HONORS INTRODUCTORY GAME THEORY
ECONOMICS 354K (UNIQUE #34495), FALL 2007

MAXWELL B. STINCHCOMBE

0. Organization

Basics: We meet Tuesday evenings, 5-8 p.m. in BRB 2.136. from 9:30-10:50 a.m. in BRB 2.136. The Unique # is 29245.

Reaching me: My office is BRB 2.118, phone 475-8515, office hours are MWF 2-3:30 p.m., and by appointment, e-mail `maxwell@eco.utexas.edu`.

Texts: The required textbook for this class is Eric Rasmusen's *Games and Information: An Introduction to Game Theory* 4'th ed., Blackwell, 2007. For the first half of the semester, we will basically cover one textbook chapter per class meeting, details below. given below. The second half of the semester will cover the textbook chapters on moral hazard and adverse selection as well as read and discuss much Carol Heimer's sociological study of the insurance industry, *Reactive Risk and Rational Action*, UC Press, 1985.

Notes, including assignments and selected exposition of material to be covered in class will be handed out as the class progresses. I also strongly recommend *Games, Strategies, and Managers* by John McMillan, Oxford University Press, 1992. The text also recommends a large number of other possible sources.

Timing for the first half of the semester

Week, date	Topic(s)	Reading
#1, Sep. 4	Examples of games and equilibria	Ch. 1
#2, Sep. 11	Information and actions	Ch. 2
#3, Sep. 18	Mixed and continuous strategies	Ch. 3
#4, Sep. 25	Dynamic games, simple information	Ch. 4
#5, Oct. 2	Reputation and repeated games	Ch. 5
#6, Oct. 9	Repeated and more general dynamic games	Ch. 5-6
#7, Oct. 16	Dynamic games, incomplete information	Ch. 6

Evaluation: There are homework assignments (45%), one take home mid-term (15%), a 5-10 page writing assignment (10%), and a final exam (30%).

In more detail:

(1) There are weekly homework assignments, due at the beginning of class meetings. The solutions will be discussed when the assignments are due, and late assignments will not be accepted.

I encourage you to work together on assignments. However, I strongly discourage you from merely copying someone else's work. If you "free ride" on others' efforts, you will not understand what is being taught sufficiently well to pass the class.

(2) The take-home mid-term will count for 15% of your grade. It will be handed out Tuesday October 16, and be due two days later, by 5 p.m., Thursday October 18.

(3) The writing assignment, worth 10%, is to choose/find an economic situation that you have observed or read about and to present an analysis of it in the form that Rasmusen uses. By mid-October, I will want to have talked to all of you about this, by mid-November, you talk through an outline with me, it is due the Tuesday November 20'th, the meeting before Thanksgiving.

(4) The final, cumulative exam will be given on Tuesday December 18, 7-10 pm, place TBA. It will count for the remaining 30% of your grade.

Background: I will assume that you have had a good introduction to microeconomics, especially the utility maximization foundation of the theory of consumer choice and the theory of the firm. I will also assume a good working knowledge of differential and integral calculus as well as familiarity with basic probability and statistics (cumulative distribution functions (cdf's), probability density functions (pdf's), and the relations between them; medians, means, variances, standard deviations; the central limit theorem and the Gaussian (or normal) distribution.

Topics: The course is in introduction to the basics of game theory as it applies to economics. We will analyze both static and dynamic situations, and pay special attention to the role of uncertainty. Examples will cover topics in insurance markets, marketing, entry deterrence, personnel economics, political science, location economics, cartels, and others.

1. Outline of First Meeting, First Homework

Tuesday, September 4, 2007

1.1. **Definitions.**

1.1.1. *Games.* $\Gamma = (A_i, u_i)_{i \in I}$. Comparison w/ PAPI.

1.1.2. *Equilibrium.* Dfn, too strong, minimalist.

1.2. **Examples.**

1.2.1. *The Advantage of Being Small.* Here $I = \{\text{Little,Big}\}$, $A_1 = A_2 = \{\text{Push, Wait}\}$. This is called a 2×2 game because there are two players with two strategies apiece. The utilities are given in the following table.

Rational Pigs

	Push	Wait
Push	$(-c, b-c)$	$(-c, b)$
Wait	$(\alpha b, (1-\alpha)b - c)$	$(0, 0)$

where $0 < \alpha < 1$ and $(1-\alpha)b > c > 0$.

Some conventions: The representation of the choices has player 1, listed first, in this case Little, choosing which row occurs and player 2, in this case Big, choosing which column; each entry in the matrix is uniquely identified by the actions a_1 and a_2 of the two players, each has two numbers, (x, y), these are $(u_1(a_1, a_2), u_2(a_1, a_2))$, so that x is the utility of player 1 and y the utility of player 2 when the vector $a = (a_1, a_2)$ is chosen.

There is a story behind the game: there are two pigs, one Little and one Big, and each has two actions. The two pigs are in a long room. There is a lever at one end, which, when pushed, gives an unpleasant electric shock, $-c$ in utility terms, and gives food at the trough across the room, worth utility b. The Big pig can shove the Little pig out of the way and take all the food if they are both at the food output together, and the two pigs are equally fast getting across the room. However, during the time that it takes the Big pig to cross the room, the Little pig can eat α of the food. With $0 < \alpha < 1$ and $(1-\alpha)b > c > 0$, the Little soon figures out that, no matter what Big is doing, Pushing gets nothing but a shock on the (sensitive) snout, so will Wait by the trough. In other words, the strategy Push is **dominated** by the strategy Wait.

Once Little has figured this out, Big will Push, then rush across the room, getting $(1-\alpha)$ of the food. The unique pure strategy Nash equilibrium is $(Wait, Push)$. Note that the Little pig is getting a really good deal. There are situations where the largest person/firm has the most incentive to provide a public good, and the littler ones have an incentive to free ride. This game gives that in a pure form.

Lemma 1.1. *If Γ is dominance solvable, then the unique strategy that survives iterated deletion of dominated strategies is the unique Nash equilibrium.*

1.2.2. *Mixed or Random Strategies.* If you have played Hide and Seek with very young children, you may have noticed that they will always hide in the same place, and that you need to search, while loudly explaining your actions, in other places while they giggle helplessly. Once they actually understand hiding, they begin to *vary* where they hide, they *mix* it up, they *randomize*. Randomizing where one hides is the only sensible strategy in games of hide and seek.

If you have played or watched a game such as tennis, squash, ping pong, or volleyball, you will have noticed that the servers, if they are any good at all, purposefully randomize where they are serving. If one, for example, always serves the tennis ball to the same spot, the opponent will move so as to be able to hit that ball in the strongest possible fashion.

Notation 1.1. *For a finite set, S, $\Delta(S) := \{p \in \mathbb{R}^S : p \geq 0,\ \sum_s p_s = 1\}$ denotes the set of probability distributions over a set S.*

Definition 1.1. *A* ***mixed strategy for*** *i is an element $\sigma_i \in \Delta_i := \Delta(A_i)$. A* ***mixed strategy for a game*** *Γ is an element $\sigma \in \times_{i\in I}\Delta_i$.*

We assume that when people pick their strategies at random, they do so **independently**. Because independent probabilities are multiplied, the expected utility of j to a strategy $\sigma = (\sigma_i)_{i\in I}$ is

$$U_i(\sigma) = \sum_a u_j(a)\Pi_{i\in I}\sigma_i(a_i). \tag{1}$$

Example 1.1. *Let $I = \{1, 2\}$, $A_1 = \{Up,\ Down\}$, $A_2 = \{Left,\ Right\}$, $\sigma_1 = (\frac{1}{3}, \frac{2}{3})$ and $\sigma_2 = (\frac{1}{3}, \frac{2}{3})$. The following three distributions over A both have (σ_1, σ_2) as marginal distributions, but only the first one has the choices of the two players independent.*

	Left	*Right*
Up	$\frac{1}{9}$	$\frac{2}{9}$
Down	$\frac{2}{9}$	$\frac{4}{9}$

	Left	*Right*
Up	$\frac{1}{3}$	0
Down	0	$\frac{2}{3}$

	Left	*Right*
Up	$\frac{1}{6}$	$\frac{1}{6}$
Down	$\frac{1}{6}$	$\frac{1}{2}$

If 1's payoffs are given by

	Left	*Right*
Up	9	4
Down	17	-3

then 1's payoffs to independent randomization with the marginals $\sigma_1 = (\frac{1}{3}, \frac{2}{3})$ and $\sigma_2 = (\frac{1}{4}, \frac{3}{4})$ are $U_1(\sigma_1, \sigma_2) = 9 \cdot (\frac{1}{3} \cdot \frac{1}{4}) + 4 \cdot (\frac{1}{3} \cdot \frac{3}{4}) + 17 \cdot (\frac{2}{3} \cdot \frac{1}{4}) - 3 \cdot (\frac{2}{3} \cdot \frac{3}{4})$.

If player 2 is playing $Left$ with probability β and Right with probability $(1-\beta)$, then 1's payoff to Up is $U_1((1,0),(\beta,(1-\beta)) = 9\beta + 4(1-\beta)$, to Down is $U_1((0,1),(\beta,(1-\beta)) = 17\beta - 3(1-\beta)$, and to playing Up with probability α, Down with probability $(1-\alpha)$ is $U_1((\alpha,(1-\alpha)),(\beta,(1-\beta)) = \alpha U_1((1,0),(\beta,(1-\beta)) + (1-\alpha)U_1((0,1),(\beta,(1-\beta))$.

For any $\sigma_{-i} \in \times_{j\neq i}\Delta_j$, a player's payoffs are linear in their own probabilities.

Lemma 1.2 (Own probability linearity)**.** *For all $\sigma \in \Delta$ and all $i \in I$, the mapping $\mu_i \mapsto U_i(\sigma\backslash\mu_i)$ is linear, i.e. $U_i(\sigma\backslash\gamma\mu_i + (1-\gamma)\nu_i) = \gamma U_i(\sigma\backslash\mu_i) + (1-\gamma)U_i(\sigma\backslash\nu_i)$.*

Proof. For i's action $b \in A_i$, let $v_i(b) = \sum_{a_{-i}} u_i(b, a_{-i}) \cdot \Pi_{j\neq i}\sigma_j(a_j)$. Fon any μ_i, $U_i(\sigma\backslash\mu_i) = \sum_b v_i(b)\mu_i(b)$. ∎

Definition 1.2. *$\sigma^* \in \Delta$ is a **Nash equilibrium for** Γ if for all $i \in I$ and all $\mu_i \in \Delta_i$, $U_i(\sigma^*) \geq U_i(\sigma^*\backslash\mu_i)$. The set of Nash equilibria for Γ is denoted $Eq(\Gamma)$.*

For $b \in A_i$, we use $b \in \Delta_i$ to denote the probability distribution on A_i that puts mass 1 on b. If we wanted to be more careful, and make this harder to read, we would instead use δ_b, the probability defined by $\delta_b(E) = 1_E(b)$.

Homework 1.1. *Using the linearity Lemma, show that σ^* is a Nash equilibrium iff for all $i \in I$ and all $b \in A_i$, $U_i(\sigma^*) \geq U_i(\sigma^*\backslash b)$.*

1.2.3. *Inspection games.* In the following game, there is a worker who can either Shirk, or put in an Effort. The boss can either Inspect or Not. Inspecting someone who is working has an opportunity cost, $c > 0$, finding a Shirker has a benefit $b > c$. The worker receives w if they Shirk and are not found out, 0 if they Shirk and are Inspected, and $w - e > 0$ if they put in the effort, whether or not they are Inspected. In matrix form, the game is

	Inspect	Don't inspect
Shirk	$(0, b-c)$	$(w, 0)$
Effort	$(w-e, -c)$	$(w-e, 0)$

Just as in the childrens' game of hide-and-seek, there cannot be an equilibrium in which the two players always choose one strategy. For there to be an equilibrium, there must be randomization.

Homework 1.2. *Referring to the inspection game just given, let α be the probability that 1 shirks and β the probability that 2 inspects.*

(1) As a function of β, 2's probability of inspecting, find 1's best response. In particular, find the critical value β^* at which 1 is indifferent between shirking and putting in an effort.
(2) As a function of α, 1's probability of shirking, find 2's best response. In particular, find the critical value α^* at which 2 is indifferent between inspecting and not inspecting.
(3) Show that $((\alpha^*, (1-\alpha^*), (\beta^*, (1-\beta^*))$ is the unique Nash equilibrium for this game.
(4) Show that the equilibrium probability of shirking goes down as the cost of monitoring goes down, but that the probability of monitoring is independent of the monitor's costs and benefits.

1.2.4. *Trust in electronic commerce.* The E-Bay auction for a Doggie-shaped vase of a particularly vile shade of green has just ended. Now the winner should send the seller the money and the seller should send the winner the vile vase. If both act honorably, the utilities are $(u_b, u_s) = (1, 1)$, if the buyer acts honorably and the seller dishonorably, the utilities are $(u_b, u_s) = (-2, 2)$, if the reverse, the utilities are $(u_b, u_s) = (2, -2)$, and if both act dishonorably, the utilities are $(u_b, u_s) = (-1, -1)$. In matrix form this is the game

		Seller	
		Honorable	Dishonorable
Buyer	Honorable	$(1, 1)$	$(-2, 2)$
	Dishonorable	$(2, -2)$	$(-1, -1)$

In many ways, this is a depressing game to think about — no matter what the other player is doing, acting dishonorably is a best response for both players. This is, again, a case of dominance.

