Homework #4, Micro III, Spring 2007

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1. Monotonic Changes in Optima: Supermodularity, and Lattices

Throughout economics, we are interested in how changes in one variable affect another variable. We answer such questions **assuming** that what we observe is the result of optimizing behavior.[1] Given what we know about rational choice theory, optimizing behavior involves maximizing a utility function (in many of the interesting models we care about). In symbols, with $t \in T$ denoting a variable not determined by the individual, we let $x^*(t)$ denote the solution(s) to the problem $P(t)$,

$$\max_{x \in X} f(x,t), \tag{1}$$

and ask how $x^*(t)$ depends on t as t varies over T.

Since the problem $P(t)$ in (1) is meant as an approximation to rather than a quantitative representation of behavior, we are after "qualitative" results. These are results that are of the form "if t increases, then $x^*(t)$ will increase," and they should be "fairly immune to" details of the approximation.[2] If $x^*(\cdot)$ is differentiable, then we are after a statement of the form $dx^*/dt \geq 0$. If $x^*(\cdot)$ is not differentiable, then we are after a statement of the form that x^* is non-decreasing in t. We're going to go after such statements three ways.

Implicit function theorem: using derivative assumptions on f when X and T are 1-dimensional intervals.

Simple univariate Topkis's Theorem: using supermodularity assumptions on f when X and T are linearly ordered sets, e.g. 1-dimensional intervals.

Monotone comparative statics: using supermodularity when X is a lattice and T is partially ordered.

Definition 1. *$(X, \precsim)$ is a **partially ordered set** if for all $x, y, z \in X$, $x \precsim x$ (reflexivity), $x \precsim y$ and $y \precsim x$ imply $x = y$ (antisymmetry), and, if $x \precsim y$ and $y \precsim z$, then $x \precsim z$ (transitivity).*

*A partially ordered $(X, \precsim)$ is a **lattice** if each pair of elements, $x, y \in X$, has a unique greatest lower bound, $\min\{x, y\}$, sometimes written $x \wedge y$, and a unique least upper bound, $\max\{x, y\}$, sometimes written $x \vee y$. These are defined by*

1. *$z = (x \wedge y)$ is the unique element of X such that, if $z \precsim x$, $z \precsim y$, and $z \precsim z'$ then either $z' \not\precsim x$ or $z' \not\precsim y$, and*
2. *$z = (x \vee y)$ is the unique element of X such that, if $x \precsim z$, $y \precsim z$, and $z' \precsim z$ then either $x \not\precsim z'$ or $y \not\precsim z'$.*

The uniqueness of $x \wedge y$ and $x \vee y$ are important.

Here is the classic example of a lattice.

[1] Part of learning to "think like an economist" involves internalizing this assumption.

[2] Another part of learning to "think like an economist" involves developing an aesthetic sense of what "fairly immune" means. Aesthetics are complicated and subtle, best gained by immersion and indoctrination in and by the culture of economists, as typically happens during graduate school.

Exercise 1. *Suppose that* $(X, \precsim) = (\mathbb{R}^n, \leq)$, $x \leq y$ *iff* $x_i \leq y_i$, $i = 1, \ldots, n$. *Show that* $(X, \precsim)$ *is a lattice.*

Notice that we can turn lattice orderings around and keep the lattice structure.

Exercise 2. *Suppose that* $(X, \precsim)$ *is a lattice. Define* $x \precsim^- y$ *if* $y \precsim x$. *Show that* $(X, \precsim^-)$ *is a lattice. [Thus,* $(\mathbb{R}^n, \leq^-)$ *is a lattice.]*

Definition 2. *A partial order that also satisfies completeness is called a* ***total (or linear) ordering****, and* $(X, \precsim)$ *is called a* ***totally ordered set****. A* ***chain*** *in a partially ordered set is a subset,* $X' \subset X$*, such that* $(X', \precsim)$ *is totally ordered.*

The classical example of a linearly ordered set is $(\mathbb{R}, \leq)$. In a total ordering, any two elements x and y in A can be compared, in a partial ordering, there are noncomparable elements. For example, $(\mathcal{P}(\mathbb{N}), \subseteq)$ is a partially ordered set with some noncomparable elements. However, the set containing $\{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}, \ldots$ is a chain in $\mathcal{P}(\mathbb{N})$.

