \in \mathbb{R}_{++})(\exists \epsilon \in \mathbb{R}_{++})[\,[d(x, x') < \epsilon] \Rightarrow [\rho(f(x), f(x')) < \delta]\,]$. Pick any $\delta \in \mathbb{R}_{++}$ and any corresponding $\epsilon$. Transfer

$$(\dagger) \quad (\forall x' \in X)[\,[d(x, x') < \epsilon] \Rightarrow [\rho(f(x), f(x')) < \delta]\,] \text{ yields}$$

$$({}^*\dagger) \quad (\forall x' \in {}^*X)[\,[d(x, x') < \epsilon] \Rightarrow [\rho(f(x), f(x')) < \delta]\,],$$

so that $d(x, x') \simeq 0$ implies $\rho(f(x), f(x')) < \delta$. Since $\delta$ was arbitrary, $\rho(f(x), f(x')) \simeq 0$.

If $f$ is not continuous at $x'$, then there exists a $\delta > 0$ such that

$$(\ddagger) \quad (\forall \epsilon \in \mathbb{R}_{++})(\exists x' \in X)[d(x, x') < \epsilon \ \& \ \rho(f(x), f(x')) \geq \delta].$$

Transfering (‡) and taking $\epsilon \simeq 0$ gives an $x' \in {}^*X$ such that $d(x, x') \simeq 0$ but $\rho(f(x), f(x')) \geq \delta$. ∎

**Lemma 6.** *If $(X, d)$ is compact, then every $x' \in {}^*X$ is nearstandard.*[12]

**Proof**: Pick an arbitrary $x' \in {}^*X$ where $X$ is compact. Let $D$ denote the diameter of $X$. To start the induction, set $A_0 = X$ and note that $x'$ belongs to ${}^*A_0$ for the closed $A_0 \subset X$ of diameter less than or equal to $D/2^0$. Suppose that $x'$ belongs to ${}^*A_n$ for some closed $A_n$ of diameter less than or equal to $D/2^n$. Take a finite cover of $A_n$ by $D/2^{n+1}$-balls and disjointify it into the sets $A_{n,k}$, $k \in \{1, \ldots, k(n)\}$. Transfer of the statement

$$(\dagger) \quad (\forall x \in X)(\exists k \in \{1, \ldots, k(n)\})[x \in A_{n,k}] \text{ gives}$$

$$({}^*\dagger) \quad (\forall x \in {}^*X)(\exists k \in {}^*\{1, \ldots, k(n)\})[x \in {}^*A_{n,k}].$$

Because $k(n)$ is finite, ${}^*\{1, \ldots, k(n)\}$ is equal to $\{1, \ldots, k(n)\}$, implying that $x'$ belongs to one of the ${}^*A_{n,k}$. Define $A_{n+1}$ as the closure of $A_{n,k}$. This induction gives a nested sequence of closed sets, $A_n$, with diameter converging down to 0 such that $x' \in {}^*A_n$ for each $n$. Because $X$ is compact, $\cap_n A_n = \{x\}$ for some $x \in X$. Since $x, x' \in {}^*A_n$ for each $n$ and the diameter of ${}^*A_n$ is less than or equal to $D/2^n$, $d(x, x') \simeq 0$, i.e. $x$ is the standard part of $x'$. ∎

A.5. **Star-finite sets, exhaustiveness, and saturation.** Of particular interest for this paper is the case where $G$ contains the class, $\mathcal{P}(X)$, of finite subsets of a compact metric space $X$. An element of ${}^*\mathcal{P}(X)$ is a **star-finite subset of** $X$. A key property of star-finite sets is their exhaustiveness — an $F \in {}^*\mathcal{P}(X)$ is **exhaustive** if for every $x \in X$, $x \in F$. Exhaustiveness requires more structure for $\mu$ and $\mathbb{M}$.

An example: Suppose that $\mathbb{M}$ is the integers, and $F = \langle F^m \rangle$ where $F^m$ is the set of $m$'th order dyadic rationals, $F^m = \{k/2^m : k = 0, 1, \ldots, 2^m\}$. For every dyadic rational $q$, $\{m : q \notin F^m\}$ is finite so that $\mu\{m : q \in F^m\} = 1$, i.e. $q \in F$. Statements true about finite sets can be transfered to give statements that are true about $F$ even though every dyadic rational belongs to $F$. To have a star-finite $F$ that contains every $x \in X$ for general $X$ requires a property called **saturation**.

A collection $A_b$, $b \in B$, has the finite intersection property (fip) if for any finite $B' \subset B$, $\cap_{b \in B'} A_b \neq \emptyset$. A subset $A$ of ${}^*G$ is **internal** if it is the equivalence class of some sequence of subsets of $G$. The purely finitely additive, $\{0, 1\}$-valued measure $\mu$ is **$\kappa$-saturated** for ordinal $\kappa$ if every collection of internal sets $A_b$,

[12] The result that every $x' \in {}^*X$ is nearstandard if and only if $X$ is compact is sometimes known as Robinson's Theorem.