Definition 1.3. *A strategy $\mu_i \in \Delta_i$* ***dominates*** *a strategy $a_i \in A_i$ and a_i is a* ***dominated strategy*** *for agent i if for all $\sigma \in \Delta$, $U_i(\sigma\backslash\mu_i) > U_i(\sigma\backslash a_i)$.*

It is possible that a mixed strategy dominates a pure strategy.

Homework 1.3. *Show that in the following game, the mixed strategy $(\frac{1}{2}, 0, \frac{1}{2})$ on (T, M, B) dominates the pure strategy $(0, 1, 0)$ for player 1, but that no pure strategy dominates it. (Only player 1's utilities are given.)*

	L	R
T	9	0
M	2	2
B	0	9

Returning to the E-Bay example above, suppose that for a (utility) cost s, $0 < s < 1$, the buyer and the seller can mail their obligations to a third party intermediary that will hold the payment until the vase arrives or hold the vase until the payment arrives, mail them on to the correct parties if both arrive, and return the vase or the money to the correct party if one side acts dishonorably. Thus, each person has three choices, send to the intermediary, honorable, dishonorable. The payoff matrix for the symmetric, 3×3 game just described is

		Seller		
		Intermed.	Honorable	Dishonorable
	Intermed.	1-s , 1-s	1-s , 1	-s , 0
Buyer	Honorable	1 , 1-s	1 , 1	-2 , 2
	Dishonorable	0 , -s	2 , -2	-1 , -1

Homework 1.4. *The first three questions are about finding the unique equilibrium, the last two lead to interpretations.*

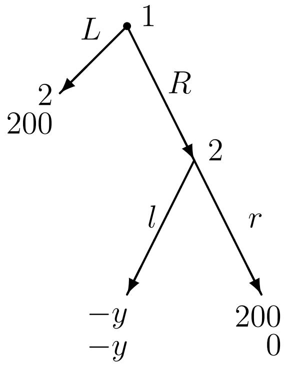
1. *(1) Show that there is no pure strategy equilibrium for this game.*
2. *(2) Show that there is no mixed strategy equilibrium involving the seller playing exactly two strategies with strictly positive probability.*
3. *(3) Find the unique mixed strategy equilibrium and its expected utility as a function of s.*
4. *(4) For what values of s is the probability of dishonorable behavior lowest? Highest?*
5. *(5) If the intermediary is a monopolist, what will they charge for their services?*

One of the really perverse aspects of this situation is that the availability of an intermediary is what makes trade possible, but people are willing to incur the cost of the intermediary only because there continues to be cheating. This is much like a monitoring game.

It is the S.E.C (Securities and Exchange Commision) and honest accounting that has, historically, made the vigor of the U.S. stock market possible. We expect to have a positive frequency of cheating, as well as the usual quota of stupidity, willful blindness, and other forms of incompetence. The complicated question is how much cheating is too much?

1.2.5. *Perfect equilibria and idle threats.* In Puccini's *Gianni Schicchi*, the wealthy Buoso Donati has died and left his large estate to a monastery. Before the will is read by anyone else, the relatives call in a noted mimic, Gianni Schicchi, to play Buoso on his deathbed, re-write the will, and then convincingly die. The relatives explain, very carefully, to Gianni Schicchi, just how severe are the penalties for anyone caught tampering with a will (at the time, the penalities included having one's hand cut off). The plan is put into effect, but, on the deathbed, Gianni Schicchi, as Buoso Donati, rewrites the will leaving the entire estate to the noted mimic and great artist, Gianni Schicchi. The relatives can expose him, *and thereby expose themselves too*, or they can remain silent.

Let player 1 be Gianni Schicchi, who has two actions while playing Donati on his deathbed, L being leave the money to the relatives, R being will it to the noted artist and mimic Gianni Schicchi. Let player 2 represent the relatives, who have two actions, l being report Gianni to the authorities, and r letting him get away with the counter-swindle. Inventing some utility numbers with $y \gg 200$ to make the point, we have



	l	r
L	$(2, 200)$	$(2, 200)$
R	$(-y, -y)$	$(200, 0)$

The box on the right gives the 2×2 representation of the extensive form game. There are two kinds of equilibria for this game.

Homework 1.5. *Show the following:*

(1) For all $\beta \geq 198/(200 + y)$, the strategy $(L, (\beta, (1 - \beta))$ is an equilibrium giving payoffs of $(2, 200)$.
(2) (R, r) is an equilibrium giving payoffs $(200, 0)$.
(3) There are no other equilibria.

The equilibria with $\beta > 0$ can be interpreted as the relatives making the threat, "If you screw us, we go to the authorities." Your analysis just showed that Gianni believing the threat is an equilibrium. However, since following through on the threat means that the relatives also suffer the loss of their right hand, you might believe that the threat is an "idle" one, that is, it is a threat that the relatives would not follow through on if Gianni "called their bluff."

One way to get at this logic is to note that $\beta > 0$ is only a best response because Gianni's choice of L means that there is no possibility that the relatives will be called on to follow through on their threat. Reinhart Selten's notion of a perfect equilibrium forces people to pay attention to all possibilities.

1.2.6. *Fisheries, the tragedy of the commons.* There are I different countries that can put out fishing fleets to catch from pelagic schools of fish. Use the number $a_i \in \mathbb{R}_+ = A_i$ to represent the number of fishing boats in the fleet of country i, $i = 1, \ldots, I$. To finish specifying the game, the utilities, $u_i : A \to \mathbb{R}$ need to be specified.

The marginal cost of a boat is constant, and equal to c. For given $a \in A$, let $n = n(a) = \sum_{i \in I} a_i$ and $n_{-i}(a) = \sum_{j \neq i} a_j$. When the total number of boats is n, the **per boat** return is $v(n)$ where $v(0) > 0$, $v'(n) < 0$, $v'(0) < c$, and $v''(n) < 0$. For country i, the benefit to putting out a fleet depends on the size of their own fleet, a_i, and the size of the other countries' fleets, $n_{-i}(s)$,

$$u_i(a_i, n_{-i}(a)) = a_i v(a_i + n_{-i}(a)) - ca_i = a_i v(n(a)) - ca_i.$$

For fixed $n_{-i} = n_{-i}(a)$, $u_i(\cdot, n_{-i})$ is concave because $\frac{\partial u_i(a_i, n_{-i})}{\partial a_i} = v(n) + a_i v'(n) - c$, implying

$$\frac{\partial^2 u_i(a_i, n_{-i})}{\partial a_i^2} = v'(n) + a_i v''(n) + v'(n) < 0.$$

The simultaneous satisfaction of the best response conditions gives the Nash equilibrium. Compare these to the FOC from any Pareto optimal point. In this game, the Nash equilibrium is inefficient compared to binding agreements to limit fleet size, and the inefficiency grows with I. Strategic considerations do not lead to socially optimal solutions.

In this game, the players' strategy sets could be taken to be interval subsets of $\mathbb{R}$, and the equilibrium, being interior to the intervals, was not at a vertex. Theorem 4.1 in some work by Anderson and Zame tells us that the set of utility functions for which all non-vertex equilibria fail to be Pareto optimal is open and finitely prevalent.

In this particular game, there is a fairly easy way to see what is involved in their argument. Let $(a_1^*, a_2^*) \gg (0, 0)$ be an equilibrium. The indifference curve of 1 in the $A_1 \times A_2$ square comes tangent, from below, to the line $a_2^* \times [0, M]$ at the eq'm point (a_1^*, a_2^*), and vice versa. Draw these two, note that when we go down and to the left, both are better off.

Crucial to this argument is the observation that neither player is indifferent, at the eq'm, to what the other is doing. This allows for us to have an area where we can push both players' actions and make both better off. What Anderson and Zame do is to show that for essentially all utility functions, at an interior pure strategy eq'm, no-one is indifferent to the actions of others.

1.3. **The First Fundamental Theorem of Welfare Economics.** This by way of comparison.

1.4. **Some additional problems.**

Homework 1.6. From ER*:* 1.3.

Homework 1.7. From ER*:* 1.6.

Homework 1.8. From ER*: Any additional problem from the end of Chapter* 1.

Homework 1.9. *Two countries put out fishing fleets of size s_1 and s_2 into international waters. The marginal cost of a boat is constant, and equal to* 10*. When the total number of boats is $s = s_1 + s_2$, the* **per boat** *return is*

$$v(s) = 100 - \frac{1}{10,000} s^2$$

so that $v(0) > 0$, $v'(s) < 0$ for $s > 0$, $v'(0) < c$, and $v''(s) < 0$. For country i, the benefit to putting out a fleet depends not only on the size of their own fleet, s_i, but also on the *size of the other country's fleet, s_j, $j \neq i$. Specifically, their utility is*

$$u_i(s_1, s_2) = s_i v(s_1 + s_2) - cs_i = s_i v(s) - cs_i.$$

(1) *Show that for fixed s_j, $u_i(\cdot, s_j)$ is concave.*

(2) *An equilibrium is a pair (s_1^*, s_2^*) such that each s_i^* solves the problem*

$$\max_{s_i \geq 0} u_i(s_i, s_j^*).$$

Find the equilibrium for this game.

(3) *Find the socially optimal fishing fleet size. Compare it to the equilibrium fishing fleet size.*

Homework 1.10. *A person accused of a crime by the District Attorney, with the help of the police department, has a probability p, $0 < p < 1$, of being guilty. At the trial,*

(1) *if the person is guilty, the evidence indicates that they are guilty with probability e_{guilty}, $\frac{1}{2} < e_{guilty} < 1$, and*

(2) *if the person is innocent, the evidence indicates that they are guilty with probability $e_{innocent}$, $0 < e_{innocent} < \frac{1}{2}$.*

Give and graph the probability, as a function of p, that the person is guilty conditional on the evidence indicating guilt. Suppose that the jury convicts every time the evidence indicates that the defendant is guilty. If $e_{guilty} = 0.98$ and $e_{innocent} = 0.01$, how reliable must the District Attorney and the police department be in order to have the false conviction rates lower than 0.1% (one in a thousand)?

2. Third Meeting, September 18, 2007

Topics: Extensive form game trees, extensive form games, Bayes Law, the Harsanyi doctrine, signaling games.

2.1. **Homeworks.** Due: Tuesday, September 25, 2007.

Homework 2.1. From ER: 2.1.

Homework 2.2. From ER: 2.2.

Homework 2.3. From ER: 2.3.

Homework 2.4. From ER: 2.5.

Homework 2.5. From ER: 2.7.

Homework 2.6. From ER: 2.8.

Homework 2.7. *Draw the extensive form game (not just the tree) for the Png settlement game in the textbook and verify that the two equilibria described on pages* 63 *and* 64 *are indeed equilibria.*

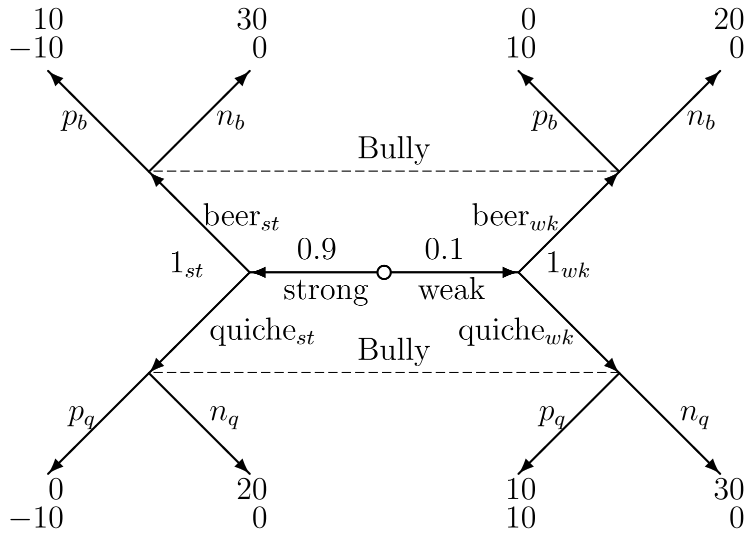
2.2. **Bayes Law.** Guilt/Innocence, Cancer/No cancer, Random Drug testing, werewolves.

States we care about, A, B, random signals X with $P(X = x\big|A)$, $P(X = x\big|B)$, observe $X = x$, want $P(A\big|X = x)$ and $P(B\big|X = x)$. Derive it, noting the need for $P(A)$ and $P(B)$. With more than two states, it's the same formula except there's a summation in the denominator.

Applications without much game theory: 4 given above; regression to the mean; cascades.

2.3. **Signaling games. Will that be beer, or quiche for breakfast?**

This game is due to Cho and Kreps (1987), who tell a version of the following story: There is a fellow who, on 9 out of every 10 days on average, rolls out of bed like Popeye on spinach. When he does this we call him "strong." When strong, this fellow likes nothing better than Beer for breakfast. On the other days he rolls out of bed like a graduate student recovering from a comprehensive exam. When he does this we call him "weak." When weak, this fellow likes nothing better than Quiche for breakfast. In the town where this schizoid personality lives, there is also a swaggering bully. Swaggering bullies are cowards who, as a result of deep childhood traumas, need to impress their peers by picking on others. Being a coward, he would rather pick on someone weak. He makes his decision about whether to pick, p, or not, n, after having observed what the schizoid had for breakfast. With payoffs, the game tree is



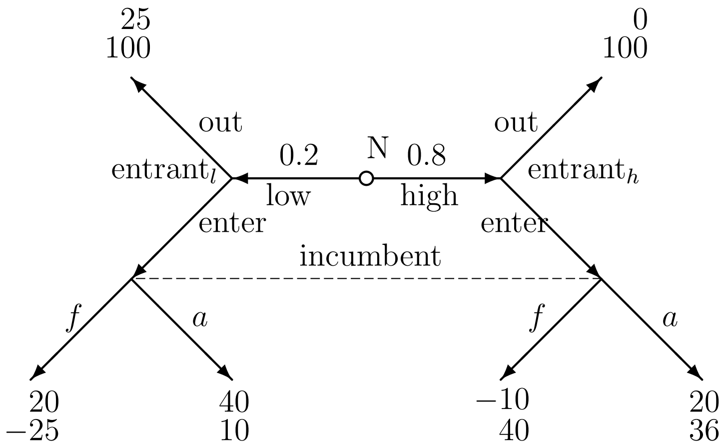
This is not only a psycho-drama, it's also a story about entry-deterrence. To get to that story, re-interpret the schizoid as an incumbent who may or may not have found a cost-saving technology change. If they have, it makes them a stronger competitor. This private information is not known to player 2, the potential entrant. The incumbent can start an aggressive advertising campaign, something they'd like better if they were strong, that is, if they have found the cost-saving change to their technology. The potential entrant would rather compete against a weak incumbent than a strong incumbent, but can condition their entry decision only on whether or not they see the aggressive advertising campaign.

Homework 2.8. *Give the* 4×4 *normal form for this game and find the two set of equilibria.*

Two other variants of entry-deterrence

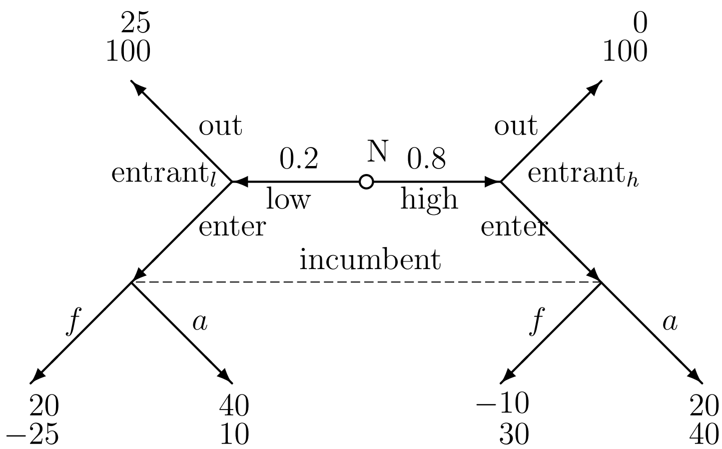
Consider a market with one incumbent firm and a potential entrant. The potential entrant has low costs with probability 0.2, and has high costs with probability 0.8. The actual costs (low or high) of the entrant are private information to the entrant, who decides whether to stay "out" or "enter." The outside option for the low cost entrant has an expected utility of 25, while the outside option for the high cost entrant has an expected utility of 0. If the potential entrant stays out, then the incumbent has an expected utility of 25. If the potential entrant enters, then the incumbent decides whether to "fight" or "acquiesce." If the incumbent fights, then the payoffs for the entrant and incumbent respectively are $(20, -25)$ when the entrant is low cost, and $(-10, 40)$ when the entrant is high cost. If the incumbent acquiesces, then the payoffs are $(40, 10)$ when the entrant is low cost, and $(20, 36)$ when the entrant is high cost.

One version of the game tree for this game is

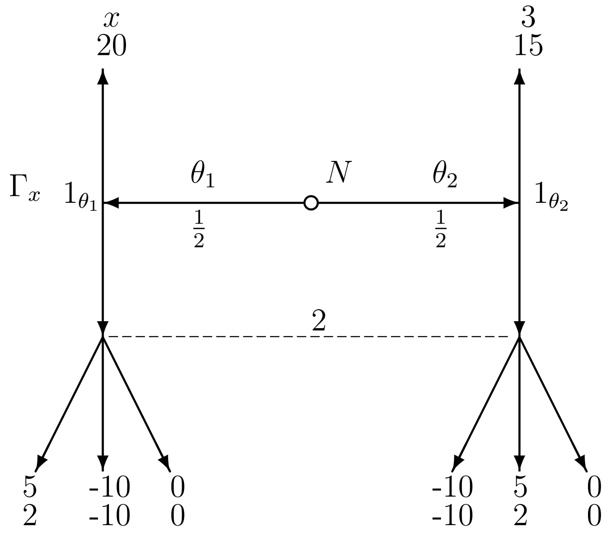


A **separating equilibrium** is one in which all the different types of Senders take different actions, thereby separating themselves from each other. A **pooling equilibrium** is one in which all the different types of Senders take the same action. The only Nash equilibria of the game just given are pooling equilibria, and all of them are sequential.

Here is a slightly changed version of the above game.

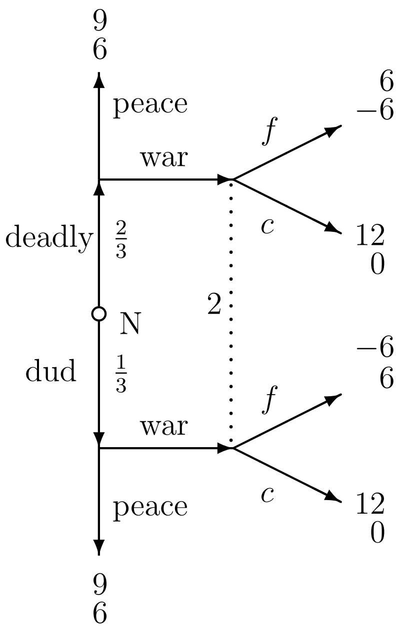


Same structure, less story



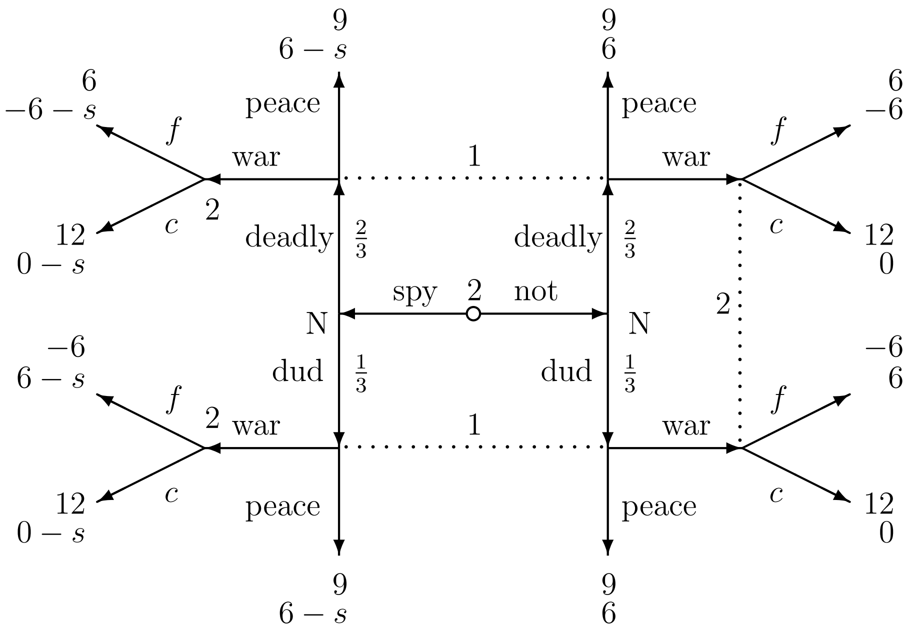
Going to war as a signal

With probability 2/3, country 1's secret military research program makes their armies deadlier (i.e. giving higher expected utility in case of war through higher probability of winning and lower losses), and with probability 1/3 the research project is a dud (i.e. making no change in the army's capacities). Knowing whether or not the research program has succeeded, country 1 decides whether or not to declare war on or to remain at peoce with country 2. Country 2 must decide how to respond to war, either *f*ighting or *c*eding territory, all of this without knowing the outcome of 1's research program. With payoffs, one version of the game tree is:



	f	c
(p, p)	$(9, 6)$	$(9, 6)$
(p, w)	$(4, 6)$	$(10, 4)$
(w, p)	$(7, -2)$	$(11, 2)$
(w, w)	$(2, -2)$	$(12, 0)$

Let us modify the previous game by adding an earlier move for country 2. Before country 1 starts its research, country 2 can, at a cost $s > 0$, insert sleepers (spies who will not act for years) into country 1. Country 1 does not know whether or not sleepers have been inserted, and if sleepers are inserted, country 2 will know whether or not 1's military research has made them deadlier. One version of the game tree is:



Later, we will evaluate, in the context of these two versions of this game, the statement, "When spying is cheap, it lowers the probability of war."

3. Fourth and Fifth Meetings, September 25 and October 2, 2007

Topics: Continuous strategy sets and mixed strategy sets; mixed equilibria in patent races, games of divided responsibility; continuous strategies in Cournot, Bertrant, Stackelberg games; supermodularity.

3.1. **Homeworks.** Due: Tuesday, October 9, 2007.

Homework 3.1. From ER: 3.5.

Homework 3.2. From ER: 3.12.

Homework 3.3. From ER: 3.14.

Homework 3.4. From ER: 3.15.

Homework 3.5. From ER: *Any other two problems from Chapter 3.*

3.2. **Games with Large Numbers of Players.** We will look at two games and find what happens as N, the number of players becomes large. For this to make sense, we need to think of players as being identical (or at least identically distributed).

3.2.1. *Divided Responsibility.* N players, will analyze as $N \uparrow$. When $N = 2$, we have the following game.

	Ignore	Phone
Ignore	$(0, 0)$	$(10, 7)$
Phone	$(7, 10)$	$(7, 7)$

Pure strategy eq'a with assigned responsibility. Role of neighborhood watches. Symmetric mixed equilibrium, find, find probability that no-one calls.

With $N > 2$ players, i's payoffs are 0 if all ignore, 10 if at least one other person phones, 7 if i phones. Again, pure strategy eq'a with assigned responsibility. Symmetric mixed equilibrium, find, find probability that no-one calls.

Summary of results: As $N \uparrow$, each individual ignores more (hoping someone else will phone), and the aggregate effect is that ignoring happens with a higher and higher probability, though the upper bound on the probabilities is less than 1.

3.2.2. *Cournot Competition.* In 1838, the first known use of what we now call Nash equilibrium was published by Augustin Cournot. His idea was to study a homogeneous good with a given demand curve and to see what individuals would do if they best responded to each other in a sequence.

Homework 3.6 (Cournot competition). *Suppose that the demand function for a product is $P = 120 - Q$ and the marginal cost of production is 12.*

(1) Find the monopoly industry price, quantity, and profits.

(2) Find the competitive industry price, quantity, and profits. Show that this production maximizes social welfare.

(3) Suppose that there are two firms that choose quantities $q_1, q_2 \geq 0$*, receiving profits* $\pi_i(q_1, q_2) = q_i(120 - (q_1 + q_2)) - 12q_i$.

(a) Show that $\partial^2\pi_i/\partial q_1 \partial q_2 < 0$.

(b) Find the best response curves for the two firms.

(c) Find the equilibrium industry price, quantity, and profits.

(4) Suppose that there are I *firms that choose quantities* $q_i \geq 0$, $i \in I$*, receiving profits* $\pi_i(q_i, q_{-i}) = q_i(120 - (\sum_j q_j)) - 12q_i$. *Show that the equilibrium industry price, quantity and profits converges to the competitive levels as* I *becomes large.*

3.3. **Slopes of Best Response Curves.** We will get to a review of monopoly social welfare arguments.

Assume that X and T are interval subsets of $\mathbb{R}$, and that f is twice continuously differentiable. f_x, f_t, f_{xx} and f_{xt} denote the corresponding partial derivatives of f. To have $f_x(x, t) = 0$ characterize $x^*(t)$, we must have $f_{xx} < 0$ (this is a standard result about concavity in microeconomics). From the implicit function theorem, we know that $f_{xx} \neq 0$ is what is needed for there to exist a function $x^*(t)$ such that

$$f_x(x^*(t), t) \equiv 0. \tag{2}$$

To find dx^*/dt, take the derivative on both sides with respect to t, and find

$$f_{xx}\tfrac{dx^*}{dt} + f_{xt} = 0, \tag{3}$$

so that $dx^*/dt = -f_{xt}/f_{xx}$. Since $f_{xx} < 0$, this means that dx^*/dt and f_{xt} have the same sign.

This ought to be intuitive: if $f_{xt} > 0$, then increases in t increase f_x; increases in f_x are increases in the marginal reward of x; and as the marginal reward to x goes up, we expect that the optimal level of x goes up. In a parallel fashion: if $f_{xt} < 0$, then increases in t decrease f_x; decreases in f_x are decreases in the marginal reward of x; and as the marginal reward to x goes down, we expect that the optimal level of x goes down.