$b \in B$, having the fip and indexed by a set $B$ satisfying $\#B < \kappa$, $\cap_{b \in B} A_b \neq \emptyset$. From e.g. [21, Theorem III.1.2], if $\mathbb{M}$ has infinite cardinality $\kappa$, there exists a $\kappa^+$-saturated $\mu$ where $\kappa^+$ is the successor of $\kappa$. In other words, if any given level of saturation is needed, it is available. $\mu$ is **polysaturated** if it is saturated for some $\kappa$ greater than the cardinality of the set $V(G)$ where $V(G)$ is inductively defined by $G^0 = G$, $G^{n+1} = G^n \cup 2^{G^n}$, and $V(G) = \cup_{n \in \mathbb{N}} G^n$.

With saturation, the existence of exhaustive star-finite sets is easy. For every $B \in \mathcal{P}(X)$, let $A_B$ denote the finite subsets of $X$ containing $B$. The collection of internal sets ${}^*A_B$ is indexed by $\mathcal{P}(X)$, a set having the same cardinality as $X$. In any model that is $\kappa$-saturated for a sufficiently large $\kappa$, e.g. polysaturated, $\cap_{B \in \mathcal{P}(X)} {}^*A_B \neq \emptyset$, and any $F \in \cap_{B \in \mathcal{P}(X)} {}^*A_B$ is exhaustive.

A.6. **Weak* limits, the standard part mapping, Loeb measures.** Soley to avoid (even more) complications, we treat only compact metric spaces.[13] $\Delta(X)$ denotes the set of Borel probability measures on the compact metric space $(X, d)$. The weak* topology on $\Delta(X)$ is compact, and can be metrized in many ways, all equivalent to $P^n \to_{w^*} P$ if and only if $\int f\, dP^n \to \int f\, dP$ for all $f \in C(X)$.

For $B \in \mathcal{P}(X)$, $\Delta(B) \subset \Delta(X)$ denotes the set of probabilities supported on the set $B$. Fix an exhaustive star-finite $F \subset {}^*X$. Any $P \in \Delta(F) \subset {}^*\Delta(X)$ has a weak* standard part, $\mathrm{st}_{\Delta(X)}(P)$. At issue is the relation between $\mathrm{st}_{\Delta(X)}(\cdot)$ and the standard part mapping from ${}^*X$ to $X$. If $P$ is point mass on some $x \in F$, then $\mathrm{st}_{\Delta(X)}(P)$ is point mass on $\mathrm{st}\,(x) \in X$ — for any $f \in C(X)$, $\int f\, d\delta_x = f(x) \simeq f(\mathrm{st}\,(x))$. In this case, it is exactly true that if we draw a point in $F$ according to $P$ and take the standard part, the resulting distribution on $X$ is $\mathrm{st}_{\Delta(X)}(P)$. For general $P \in \Delta(F)$, it is difficult to speak of drawing a point according to $P$ because we have neither a $\sigma$-field nor a countably additive, $[0, 1]$-vauled probability. This is (one place) where Loeb measures arise.

Let $\mathcal{X}$ denote the Borel $\sigma$-field on $X$. ${}^*\mathcal{X}$ is a field of internal subsets of $X$. The next Lemma shows that ${}^*\mathcal{X}$ is not closed under countable union (unless $X$ is finite).

**Lemma 7** (Loeb)**.** *If $A_n$, $n \in \mathbb{N}$, is a sequence of internal sets, $A = \cup_n A_n$ is internal if and only if $A = \cup_{n \leq N} A_n$ for some integer $N$.*

**Proof**: The finite union of internal sets being internal is easy, so suppose that $A$ is internal. $A \backslash A_n$, $n \in \mathbb{N}$, is also a sequence of internal sets, and it satisfies $\cap_n (A \backslash A_n) = \emptyset$. By saturation, the collection $(A \backslash A_n)$ must fail to have the fip, i.e. $\cap_{n \leq N} (A \backslash A_n) = \emptyset$ for some $N$, implying $A = \cup_{n \leq N} A_n$. ∎

The mapping ${}^\circ P : {}^*\mathcal{X} \to [0, 1]$ is a finitely additive probability on a field of sets. By Lemma 7 and Caratheodory's extension theorem, ${}^\circ P$ has a unique extension to

[13] [3] and [23] contain more complete treatments.

$\sigma({}^*\mathcal{X})$. This extension, due to Loeb [22], is called a **Loeb measure** and denoted by $L(P)$. The relevant facts are (1) the mapping $\text{st} : {}^*X \to X$ is measurable, and (2) for every $E \in \mathcal{X}$, $L(P)(\text{st}^{-1}(E)) = \text{st}_{\Delta(X)}(P)(E)$. Thus, drawing a point in $(F, \sigma({}^*\mathcal{X}))$ according to $L(P)$ and taking the standard part is the same as drawing a point in $X$ according to the weak* standard part of $P$.

A.7. **Connections of nonstandard analysis to this paper.** In this paper, infinite sets of actions and signals in games are replaced by exhaustive star-finite sets. The resulting nonstandard games have equilibria by transfer of Nash's existence theorem or the more refined existence theorems that build on Nash's result. These equilibria give rise to a distribution, $P$, on a star-finite set of histories. The standard part of these distributions are the finitistic expansion equilibrium outcomes. If $h$ is a history for the game and $h$ is not isolated, $\text{st}^{-1}(h)$ will contain many elements in addition to $h$. These elements provide the extra signals and actions needed to make the theory of games well-behaved.

## Appendix B. Compactifications of $\Omega$

Imbed $\Omega \times H$ in $\overline{\Omega} \times H$ in the usual fashion, extend the utility functions (being measurable that is easy), and make sure that $\overline{\Omega}$ is compact.