There are two versions of Bertrand competition: Bertrand's, which was a strong critique of Cournot's reasoning about demand for goods that are homogeneous; and differentiated good Bertrand.

Example 3.1 (Bertrand's response to Cournot)**.** *Suppose (as above) that the demand function for a product is* $P = 120 - Q$ *and the marginal cost of production is* 12. *Suppose that prices are the strategies and that profits are given by* $\pi_i(p_i, p_j) = (p_i - 12)(120 - p_i)$ *if* $p_i < p_j$, $\pi_i(p_i, p_j) = \frac{1}{2}(p_i - 12)(120 - p_i)$ *if* $p_i = p_j$*, and* $\pi_i(p_i, p_j) = 0$ *if* $p_i > p_j$. *There is a unique equilibrium for this game, it involves the competitive price and quantity as soon as* $N = 2$.

Example 3.2 (Differentiated Bertrand competition)**.** *$I = \{1, 2\}$, $\pi_i(p_i, p_j) = (24 - 2p_i + p_j)(p_i - c)$, $\partial^2 \pi_i / \partial p_i \partial p_j > 0$, find best response curves. Do iterated deletion of dominated strategies.*

This idea of increasing marginal utilities has a more general formulation.

Definition 3.1. *For $X, T \subset \mathbb{R}$, a function $f : X \times T \to \mathbb{R}$ is **supermodular** if for all $x' \succ x$ and all $t' \succ t$,*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t),$$

equivalently

$$f(x', t') - f(x', t) \geq f(x, t') - f(x, t).$$

*It is **strictly supermodular** if the inequalities are strict.*

Notice that we wrote "$t' \succ t$" instead of "$t' > t$." This is not just a notational quirk. We will return to it just below. For right now, this may seem to be a cryptic remark.

At t, the benefit of increasing from x to x' is $f(x', t) - f(x, t)$, at t', it is $f(x', t') - f(x, t')$. This assumption asks that benefit of increasing x be increasing in t. A good verbal shorthand for this is that f **has increasing differences in** x **and** t. The sufficient condition in the differentiable case is $\forall x, t,\ f_{xt}(x, t) \geq 0$.

Theorem 3.1. *If $f : X \times T \to \mathbb{R}$ is supermodular and $x^*(t)$ is the largest solution to $\max_{x \in X} f(x, t)$ for all t, then $[t' \succ t] \Rightarrow [x^*(t') \succsim x^*(t)]$.*

If there are unique, unequal maximizers at t' and t, then $x^*(t') \succ x^*(t)$.

Proof: Suppose that $t' \succ t$ but that $x' = x^*(t') \prec x = x^*(t)$. Because $x^*(t)$ and $x^*(t')$ are maximizers, $f(x', t') \geq f(x, t')$ and $f(x, t) \geq f(x', t)$. Since x' is the largest of the maximizers at t' and $x \succ x'$, we know a bit more, that $f(x', t') > f(x, t')$. Adding the inequalities, we get $f(x', t') + f(x, t) > f(x, t') + f(x', t)$, or

$$f(x, t) - f(x', t) > f(x, t') - f(x', t').$$

But $t' \succ t$ and $x \succ x'$ and supermodularity imply that this inequality must go the other way. ∎

Clever choices of T's and f's can make some analyses criminally easy.

Example 3.3 (Review Monopolies and Social Welfare)**.** *Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$ so that $CS(p) = \int_p^\infty x(r)\, dr$ is the consumer surplus when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.*

The monopolist's profit when they produce x is $\pi(x) = x \cdot p(x) - c(x)$ where $c(x)$ is the cost of producing x. The maximization problem for the monopolist is

$$\max_{x \geq 0}\ \pi(x) + 0 \cdot CS(p(x)).$$

Society's surplus maximization problem is

$$\max_{x \geq 0} \ \pi(x) + 1 \cdot CS(p(x)).$$

Set $f(x,t) = \pi(x) + tCS(p(x))$ *where* $X = \mathbb{R}_+$ *and* $T = \{0,1\}$*. Because* $CS(p(x))$ *is nondecreasing,* $f(x,t)$ *is supermodular.*[1] *Therefore* $x^*(1) \geq x^*(0)$*, the monopolist always (weakly) restricts output relative to the social optimum.*

Here is the externalities intuition: increases in x increase the welfare of people the monopolist does not care about, an effect external to the monopolist; the market gives the monopolist insufficient incentives to do the right thing.

Back to the cryptic remark about "$\succ$" rather than "$>$."

Example 3.4. *Return to the Cournot game, define* $q_1' \succ q_1$ *if* $q_1' > q_1$ *and define* $q_2' \succ q_2$ *if* $q_2' < q_2$*. Now each* π_i *is supermodular.*

3.4. **Stackelberg Games.** The classical Stackelberg game could be called a "sequential" version of Cournot. Suppose that firm 1 is the leader, choosing $q_1 = q_1^\circ$, which is then observed by 2, who chooses q_2 to maximize $\pi_2(q_1^\circ, q_2)$. Let $q_2^*(q_1^\circ)$ be the solution to this problem. From what has come before, we know this is a decreasing function (for the demand curve $P = 120 - Q$). This means that 1's problem is $\max_{q_1} \pi_1(q_1, q_2^*(q_1))$.

Homework 3.7. *Assuming the demand function* $P = 120 - Q$ *and marginal costs of* 12*, find the Stackelberg equilibrium quantities. Is industry supply larger or smaller than in the Cournot case? Are (the leader) firm* 1*'s profits higher or lower than (the follower) firm* 2*? If higher, then there is a* ***firm mover advantage****, if lower there is a* ***second mover advantage****. Are the leader/follower profits higher or lower than the Cournot profits?*

There is also a sequential version of differentiated Bertrand competition.

Homework 3.8. *Returning to the differentiated Bertrand game above, we have* $I = \{1,2\}$*,* $\pi_i(p_i, p_j) = (24 - 2p_i + p_j)(p_i - c)$*. Suppose that* 1 *moves first, choosing* p_1°*,* 2 *observes this and chooses* $p_2^*(p_1^\circ)$*. Find the equilibrium prices and quantities, compare them to the Bertrand and to the competitive outcomes. Is there a first mover or a second move advantage?*

3.5. **Some Dynamic Games.** We'll begin with what are called wars of attrition, which are related to the game called Chicken, then turn to two patent race models. In each of these three dynamic games, we will look at the comparison with the social optimum.

3.5.1. *Chicken and Wars of Attrition.* Think testosterone poisoning for this one — two adolescent boys run toward each other along a slippery pier over the ocean, at predetermined points, they jump onto their boogie boards, each has the option of "chickening out" and ending up in the ocean, or going through, since they are going headfirst at

[1]This is an invitation/instruction to check this last statement.

full running speed, if they both decide to go through, they both end up with concussions, since chickening out loses face (the other boys on the beach laugh at them), the payoffs are

	Chicken	Thru
Chicken	$(0,0)$	$(-2,10)$
Thru	$(10,-2)$	$(-9,-9)$

Sometimes, one thinks about lockout/agreement and strike/agreement problems using a game of chicken.

There are no dominant strategies for this game. There are two pure strategy equilibria, and one, rather disastrous (for the parents of the sense-challenged young men) mixed strategy equilibrium. Here, the complementarities are negative, in terms of α, β, the probabilities that the two play Chicken,

$$U_1(\alpha,\beta) := E\,u_1 = \text{mess} + \alpha\beta[2-9-10], \text{so } \partial^2 U_1/\partial\alpha\partial\beta < 0.$$

Increases in β decrease 1's utility to α, a decreasing best response set.

More seriously, there are two firms in an industry. With two firms in an industry, profits are negative, say -1 per firm, with only one firm, profits are positive, say 3. There are prohibitively large costs to coming back into the industry after leaving. Per period payoffs are

	Exit	In
Exit	$(0,0)$	$(0,3)$
In	$(3,0)$	$(-1,-1)$

Asymmetric and symmetric equilibria for this game, with discount factor $\delta = 1/(1+r)$. Expected payoffs to any eq'm for this game must be at least 0. Indeed, exactly 0, but this is slightly messy calculation or relatively simple calculation plus clever argumentation. This is called **rent dissipation**.

3.5.2. *A First Patent Race Model.* The first model has an investment in each period of x leading to a probability $P(x)$, $P'(\cdot) > 0$, $P''(\cdot) < 0$, of making the breakthrough that leads to a patentable, profitable new market with value V. A monopolist would solve one version of this problem, a pair of contestants another and they would find the invention earlier, though at a greater expenditure of money, $I > 2$ contestants yet more investment and yet earlier discovery at a higher cost. Social tradeoffs.

3.5.3. *A Second Patent Race Model.* Another model has $T(x)$ being the time of discovery if x invested with $T'(\cdot) < 0$. The payoffs are discontinuous with a discontinuous "devil take the hindmost" feel to them. $I = 3$. No pure equilibrium because of discontinuities. Symmetric mixed equilibrium with support on the interval $[0, V]$ has a unique cdf.

4. Sixth Meeting, October 9, 2007

Topics: Perfection in Dynamic games. Applications to the money burning, reasonable beliefs, sunk costs, nuisance suits, chain store paradox.

4.1. Homeworks.

Due: Tuesday, October 16, 2007.

Homework 4.1. From ER: *4.1.*

Homework 4.2. From ER: *4.2.*

Homework 4.3. From ER: *Either 4.4 or 4.5.*

4.2. Stackelberg Eq'm, Commitment, Perfectness.

Revisit Stackelberg, consider the story of the idiot son-in-law. Then go to Follow-the-Leader I, which has the simultaneous move matrix

	Large	Small
Large	$(2, 2)$	$(-1, -1)$
Small	$(-1, -1)$	$(1, 1)$

Play this sequentially, add perfectness to the normal form. Interpret this as subgame perfectness here.

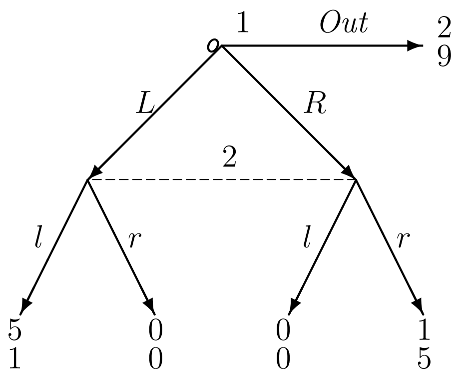
Definition 4.1. *For $\epsilon > 0$, a strategy $\sigma^\epsilon \gg 0$ is an **ϵ-perfect equilibrium** if for all $i \in I$, $\sigma_i(Br_i(\sigma^\epsilon)) \geq 1 - \epsilon$. A strategy is a **perfect equilibrium** if it is the limit, as $\epsilon_n \to 0$, of a sequence σ^{ϵ_n}, of ϵ_n-perfect equilbria.*

4.3. Money Burning.

In the following game, player 1 has an outside option, a safety point if you will. In the extensive form game, if player 2 gets to play, it means that 1 has forsaken her/his outside option. One could argue that this means that 2 should believe that if s/he gets to play, it is because 1 believes that the equilibrium that 1 and 2 will play together is better for 1 that the outside option. This is called a "money burning" game because player 1 has forsaken something expensive. An example of forsaking something expensive is burning money.

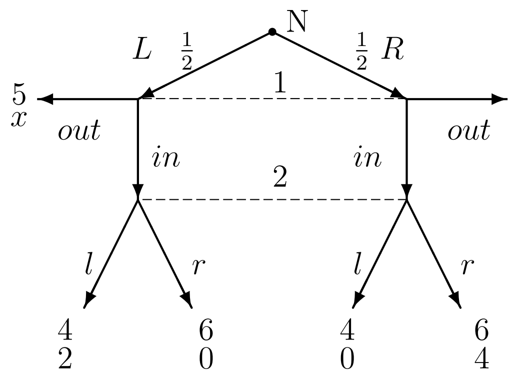
The iterated deletion of weakly dominated strategies that delivers this conclusion also applies to the version of the game in which 2 moves first and then 1 picks, not knowing what 2 has picked.

Homework 4.4. *Consider the following version of the money-burning game, Γ,*



(1) Give the normal form for Γ.
(2) Give the extensive form of this game in which 2 *move first.*
(3) Find $Eq(\Gamma)$.
(4) Find the subset of $Eq(\Gamma)$ *that survives iterated deletion of weakly dominated strategies.*

Perfectness restricts reasonable beliefs.



	l	r
in	$(4,1)$	$(6,2)$
out	$(5,x)$	$(5,x)$

Homework 4.5. *Show that the given* 2×2 *game is the normal form, and give the unique perfect equilibrium.*

4.4. **Nuisance Suits and Sunk Costs.** Putting an expensive firm of lawyers on a non-refundable retainer and publicizing it discourages people from suing you. Whether or not this is a good idea for society is a different question.

Figure 4.4 (p. 114) is the basic game. $\exists!$perfect eq'm.

Put in an initial stage in which the plaintiff can sink a cost of p into a non-refundable lawyer's fee, basically, burn money, in such a fashion that the defendant sees it.

What about a pre-initial stage, in which the defendant sinks a cost d? If plaintiff knows they face a fight, ... Legal insurance?

4.5. **The Chain Store Paradox.** The potential entrant decides whether or not to enter, if they do enter, the incumbent must choose to fight or accomodate/collude. The 2×2 payoffs are (e.g.)

	Fight	Collude
Enter	$(-10, 0)$	$(40, 50)$
Stay out	$(0, 300)$	$(0, 300)$

There is a unique perfect eq'm for this game. There is a very different Nash equilibrium outcome.

The chain store version: the incumbent is active in 2 markets, we just analyzed market 1, after seeing what happens there, a potential entrant in market 2 decides what to do. There is a unique perfect eq'm for this game, and a very different Nash eq'm outcome.

What if instead of 2 markets, it is 3 markets? What about 20?

Selten's answer, an early version of what we now call "behavioral economics," a name I hate. There is a different answer, based on reputation and uncertainty about others' payoffs. Think about the value of having a reputation of being crazy, being willing to do anything to defend, say, one's "turf," one's "honor."

Topics: Repeating games finitely many times. Games with random end times and discounting. Minmax'ing people. ESS's and a different take on equilibrium.

The chain store paradox revisited. Repeated Bertrand and Cournot competition.

5.1. **Homeworks.** Due: Tuesday, October 23, 2007.

From ER: 5.1, 5.3, 5.4 and 5.5.

Plus some problems below.

5.2. **Finite Repetition.** We've seen a bit of this, entrance in two markets rather than entrance in one. In that game there was only one perfect equilibrium and many Nash equilibria. Sometimes there is only one Nash equilibrium. Consider the prisoners' dilemma, Γ, given by

	Confess	Deny
Confess	$(0,0)$	$(10,-5)$
Deny	$(-5,10)$	$(5,5)$

Draw in utility space.

Γ^T, with discounting $0 < \delta \leq 1$ has payoffs for history $s_0, s_1, s_2, \ldots, s_{T-1}$, $U_i = \frac{1-\delta}{1-\delta^T} \sum_{t=0}^{T-1} \delta^t u_i(s_t)$. The normalization is to keep things in the same region of utility space. E.g. $T = 2, 3$.

Higher δ correspond to more patience, that is, more regard for future.

Prisoners' dilemma claim: For $0 < \delta \leq 1$, there exists a unique **Nash** equilibrium outcome for Γ^T for any finite T.

Notice the "outcome" in the above. Compare perfect/SGP equilibria.

Consider the following 3×3 game played $T \geq 2$ times.

	High	Medium	Low
High	$(10,10)$	$(-1,-12)$	$(-1,15)$
Medium	$(-12,-1)$	$(8,8)$	$(-1,-1)$
Low	$(15,-1)$	$(-1,-1)$	$(0,0)$

No Nash equilibrium has (High,High) in all two/three periods. Use good/bad Nash reversion threats to support (High,High) in some of the early periods.

5.3. **"Infinite" Repetition.** For a bounded sequence of payoffs $s_0, s_1, s_2, \ldots$ and $0 < \delta < 1$, the δ-discounted valued is $\sum_{t=0}^{\infty} s_t \delta^t = s_0 + \delta^1 s_1 + \delta^2 s_2 + \cdots$.

Homework 5.1. *Show/calculate the following:*

(1) For $\delta \neq 1$*,* $\sum_{t=0}^{T-1} \delta^t = \frac{1-\delta^T}{1-\delta}$.

(2) $\lim_{\delta \uparrow 1} \frac{1-\delta^T}{1-\delta} = T$.

(3) For $|\delta| < 1$*,* $\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$.

(4) *For the sequence* $7, 3, 3, 3, 3, \ldots$, *find the* δ*-discounted value. For what values of* δ *does the sequence* $4, 4, 4, \ldots$ *have a higher* δ*-discounted value?*

(5) *For what values of* T *and* δ *does the sequence* $4, 4, 4, \ldots$ *have a higher* δ*-discounted value than the sequence* $7, \underbrace{3, 3, \ldots, 3}_{\tau \text{ times}}, 4, 4, 4, \ldots$*?*

Discounting as uncertainty about whether or not tomorrow will arrive for this game: I know, right now, that today's actions will give me a payoff, $u_i(s_t)$, with probability δ, tomorrow will arrive for this game, with probability $(1-\delta)$, it will not, the world will end. The randomness is iid. The expected payoffs are

$$u_i(s_0) + (1-\delta)\cdot 0 + \delta[u_i(s_1) + (1-\delta)\cdot 0 + \delta[u_i(s_2) + \cdots \ , \tag{4}$$

which can be conveniently re-written as $\sum_{t=0}^{\infty} \delta^t u_i(s_t)$.

If, in addition to the uncertainty, there is also pure time discounting, meaning that consumption tomorrow is simply worth less to me, perhaps because my taste buds and brain will be a bit more decayed, then we have a smaller δ.

Let τ be the random number of further periods that the game will last. Find $E\,\tau$. Find $E\left(\tau \middle| \tau \geq t\right)$.

Call the game with this uncertainty Γ_δ, or Γ_δ^∞ if you want to match the previous notation a bit better.

Find the range of δ for which the grim trigger strategy is an equilibrium. Find the range of δ's and T's for which T periods of penance followed by return to cooperation is an equilibrium.

Homework 5.2. *Suppose that there are two firms, the demand curve is* $p = 1 - q$ *or* $q = 1 - p$*, and that costs of production are* $C(q_i) = c \cdot q_i$ *for some constant marginal cost* $0 \leq c < 1$*.*

(1) *Find the largest possible profits that can be made in the industry.*

(2) *Consider the strategies: "start by playing* $\frac{1}{2}$ *of the monopoloy quantity, do this at all periods in which everyone has played this, and play the one-stage Cournot quantities otherwise." Find the range of* δ*'s for which this is a perfect equilibrium.*

(3) *Consider the strategies: "start by playing the monopoloy price, do this at all periods in which everyone has played this, and play the one-stage Bertrand prices otherwise." Find the range of* δ*'s for which this is a perfect equilibrium.*

5.4. **Minmax'ing people, or Threats in finitely repeated games.** Put equilibrium considerations in the back of your mind for just a little bit, and cast your memory back (fondly?) to those school year days spent in terror of threats by older, larger kids. Consider the question, "What is the worst that players $j \neq i$ can do to player i?" Well, they can call him/her a booby or a nincompoop, but these are not sticks nor stones, and given that we are now adults, we are not supposed to believe that this hurts too much. However, stealing your lunch money and stuffing you into a garbage can, now that hurts. What $j \neq i$ can do is get together and agree to take those actions that make i so miserable as possible. This is a threat with some teeth to it. Now, i has some protection against this behavior — knowing that the others are ganging up, i can plan accordingly to maximize i's utility against the gang-up behavior. There are three "safe" utility levels that one might imagine i being able to guarantee $i$self,

$$\underline{v}_i^{pure} = \min_{a_{-i} \in \times_{j \neq i} A_j} \max_{t_i \in A_i} u_i(a_i, s_{-i}),$$

$$\underline{v}_i^{mixed} = \min_{\sigma_{-i} \in \times_{j \neq i} \Delta(A_j)} \max_{t_i \in A_i} u_i(a_i, \sigma_{-i}).$$

Since $\times_{j \neq i} A_j \subset \times_{j \neq i} \Delta(A_j)$,

$$\underline{v}_i^{pure} \geq \underline{v}_i^{mixed}.$$

Give games where the inequalities are strict.

The first corresponds of the worst that dolts who do not understand randomization can do to i, and the second corresponds of the worst that enemies who do understand independent randomization can do to i. The $\underline{v}_i$'s are called "safety levels." Here is one of the reasons.

Lemma 5.1. *For all $i \in I$ and for all δ, if σ is an equilibrium for Γ_δ^∞, then $U_{\delta,i}^\infty(\sigma) \geq \underline{v}_i^{mixed}$.*

This lemma is ridiculously easy to prove once you see how. Suppose that other players are playing some strategy σ_{-i}. In period 1, have i play a myopic, that is, one period best response to the distribution over A_{-i} induced by σ_{-i}^1, $\sigma_i \in Br_i(\sigma^1(h^0))$. More generally, after any h^{t-1}, have i play $\sigma_i^t \in Br_i(\sigma(h^{t-1})$. In each period, it must be the case that $u_i(s^t) \geq \underline{v}_i^{mixed}$.

Find the minmax levels for Cournot, for Bertrand, for entry deterrence.

Let us also look at the following 2×2 game,

	L	R
T	(2,9)	(-20,-80)
B	(10,0)	(-30,-100)

For this game, $Eq(\Gamma) = (B, L)$ and $u(Eq(\Gamma)) = (10, 0)$.

Claim: $\mathbb{O}(Eq(\Gamma^2))$ with $\delta = 1$ contains the history $h = ((T, L), (B, L))$.

It is important to note that (T, L) is nothing like the unique equilibrium of Γ. The claim can be seen to be true by considering the strategies $\sigma^1(h^0) = (T, L)$, $\sigma_1^2(h^1) \equiv B$, and

$$\sigma_2^2(h^1) = \begin{cases} L & \text{if } h^1 = (T, L) \\ R & \text{otherwise} \end{cases}.$$

These are in fact Nash equilibria, just check the mutual best response property. They are not subgame perfect equilibria, just check that they call for play of a dominated strategy in the case of "otherwise." Show that the $\underline{v}_i^{pure}$ are $(-20, 0)$ for this game.

A more subtle observation is that for 2 to min-max 1, 2 must suffer a great deal. Therefore, to have a subgame perfect equilibrium in which 1's utility is held down, we must have strategies in which it regularly happens that some s_i^t giving 2 $u_2(s^t) < \underline{v}_2^{pure}$ happens.

5.5. **Threats in infinitely repeated games.** The third reason to call the $\underline{v}_i$'s safety levels appears in the following result, which we will not prove, though we will talk about it.[2]

Folk Theorem: Suppose that $\mathbf{co}\,(u(S))$ has non-empty interior. Let v be a vector in $\mathbf{co}\,(u(S))$ such that for all $i \in I$, $v_i > \underline{v}_i^{mixed}$. For all $\epsilon > 0$, there exists a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$,

$$B(v, \epsilon) \cap U^\delta(SGP(\Gamma_\delta^\infty)) \neq \emptyset.$$

Before discussing how the proof works, let look at an example violating the condition that $\mathbf{co}\,(u(S))$ have non-empty interior, in particular, let us look at the Matching Pennies game,

	H	T
H	$(+1, -1)$	$(-1, +1)$
T	$(-1, +1)$	$(+1, -1)$

Claim: For all $\delta \in (0, 1)^I$, if $\sigma^\infty \in Eq(\Gamma_\delta^\infty)$, then $\mathbb{O}(\sigma^\infty)$ is the i.i.d. distribution putting mass $\frac{1}{4}$ on each point in S in each period. In this game, there is no $v \in \mathbf{co}\,(u(S))$ that is greater than the threat point vector $(0, 0)$.

5.6. **ESS and the one population model.** Here's the class of games to which these solution concepts apply.

Definition 5.1. *A two person game* $\Gamma = (S_i, u_i)_{i=1,2}$ *is* **symmetric** *if*

(1) $S_1 = S_2 = S = \{1, 2, \dots, n, \dots, N\}$,
(2) for all $n, m \in S$, $u_1(n, m) = u_2(m, n)$

We have a big population of players, typically $\Omega = [0, 1]$, we pick 2 of them independently and at random, label them 1 and 2 **but do not tell them the labels**, and they pick $s_1, s_2 \in S$, then they receive the vector of utilities $(u_1(s_1, s_2), u_2(s_1, s_2))$. It is very important, and we will come back to this, that the players do not have any say in who they will be matched to.

Let σ_n be the proportion of the population picking strategy $n \in S$, and let $\sigma = (\sigma_1, \dots, \sigma_N) \in \Delta(S)$ be the summary statistic for the population propensities to play different strategies. This summary statistic can arise in two ways: **monomorphically**, i.e. each player ω plays the same σ; or **polymorphically**, i.e. a fraction σ_n of the population plays pure strategy n. (There is some technical mumbo jumbo to go through at this point about having uncountably many independent choices of strategy in the

[2]We will not even state the most general version of this result, for that see Lones Smith (19??, Econometrica).

monomorphic case, but there are both nonstandard analysis and other ways around this problem.)

In either the monomorphic or the polymorphic case, a player's expected payoff to playing m when the summary statistic is σ is

$$u(m, \sigma) = \sum_{n \in S} u(m, n)\sigma_n,$$

and their payoff to playing $\tau \in \Delta(S)$ is

$$u(\tau, \sigma) = \sum_{m \in S} \tau_m u(m, \sigma) = \sum_{m,n \in S} \tau_m u(m, n)\sigma_n.$$

From this pair of equations, if we pick a player at random when the population summary statistic is σ, the expected payoff that they will receive is $u(\sigma, \sigma)$.

Now suppose that we replace a fraction ϵ of the population with a "mutant" who plays m, assuming that $\sigma \neq \delta_m$. The new summary statistic for the population is $\tau = (1 - \epsilon)\sigma + \epsilon\delta_m$. Picking a non-mutant at random, their expected payoff is

$$v^{\epsilon}_{n-m} = u(\sigma, \tau) = (1 - \epsilon)u(\sigma, \sigma) + \epsilon u(\sigma, \delta_m).$$

Picking a mutant at random, their expected payoff is

$$v^{\epsilon}_{m} = u(m, \tau) = (1 - \epsilon)u(m, \sigma) + \epsilon u(m, m).$$

Definition 5.2. *A strategy σ is an* **evolutionarily stable strategy (ESS)** *if there exists an $\underline{\epsilon} > 0$ such that for all $\epsilon \in (0, \underline{\epsilon})$, $v^{\epsilon}_{n-m} > v^{\epsilon}_{m}$.*

An interpretation: a strategy is an ESS so long as scarce mutants cannot succesfully invade. This interpretation identifies success with high payoffs, behind this is the idea that successful strategies replicate themselves. In principle this could happen through inheritance governed by genes or through imitation by organisms markedly more clever than (say) amœbæ.

Homework 5.3 (Optional)**.** *The following three conditions are equivalent:*

(1) σ is an ESS.
(2) For all $\tau \neq \sigma$, $u(\sigma, \sigma) > u(\tau, \sigma)$ or $u(\sigma, \sigma) = u(\tau, \sigma)$ and $u(\sigma, m) > u(m, m)$.
(3) $(\exists \underline{\epsilon} > 0)(\forall \tau \in B(\sigma, \epsilon)\ \tau \neq \sigma)[u(\sigma, \tau) > u(\tau, \tau)]$.

5.7. **ESS, or nature, long of fang and red of claw.** Here is an old puzzle: when animals fight over nesting territories or mates, it is rarely to the death. Poisonous snakes wrestle for dominance but almost never bite each other. Wolves do not tear the throat out of the loser who offers their throat. Stags push and shove at each other till exhaustion sometimes, but rarely use their antlers to gore or cripple. The reason that this is a bit of puzzle is that, by getting rid of competitors for mating opportunities, the users of deadly force would expand the opportunities for their own genes to be passed on. Wolves are pack animals, and seem quite smart. Perhaps we could believe that the remaining wolf pack would gang up and kill the fighter using deadly force, wolf justice perhaps. Snakes and many stags are solitary creatures though, so this is not what's at work in any generality.

Stripping away a huge amount of detail, suppose that there are two strategies that the males could use, Deadly Force and Posturing. The idea of "Posturing" is that you make lots of noise, perhaps wrassle, but back down against Deadly Force. Payoffs might then be

	Deadly Force	Posturing
Deadly Force	$(\frac{g-c}{2}, \frac{g-c}{2})$	$(g, 0)$
Posturing	$(0, g)$	$(\frac{g}{2}, \frac{g}{2})$

Here, we're thinking about $g > 0$ as the gain, and $c > 0$ as the cost to using deadly force against another creature using it. Further, when two creatures meet each other, they are, on average, evenly matched, hence the division by 2 along the diagonal in the matrix. Let us set $g = 2$ and look at what happens to the unique symmetric equilibrium as $c \uparrow$, that is, as the behavior is more and more deadly. The symmetric, mixed equilibrium, regarded as a population summary statistic, is $(\frac{g}{c}, 1 - \frac{g}{c})$ on Deadly Force and Posturing respectively. The payoffs in this equilibrium are $\frac{g}{2}(1 - \frac{g}{c})$

In particular, when c is large relative to g, there is very little Deadly Force in the population. The reason is that the offspring of the users of Deadly Force will rip each other's throats out, or poison each other, thereby making the world a safer place for the Posturers. Back to the puzzle, Deadly Force works against a population of mostly Posturers, but it doess pretty badly against other users of Deadly Force, so the Posturers have advantages in a population of predominantly Deadly Force users (e.g. they don't die).

6. Ninth Meeting, October 30, 2007

Topics: Overview of fixed versus reactive risk in insurance industry; techniques and results for fixed risks; demand for insurance against fixed risk.

6.1. **Homeworks.** Due: Tuesday, November 6, 2007.

Read Heimer Chapters 1 through 3, Rasmussen 8.5. In reading Heimer's coverage of the industries, keep looking for the game theoretic and economic aspects of what she is talking about.

Problems given below.

6.2. **Fixed vs. Reactive Risks.**

(1) Hurricane insurance (fixed risk).
(2) Maritime insurance (storms happen, this is a fixed risk; sailing out into stormy weather, not keeping shipshape, deciding to cast away the ship, these are reactive risks).
(3) Fire insurance (some unavoidable risk, lowering/managing risks is a crucial part of the industry, but, there is an interesting bit of history of mutual insurance companies versus stockholder-owned insurance companies).
(4) Divorce insurance, not purchasable (moral hazard reactive risk).
(5) Car insurance (deductibles as incentives to drive better, park in safer areas, strategies to manage the reactive risk).
(6) Health insurance (possibility of adverse selection of a riskier pool, pre-screening, genetic testing, previous conditions).
(7) Surety bonding of employees (when there is little control by person buying the insurance against theft by her/his employees, you have one kind of contract, when there is more control, you have another. This is managing the reactive risk).

Reactive risks are, essentially, game theoretic in nature. Fixed risks are, at least potentially, actuarially identifiable.

6.3. **Techniques for Analyzing Fixed Risks.**

Basically, we are going to look at the basic ideas in expected utility preferences over random variables taking values in an interval $[a, b]$. A **lottery** is a probability distribution on $[a, b]$.

A short list of the topics:

(1) Compound lotteries.
(2) Basic axiom: the Independence Axiom, for all lotteries L, L', L'' and all $0 < \alpha < 1$,
$$[L \succsim L'] \text{ iff } \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$
(3) The Expected utility theorem.
(4) Increasing $u(\cdot)$ and First Order Stochastic Dominance.
(5) Concave $u(\cdot)$, certainty equivalents, and risk aversion.
(6) Concave $u(\cdot)$ and Second Order Stochastic Dominance.

6.3.1. *Compound lotteries.* Combinations of lotteries are called **compound lotteries**, and are defined by $(\alpha L + (1-\alpha)L')(E) = \alpha L(E) + (1-\alpha)L'(E)$. Flip an independent coin to pick between the lotteries, this is the right idea.

Homework 6.1. *If* $L(10) = L(20) = L(30) = \frac{1}{3}$ *and* $L'(5) = 0.10$*,* $L'(10) = 0.45$*,* $L'(20) = 0.45$*, give the cdf's of the following compound lotteries:*

(1) $0.1L + 0.9L'$*;*
(2) $0.3L + 0.7L'$*;*
(3) $0.5L + 0.5L'$*;*
(4) $0.7L + 0.3L'$*; and*
(5) $0.9L + 0.1L'$*.*

We assume that people's preferences depend only on the lottery, not how it is presented. Thus, if $L(10) = 1$ and $L'(20) = 1$, then $\frac{1}{2}L + \frac{1}{2}L'$ is the same as the lottery $L''(10) = \frac{1}{2}$, $L''(20) = \frac{1}{2}$.

6.3.2. *The expected utility theorem.* The basic axiom is called the **Independence Axiom**.

It asks that preferences over $\Delta[a, b]$ satisfy, for all lotteries L, L', L'' and all $0 < \alpha < 1$,

$$[L \succsim L'] \text{ iff } \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

The only difference between $\alpha L + (1-\alpha)L''$ and $\alpha L' + (1-\alpha)L''$ is what happens after the α part of the mixture. This makes the axiom make sense.

The expected utility theorem says that: A preference ordering $\succsim$ over $\Delta[a, b]$ is continuous and satisfies the independence axiom iff $[L \succsim L'] \Leftrightarrow [\int u(x)\, dL(x) \geq \int u(x)\, dL'(x)]$ for some continuous utility function $u : [a, b] \to \mathbb{R}$.

The $u(\cdot)$ is called a **Bernoulli** or a **von Neumann-Morgenstern** utility function.

What is interesting is how properties of $u(\cdot)$ map to properties of the preferences.

6.3.3. *First and second order stochastic dominance.* $C[a, b]$ is the set of continuous functions on $[a, b]$, C_{con} the set of concave functions, ND the set of non-decreasing functions, whether or not they are continuous,

Definition 6.1. *For* $L, L' \in \Delta$*,*

(1) L ***first order stochastically dominates*** L'*,* $L \succsim_{FOSD} L'$*, if for all* $u \in ND$*,* $\int u\, dL \geq \int u\, dL'$*,*
(2) L ***is riskier than*** L' *if if for all* $u \in C_{con}$*,* $\int u\, dL \leq \int u\, dL'$ *(note the change in the direction of the inequality here), and*
(3) for L *and* L' *with the same expectation,* L ***second order stochastically dominates*** L'*,* $L \succsim_{SOSD} L'$*, if for all* $u \in (C_{con} \cap ND)$*,* $\int u\, dL \geq \int u\, dL'$*.*

That's pretty abstract. With cdf's, maybe FOSD makes more sense — $F_L(x) \leq F_{L'}(x)$ is FOSD.

Homework 6.2. *Let us restrict attention to lotteries taking on only the values* 10, 20, *and* 30. *Let* $L(10) = L(20) = L(30) = \frac{1}{3}$. *Find and graph in the simplex,*

(1) the set of probabilities that FOSD L,
(2) the set of probabilities that SOSD L,
(3) the set of probabilities that are FOSD'd by L,
(4) the set of probabilities that are SOSD'd L,
(5) the set of probabilities that give higher expected utility for the von Neumann-Morgenstern utility function $u(r) = \sqrt{r}$, *and*
(6) the set of probabilities that give higher expected utility for the von Neumann-Morgenstern utility function $u(r) = r$.

6.3.4. *Certainty equivalents and Jensen's inequality.* Jensen: If $u(\cdot)$ is concave and X is a random variable, then $u(E\,X) \geq E\,u(X)$. Equivalently, if $r = \int x\,dL(x)$, then $u(r) \geq \int u(x)\,dL(x)$.

Examples: two point, uniform.

Proof.

Definition 6.2. *The* ***certainty equivalent*** *of* L *is that number* r *with* $u(r) = \int u(x)\,dL(x)$.

By Jensen's inequality, if $u(\cdot)$ is concave, then the certainty equivalent is less than the mean of a random variable.

For lottery L, its expectation is $r = \int x\,dL(x)$ and δ_x is the lottery that gives x for sure.

Definition 6.3. $\succsim$ *is* ***risk averse*** *if for all lotteries* L, $\delta_r \succsim L$ *where* r *is the expectation of* L.

Theorem 6.1. *If* $\succsim$ *satisfies continuity and the Independence Axiom, then* u *is concave iff* $\succsim$ *is risk averse.*

Homework 6.3. *For the following random variables/lotteries and vN-M utility functions, find the certainty equivalents:*

(1) $L(10) = L(30) = \frac{1}{2}$, $u(r) = \sqrt{r}$,
(2) $L(10) = L(30) = \frac{1}{2}$, $u(r) = r$,
(3) $L(10) = L(30) = \frac{1}{2}$, $u(r) = r^2$ *(notice that this is convex not concave),*
(4) $L = U[0, 1]$, $u(r) = \sqrt{r}$,
(5) $L = U[10, 100]$, $u(r) = \ln r$, *and*
(6) $L = U[0, 1]$, $u(r) = 1 - e^{-\lambda r}$.

6.4. **Demand for Insurance.** We begin with the observation that, if you charge an actuarially fair price for insurance, the risk averse will choose to fully insure. This has implications for the market for insurance.

Homework 6.4. *Mary, through hard work and concentration on her education, had managed to become the largest sheep farmer in the county. But all was not well. Every* *month Mary faced a 50% chance that Peter, on the lam from the next county for illegal confinement of his ex-wife, would steal some of her sheep. Denote the value of her total flock as w, and the value of the potential loss as L, and assume that Mary is a expected utility maximizer with a Bernoulli utility function*

$$u(w) = \ln w.$$

(1) *Assume that Mary can buy insurance against theft for a price of p per dollar of insurance. Find Mary's demand for insurance. At what price will Mary choose to fully insure?*

(2) *Assume that the price of insurance is set by a profit maximizing monopolist who knows Mary's demand function and the true probability of loss. Assume also that the only cost to the monopolist is the insurance payout. Find the profit maximizing linear price (i.e. single price per unit) for insurance. Can the monopolist do better than charging a linear price?*

7. Tenth and Eleventh Meetings, November 6 and 13, 2007

Topics: Finishing up topics on fixed risks. Reactive risks, types of insurance contracts; market for lemons; adverse selection; beginnings of signaling models.

7.1. **Efficient Risk Sharing.** Remember, the word "efficient" is different when an economist uses it.

7.1.1. *Risk sharing between two people.* Revisit Mary's problem as an efficient division of risk between a risk averse buyer of insurance and a risk neutral insurance company.

Homework 7.1. *There are two possible states of the world,* 1 *and* 2*, with probabilities* $\frac{1}{2}$ *each. State* 1 *is good for person* A*, they have* 10 *while person* B *has* 6*. In state* 2*, things are reversed,* A *has* 6 *while* B *has* 10*. Person* A*'s vN-M utility function is* $u_A(r) = \sqrt{r}$ *while* B*'s is* $u_B(r) = 1 - e^{-r}$*.*

An ***allocation*** *or sharing rule is a pair of pairs,* $((x_{A,1}, x_{A,2}), (x_{B,1}, x_{B,2}))$ *where* $x_{A,1}$ *is what* A *gets in state* 1 $x_{B,2}$ *is what* B *gets in state* 2*, and so on. Thus, the initial allocation is* $((10, 6), (6, 10))$*. The restriction that must be satisfied* $x_{A,1} + x_{B,1} \leq 10 + 6$ *and* $x_{A,2} + x_{B,2} \leq 6 + 10$*.*

1. *Give the expected utilities,* $E\, u_A$ *and* $E\, u_B$*, of the two people at the initial allocation.*
2. *Find the allocation that maximizes* A*'s expected utility subject to* B *having the expected utility of their initial allocation,* $E\, u_B$*.*
3. *Find the allocation that maximizes* B*'s expected utility subject to* A *having the expected utility of their initial allocation,* $E\, u_A$*.*
4. *Find all of the other efficient allocations that give each at least their initial expected utility.*

7.1.2. *Risk sharing between many.* A large population is divided into two types, L and H. People know their types. Type L is α of the population, H is $1 - \alpha$. Type L has a probability p of having an accident, H has a probability q, $0 < p < q < 1$. For type L the cost of recovering from the accident is c, for type H the cost is C, $0 < c < C$. In other respects, the types are identical risk averters with income w and strictly concave von Neumann-Morgenstern utility functions $u(\cdot)$.

a. Give $\overline{C}$, the expected per capita cost of recovering from the accident, in terms of the above parameters.
b. A single payer hospital system or a mutual protection fund can pool the risk across the whole population and make 0 profit. Show that the H types will always prefer population wide pooling while the L types may or may not prefer it.
c. Show that for low enough p and q, both types will will prefer population wide pooling to no pooling.
d. Suppose that there is a genetic marker for types that can be identified at 0 cost. Show that both types would prefer pooling across their own type to no pooling at all, but

that one of the types would prefer pooling across the whole population to pooling by type.

e. In terms of maximizing population wide *ex ante* (that is, before type is known) expected utility, which is better, pooling across the whole population or pooling across types? Briefly explain your answer.

Some background for and extensions to this problem: For the U.S. population as a whole there is roughly a 0.5% chance of contracting cancer in any given year. Women, 51% of the population, have roughly a 0.4% chance, while men, 49% of the population, have roughly a 0.6% chance. Women's hospital stays for cancer average roughly 7 days, men's hospital stays roughly 8.5 days (this may be for cultural rather than medical reasons — women have a reputation of being tougher). The problem does not consider survival probabilities being less than 1. Death rates from cancer are much higher for older people, slightly higher for middle aged women than for middle aged men, and much higher for older men than for older women. Pooling by age is another possibility in the above problem. The problem also does not consider that access to medical care differs between groups.

7.2. **Deductibles.** A potential purchaser of insurance faces a random loss, $L \geq 0$ with cdf F_L. We allow, and may even believe that it is most usual that $P(L = 0) > 0$.

Facts: $E\,L = \int_0^\infty [1 - F_L(x)]\,dx$, and $E \min\{D, L\} = \int_0^D [1 - F_L(x)]\,dx$. The reason for this also explains FOSD.

One starts with a random income $w - L$ (remember, $L = 0$ is possible). If one buys, at a price $p(D)$, an insurance policy that covers all losses greater than or equal to D, i.e. one with a deductible D, then one's random income is $[w - p(D)] - \min\{D, L\}$.

Homework 7.2. *Find the function $p(D)$ for the following cases, supposing that it is equal to the insurance company's expected payouts.*

(1) $F_L(x) = 0.8$ for $0 \leq x < 20$, and $F_L(x) = 1$ for $20 \leq x$.
(2) $F_L(x) = 0.9 + 0.1 \cdot \frac{x}{20}$ for $0 \leq x \leq 20$, and $F_L(x) = 1$ for $20 \leq x$.
(3) $F_L(x) = 0.80 + 0.20(1 - e^{-\lambda x})$ for $0 \leq x$.

In general, the optimization problem when facing a menu of actuarially fair deductible policies is pretty gruesome looking. However, using SOSDominance arguments again, if $p(D)$ is actuarially fair, i.e. equal to the insurance company's expected payout, then for concave vN-M utility function, the problem

$$\max_{D \geq 0} \int u([w - p(D)] - \min\{D, x\})\, dF_L(x) \tag{5}$$

always has, as a solution, $D = 0$, that is, full insurance once again.

Homework 7.3. *Suppose that the loss faced by someone with an income of $w = 100$ has cdf $F_L(x) = 0.5$ for $0 \leq x < 36$ and $F_L(x) = 1$ for $36 \leq x$. Suppose further that their vN-M utility function is $u(r) = 2\sqrt{r}$.*

(1) *Find the maximum that the person would be willing to pay for the policy with deductible* $D = 0$.
(2) *Find the maximum that the person would be willing to pay for the policy with deductible* $D = 5$.
(3) *Find the maximum that the person would be willing to pay for the policy with deductible* $D = 10$.
(4) *Find the maximum that the person would be willing to pay for the policy with deductible* $D = 36$.

7.3. **A First Look at Reactive Risk.** Here the story is very simple: after buying insurance, incentives to avoid loss change. In the first part of the following, they change their action pretty drastically.

Homework 7.4. *A potential consumer of insurance faces an initial risky situation given by* $w - L$ *where the distribution of* L *depends on* e*, the effort, measured in dollars. Specifically,* $P(L = 0) = f(e)$ *is an increasing concave function of* e *with* $\lim_{e \downarrow 0} f'(e) = \infty$. *This means that* $P(L = 10) = 1 - f(e)$ *is a decreasing convex function. The consumer has a smooth, concave von Neumann-Morgenstern utility function,* u. *Throughout, we assume that* e *is* ***not*** *observable by the insurance company. In game theory terms, this is a case of hidden action.*

(1) *Set up the consumer's maximization problem and give the FOC when insurance is not available.*
(2) *Now suppose that the consumer is offered an actuarially fair full coverage insurance policy, that is, one that is Pareto optimal when there is no reactive risk. Suppose also that the effort* e *is not observable by the insurance company. Find the consumer's optimal choice of actions after they have bought the insurance.*
(3) *How might a deductible insurance policy partly solve the problem you just found?*
(4) *Suppose that* e *was observable and could be written into the contract with the insurance company. What would be the optimal level of effort?*
(5) *Suppose now that a monopolistic risk-neutral insurance company offered the consumer a policy with deductible* D^* *at a price* p^* *that maximized the insurance company's expected profits taking into account that the probability of loss is* $1 - f(e^*)$ *where* e^* *solves the consumer maximization problem. I.e. suppose we had a perfect equilibrium in the game where the insurance company offers any* (p^*, D^*)*, the consumer accepts or rejects the contract, and if they accept it, solves their maximization problem.*

    Show that the perfect equilibrium $((p^*, D^*), e^*)$ *gives an inefficient allocation of risk and effort.*

7.4. **A Second Look at Reactive Risk.** We now introduce partial control.

Homework 7.5. *A strictly risk averse person with wealth W and a twice continuously differentiable von Neumann-Morgenstern utility function $u(\cdot)$ depending on income spends $e \geq 0$ on loss prevention. After spending e, their random income is $Y = X - e$ where Y is their net income, and X is their gross income. X is a random variable whose distribution, R_e, is a convex combination of two distributions μ and Q_e, $R_e = cQ_e + (1-c)\mu$, $c \in [0,1]$. Here, μ is an arbitrary distribution on the non-negative reals that does not depend on e, and Q_e is a two-point distribution with probability $f(e)$ at W and $1 - f(e)$ at $W - L$.*

We interpret $c \in [0,1]$ as the consumer's level of **c***ontrol — if $c = 0$, then a makes no difference in the distribution, as c rises to 1, the control over the distribution increases.*

The function $f(\cdot)$ satisfies the same conditions as in the previous problem.

(1) *Write out the person's expected utility and their expected income as a function of e.*
(2) *Give the derivative conditions that must hold when e is chosen optimally, and verify that the second derivative conditions hold.*
(3) *How does e^* depend on c?*

The previous problem indicated that the lower the degree of control, the smaller the optimal spending on loss prevention. Further, the smaller the degree of control, the less effect insurance has on the incentives for loss control. This means that we expect that, when there is less control, there is less of a problem with reactive risk.

7.5. **Mutual Companies.** Public goods are under-provided by markets because there are privately borne costs and publicly enjoyed benefits. This has implications for the difference between mutual companies and insurance companies.

For the next bit of analysis, we are thinking of e as being effort spent on research on sprinkler systems, other flame retardants, flame-proof building materials, best practices for handling flammable materiel, and so on.

Homework 7.6. *Suppose that spending effort e on improving overall safety has a benefit $b(e)$ and a cost $c(e)$ where $b(e) = e^\alpha$ and cost $c(e) = e^\beta$ where $0 < \alpha < 1 < \beta$.*

(1) *Suppose that a single firm must solve $\max_{e \geq 0} b(e) - c(e)$. Find the solution in terms of α and β.*
(2) *Now suppose that 2 firms can pool their efforts and solve the problem $\max_{e \geq 0} 2 \cdot b(e) - c(e)$. Find the solution in terms of α and β.*
(3) *Now suppose that J firms can pool their efforts and solve the problem $\max_{e \geq 0} J \cdot b(e) - c(e)$. Find the solution in terms of α and β.*

More generally, suppose that firm i receives benefits $b(e,i)$ from efforts e and that for all firms i, $e' > e$ implies that $b(e',i) > b(e,i)$. Let $c(e) \geq 0$ be a cost function for effort.

Let $e^*(I)$ be the solution to

$$\max_{e\geq 0}[\sum_{i=1}^{I} b(e,i)] - c(e). \tag{6}$$

From our work with supermodularity before, we know that $e^*(I)$ increases with I because $f(e, I) := [\sum_{i=1}^{I} b(e,i)] - c(e)$ has increasing differences in e and I.

If we think of I as the membership of a mutual insurance company, this says that higher I will entail higher spending on research on preventing losses to the owners of the company, that is, to the firms that belong to the insurance company. If the research has positive spill-over to companies that do not belong to the mutual company, then we still have under-investment in this public good. This is a situation typical of what are called **non-excludable** goods. Knowledge is often thought of as being somewhat non-excludable.

The intuition, once again, is that the benefits are enjoyed by all, and if the costs can be shared, then more effort will be spent.

In the U.S., there have been waves of de-mutualization of insurance companies, they have become stock-holder owned companies instead of being owned by a coalition of companies with the same kinds of risks. One disadvantage of being a mutual insurance company is that one cannot acquire large amounts of capital by selling stock. In any case, after de-mutualization, the benefits no longer go fully to the companies that are insured, some proportion, call it γ, goes to the stock-holders (and the new class of professional managers). If I is the set of firms still getting insurance, the new research problem is

$$\max_{e\geq 0}\gamma[\sum_{i=1}^{I} b(e,i)] - c(e). \tag{7}$$

Again, from our work with supermodularity before, we know that $e^*(\gamma)$ increases with γ, hence decreases with demutualization.

Homework 7.7. *Suppose that* $b(e) = e^{\alpha}$ *and* $c(e) = e^{\beta}$ *where* $0 < \alpha < 1 < \beta$. *Examine, as a function of* γ, *the solution to the problem* $\max_{e\geq 0}\gamma[J \cdot b(e)] - c(e)$.

This is not to say that stock-holder-owned insurance companies do not support research into safe practices, they fund research that informs Fire Marshalls, and this is a public good.

7.6. **Adverse Selection.** Suppose that individuals i have income w_i, face a loss L_i with $\mu_i = E\,L_i$. The certainty equivalent c_i is defined by $u(w_i - c_i) = E\,u(w_i - L_i)$. We're going to assume that people are risk averse, i.e. that $c_i > \mu_i$, and that c_i is an increasing function of μ_i. In other words, if we rank people by their expected losses, from left to right on the horizontal axis, their certainty equivalents rank them the same way on the vertical axis. For expected loss r, we have certainty equivalent $c(r)$.

For any number r, let $\mu(r)$ be the average loss of everyone whose loss is greater than r. Suppose that we consider pooling arrangements. The constraint is that only i knows what μ_i is.

Offer a risk-pooling arrangement in which everyone puts in the expected population wide losses, i.e. puts in $m = \frac{1}{I}\sum_{i=1}^{I}\mu_i$ and any losses are fully covered (which we can come very close to if I is fairly large). Everyone with $c(\mu_i) > m$ would want to be part. Let $r = c^{-1}(m)$. Provided that some part of the population has $c(\mu_i) < m$, the average loss of everyone who wants to take part is $\mu(r) > m$. So, without some kind of modification, a pool of the whole will not work in this case.

This is called **adverse selection**, only those with higher risks want to be part of the arrangement.

8. Twelfth Meeting, November 27, 2007

Topics: Adverse selection and the market for lemons. Adverse selection and the market for insurance.

Readings: Rasmusen §9.1 through 9.4, and 9.6.

8.1. **Homeworks.** Due: Thursday, December 6, 2007.

Homework 8.1. From ER*:* 9.1.

Homework 8.2. From ER*:* 9.5.

A couple of problems below.

8.2. **Adverse Selection and the Market for Lemons.**

"I refuse to join any club that would have me as a member." (Groucho Marx)

We are going to look at several versions of a story about trying to sell a used car. The basic problem is that the buyer does not know the quality of the car as well as the seller does. Therefore, if we have exchange happening at a given price p, all sellers whose value is less than p want to take part. This means that the value to the buyers is the average of their values of cars that sellers who value them below p have. This can cause problems, the selection of sellers willing to part with their cars is adverse to the functioning of the market. This set of ideas comes from George Akerlof, who received the Nobel Prize in Economics several years ago.

Version I: The seller of a used car knows more about its quality, call it θ, than a potential buyer. The buyer makes an offer to buy at a price p. If the offer is reject, $u_{seller} = u_{buyer} = 0$. If accepted,

$$u_{seller} = p - \theta, \quad \text{and} \quad u_{buyer} = \theta - p.$$

What is distinctive about this case is that both the buyer and the seller value the car the same way, so efficiency may leave the car with the seller or involve a transfer.

Some distributions to work with:

(1) $P(\theta = 10,000) = \alpha$, $P(\theta = 6,000) = 1 - \alpha$.
(2) $\theta \sim U[0, 10,000]$.
(3) $\theta \sim F$, where $F(a) = 0$, $F(b) = 1$, and F has a strictly positive density, f, on the interval $[a, b)$. $0 \leq a < b \leq \infty$.

Graph expected quality on the horizontal axis, price on the vertical, the 45° line gives willingness to pay of buyers, the pairs $(\bar{\theta}(p), p)$ with $\bar{\theta}(p) = E\left(\theta \middle| \theta \leq p\right)$ gives a line that is above the 45° line except at the bottom of the support. Work through this for the three distributions just given.

The fact that $g(x) := E\left(\theta \middle| \theta \leq x\right) \leq x$ is very intuitive — if you average all of your grades below x, or your scores in a video game below x, or your bowling scores below x, the average must be less than x. Slightly more formally, averaging over things less than or equal to x **must** give a number smaller than x, or equal to x in the extreme case that there is nothing smaller than x.

Slightly less intuitive, but still true is that $g(x)$ is an increasing (or at least non-decreasing) function. When you average all your scores below x and compare it to all your scores below $x+1$, the only thing that can have happened is that the average rises. The following is the math version of the argument.

Homework 8.3. *Suppose that $\theta \sim F$, where $F(a) = 0$, $F(b) = 1$, and F has a strictly positive density, f, on the interval $[a, b)$. $0 \leq a < b \leq \infty$.*

1. *Show that the function $g(x) := E\left(\theta \middle| \theta \leq x\right)$ is strictly increasing.*
2. *Show that $g(x) < x$ for $a < x < b$.*
3. *Explicitly calculate $g(x)$ when the cdf is $F(x) = (x - 20)/40$ for $20 \leq x \leq 60$, $F(x) = 0$ for $x < 20$ and $F(x) = 1$ for $x > 60$, and show that $g(x) < x$ for $20 < x < 60$.*
4. *Explicitly calculate $g(x)$ when the cdf is $F(x) = 1 - e^{-x}$, and show that $g(x) < x$ for $0 < x$.*

We assumed that the buyer made a price offer and the seller accepted or rejected, and then the game ended. We might imagine a much more complicated set of haggling strategies. Something called the **revelation principle** comes into play here. Inventors/early users of the revelation principle received this year's Nobel Prize in Economics. It's a very easy idea with rather shocking implications in this situtation — there is no game between the buyer and the seller that gets us to trade except at the bottom of the support of the distribution of values.

Version II: Same as above, except

$$u_{seller} = p - \theta, \quad \text{and} \quad u_{buyer} = 1.2 \cdot \theta - p.$$

Now the buyer always values the car more than the seller. If efficiency is to happen, we must have trade always taking place. George Akerlof's observation is that we will, often, not get this.

Work through the three distributions above once more.

Homework 8.4. *Suppose that $\theta \sim U[20, 60]$, that $u_{seller} = p - \theta$, and $u_{buyer} = 1.3 \cdot \theta - p$.*

1. *What cars trade? At what price?*
2. *If you invent a machine that flawlessly and honestly identifies θ for any car, how much could you charge for its use? What would your expected revenues be?*

8.3. **Adverse Selection and Competition in the Market for Insurance.** Mike Rothschild and Joe Stiglitz were, with George Akerlof, part of the leaders of a large group of economists that worked at trying to understand how differential information affected markets. Stiglitz shared the Nobel with Akerlof, Rothschild has not received a Nobel Prize yet, but I am hopeful.

We are going to follow ER §9.4 pretty closely here. There is one potential purchaser, Mike, of insurance and two competing insurance companies. Mike may be a safe fellow or an unsafe fellow, say with probabilities 0.6 and 0.4. The insurance companies compete by offering contracts (x, y) where Mike pays x and receives y in the event his house burns down. (We will return to the reactive risk aspect of this.) Mike then picks his favorite contract. If Mike is the safe fellow, with probability 0.5 his house burns down, if unsafe, with probability 0.75 his house burns down.

Expected utilities for Mike are: $E\, u_{safe} = 0.5u(12 - x) + 0.5u(0 + y - x)$, $E\, u_{unsafe} = 0.75u(12 - x) + 0.25u(0 + y - x)$.

Expected profits for the insurance company are: fill in in the four cases.

Benchmark: insurance companies observe whether Mike is safe or unsafe.

More interesting case: insurance companies do not observe. Single crossing property of indifference surfaces in the state-space diagram, with the unsafe types having the shallower curves.

9. Last Meeting, December 4, 2007

Topics: Price discrimination and efficiency, information rents. Another look at public goods.

Readings: Rasmusen §10.1, 10.4, and 10.5

9.1. Homeworks.

Due: Tuesday, December 11, 2007.

Homework 9.1. From ER*:* 10.4.

Homework 9.2. From ER*:* 10.5.

Homework 9.3. From ER*:* 10.6.

Homework 9.4. From ER*:* 10.7.

One problem, directly below.

9.2. Review of Monopoly and Efficiency.

Suppose that consumers $i = 1, \ldots, I$ have willingness-to-pay for a single unit of consumption that is equal to v_i. These are, say, the Big Daddy beef rib platters which resemble the rack of ribs that tips over Fred Flintstone's car, and no-one, not ever Fred, can eat more than one. The fancy way to say this is that the consumers all have "unit demand."

For any price, p, charged by a monopolist, the quantity demanded is $Q(p) = \#\{i : v_i \geq p\}$, a decreasing (or at least non-increasing) function. Suppose that the cost of producing q is $c(q)$, where $c(\cdot)$ is an increasing (or at least non-increasing) function. The consumer surplus at price p is

$$CS(p) = \left[\sum_{\{i:v_i \geq p\}}(v_i - p)\right],$$

the producer's profit is

$$\Pi(p) = p \cdot Q(p) - c(Q(p)),$$

and the total surplus is

$$TS(p) = \Pi(p) + CS(p).$$

Homework 9.5. *Suppose that p^*_M is the price that maximizes the producer's profit and p^*_S the price maximizes social surplus, that q^*_M and q^*_S are the associated quantities, and that $p^*_M \neq p^*_S$.*

(1) Show that $p^*_M > p^*_S$.

(2) Show that $q^*_M < q^*_S$.

(3) Now suppose that the monopolist can ***perfectly discriminate*** *between people according to their value, that is, that the monopolist knows each v_i and can charge different prices to different people. Show that q^*_D, the profit maximizing quantity for the monopolist to sell, is equal to q^*_S.*

For present purposes, the thing to notice is that the consumers with higher values benefitted more from having their values be private information to them. That kind of benefit is called an **information rent**, for Ricardian reasons. We are going to look at information rents today.

9.3. **Nonlinear Pricing.** Linear pricing is the system where, if a consumer wants to purchase a quantity q, they have to pay $p \cdot q$. That is, the price paid is a linear function of the quantity. Quantity discounts/premia and two-part tariffs are classic examples of non-linear pricing.

9.3.1. *Two-part Tariffs.* Disneyland and other commercial theme parks are good at charging different people different prices. The idea is that you offer a menu of contracts and people with different values for things sort themselves. This is not perfect discrimination, but it can be pretty good discrimination. We saw this idea when Mary was buying insurance against Mr. Pumpkineater's thievery — the monopoly insurance company could charge a lump sum L to Mary to buy any insurance, and then let her choose how much coverage she wanted at the actuarially fair rate. Carefully choosing L had the effect of getting Mary down to her certainty equivalent.

There are two things to understand: that a two-part tariff can strictly increase profits; and that people sorting themselves means that there is room for information rents.

Suppose that a consumer has income w and utility $u(x) + \$$ when they consume x of the good produced by the monopolist and have \$ left to spend on the rest of their consumption. Further suppose that $u'(0) > c$ and $u''(x) < 0$. Suppose also that the monopolist's costs of production or $C(x) = c \cdot x$ for some positive, constant marginal cost c.

(1) When the monopolist charges p per unit of consumption, i.e. uses linear pricing, $x^*(p)$, the consumer's demand, solves $u'(x) = p$. Put another way, the demand curve with x on the horizontal axis and p on the vertical axis is $D(q) = u'(q)$.
(2) At the optimum q^*, the consumer's marginal utility is larger than c, i.e. there is still room for profitable trade between the monopolist and the consumer, and the consumer surplus is strictly positive. [Diagram in (x, w) space does well here.]
(3) Solve the monopolist's two-part tariff problem,

$$\max_{p,L} L + (p - c)x \text{ subject to } u(x) + w - px - L \geq u(0) + w.$$

Defining $v(x) = u(x) - u(0)$, we see what normalizing utility to 0 is about. [Same diagram, or Lagrangean multipliers.]

9.3.2. *Bulk Discounts/Premia.* Now let us suppose that people are different, say half of them have the utility function $\theta_1 u(x) + \$$, and the other half have the utility function $\theta_2 u(x) + \$$ where $0 < \theta_1 < \theta_2$. Assume $\theta_1 u'(x) > c$. The monopolist offers two price-quantity packages, (r_i, x_i), $i = 1, 2$, and resale is not allowed, e.g. airlines no longer allow people to use tickets sold to others, year-long passes to theme parks require official i.d.

that matches the name on the pass. (Geng, Whinston, Wu, resellable tickets as options.) The monopolist's problem is

$$\begin{aligned}
\max_{(r_1,x_1),(r_2,x_2)} (r_1+r_2) &- c(x_1+x_2) \text{ subject to} \\
\theta_1 u(x_1) + w - r_1 &\geq \theta_1 u(0) + w \\
\theta_2 u(x_2) + w - r_2 &\geq \theta_2 u(0) + w \\
\theta_1 u(x_1) + w - r_1 &\geq \theta_1 u(x_2) + w - r_2 \\
\theta_2 u(x_2) + w - r_2 &\geq \theta_2 u(x_1) + w - r_1
\end{aligned}$$

With $v(x) := u(x) - u(0)$ so that $v(0) = 0$, these are

$$\begin{aligned}
\max_{(r_1,x_1),(r_2,x_2)} (r_1+r_2) &- c(x_1+x_2) \text{ subject to} \\
\theta_1 v(x_1) - r_1 &\geq 0 \\
\theta_2 v(x_2) - r_2 &\geq 0 \\
\theta_1 v(x_1) - r_1 &\geq \theta_1 v(x_2) - r_2 \\
\theta_2 v(x_2) - r_2 &\geq \theta_2 v(x_1) - r_1
\end{aligned}$$

Observations:

(1) At the optimum, some of the constraints will bind.
(2) If the optimum $(r_1^*, x_1^*), (r_2^*, x_2^*)$ is separating, then it is the high value consumer who is getting more than their reservation utility, while the low value consumer is getting their reservation utility.

This last point is the re-emergence of information rents, again for the higher value types. It arises because the high value consumer is worth more to the monopolist, and the monopolist must induce the high value consumer to separate her/himself from the low value one.

The two binding constraints are $\theta_1 v(x_1) - r_1 = 0$ and $\theta_2 v(x_2) - r_2 = \theta_2 v(x_1) - r_1$, look at the FOC and the diagram. Note that $x_1^* < x_2^*$. Much of the time, but not all, this corresponds to bulk discounts, sometimes one sees bulk premia.

9.4. **The Vickrey-Clark-Groves Mechanism for Public Goods.** In looking at mutual insurance companies, the benefits of the research on hazard reduction typically have the property that one firm using the results does not impede another firm from using them. That is, the research is **non-excludable**. The benefits should therefore be added across the firms involved, and the costs split between them.

Now suppose that the benefits of the public good are variable, and known to the people/firms but not the organization incurring the costs. A classic version of this is the crossing guard problem — where, no joke intended, the problem is one of **free riding**.

Vickrey auction (aka second price, Dutch, ascending price), and truth-telling as a dominant strategy equilibrium. The idea is that one's "report" of one's willingness to pay, here in the form of a bid, has no impact on the price you actually pay.

The V-C-G mechanism for funding the crossing guard asks everybody their willingness to pay, w_i, with the understanding that they will pay:

(1) their reported willingness to pay if their report does not make a difference between whether or not the public good is provided;

(2) the difference between the total cost of the good and the sum of everyone else's reported valuations if the person's report is **pivotal**, that is, if $\sum_{j \neq i} \theta_j < C$ and $\theta_i + \sum_{j \neq i} \theta_j \geq C$; and

(3) nothing if the sum of willingnesses to pay is less than or equal to C.

Same idea, truth-telling is the dominant strategy.