Exercise 3. *Show that if* $A \subset B$ *and* B *is totally ordered, then* A *is totally ordered.*

1.1. **The implicit function approach.** Assume that X and T are interval subsets of $\mathbb{R}$, and that f is twice continuously differentiable. f_x, f_t, f_{xx} and f_{xt} denote the corresponding partial derivatives of f. To have $f_x(x, t) = 0$ characterize $x^*(t)$, we must have $f_{xx} < 0$ (this is a standard result about concavity in microeconomics). From the implicit function theorem, we know that $f_{xx} \neq 0$ is what is needed for there to exist a function $x^*(t)$ such that

$$f_x(x^*(t), t) \equiv 0. \tag{2}$$

To find dx^*/dt, take the derivative on both sides with respect to t, and find

$$f_{xx}\tfrac{dx^*}{dt} + f_{xt} = 0, \tag{3}$$

so that $dx^*/dt = -f_{xt}/f_{xx}$. Since $f_{xx} < 0$, this means that dx^*/dt and f_{xt} have the same sign.

Exercise 4. *Let* $X = T = \mathbb{R}_+$, $f(x, t) = x - \frac{1}{2}(x - t)^2$. *Find* $x^*(t)$ *and verify directly that* $dx^*/dt > 0$. *Also find* f_x, f_{xx}, *and* f_{xt}, *and verify, using the sign test just given, that* $dx^*/dt > 0$. *If you can draw three dimensional figures (and this is a skill worth developing), draw* f *and verify from your picture that* $f_{xt} > 0$ *and that it is this fact that make* $dx^*/dt > 0$. *To practice with what goes wrong with derivative analysis when there are corner solutions, repeat this problem with* $X = \mathbb{R}_+$, $T = \mathbb{R}$, *and* $g(x, t) = x - \frac{1}{2}(x + t)^2$.

This ought to be intuitive: if $f_{xt} > 0$, then increases in t increase f_x; increases in f_x are increases in the marginal reward of x; and as the marginal reward to x goes up, we expect that the optimal level of x goes up. In a parallel fashion: if $f_{xt} < 0$, then increases in t decrease f_x; decreases in f_x are decreases in the marginal reward of x; and as the marginal reward to x goes down, we expect that the optimal level of x goes down.

Example 1. *The amount of a pollutant that can be emitted is regulated to be no more than* $t \geq 0$. *The cost function for a monopolist producing* x *is* $c(x, t)$ *with* $c_t < 0$ *and*

$c_{xt} < 0$. *These derivative conditions means that increases is the allowed emission level lower costs and lower marginal costs, so that the firm will always choose* t. *For a given* t, *the monopolist's maximization problem is therefore*

$$\max_{x \geq 0} f(x,t) = xp(x) - c(x,t) \tag{4}$$

where $p(x)$ *is the (inverse) demand function. Since* $f_{xt} = -c_{xt}$, *we know that increases in* t *lead the monopolist to produce more,* ***provided*** $f_{xx} < 0$.

The catch in the previous analysis is that $f_{xx} = xp_{xx} + p_x - c_{xx}$, so that we need to know $p_{xx} < 0$, concavity of inverse demand, and $c_{xx} > 0$, convexity of the cost function, before we can reliably conclude that $f_{xx} < 0$. The global concavity of $f(\cdot, t)$ seems to have little to do with the intuition that it is the lowering of marginal costs that makes x^* depend positively on t. However, global concavity of $f(\cdot, t)$ is **not** what we need for the implicit function theorem, only the concavity of $f(\cdot, t)$ in the region of $x^*(t)$. This local concavity is an **implication** of the second order derivative conditions for $x^*(t)$ being a strict local maximum for $f(\cdot, t)$. What supermodularity does is to make it clear that the local maximum property is all that is being assumed, and to allow us to work with optima that are non-differentiable.

1.2. **The simple supermodularity approach.** The simplest case has X and T being linearly ordered sets. The most common example has X and T being intervals in $\mathbb{R}$ with the usual less-than-or-equal-to order. However, nothing rules out the sets X and T being discrete.

Definition 3. *For linearly ordered* X *and* T, *a function* $f : X \times T \to \mathbb{R}$ *is* ***supermodular*** *if for all* $x' \succ x$ *and all* $t' \succ t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t),$$

equivalently

$$f(x', t') - f(x', t) \geq f(x, t') - f(x, t).$$

It is ***strictly supermodular*** *if the inequalities are strict.*

At t, the benefit of increasing from x to x' is $f(x', t) - f(x, t)$, at t', it is $f(x', t') - f(x, t')$. This assumption asks that benefit of increasing x be increasing in t. A good verbal shorthand for this is that f **has increasing differences in** x **and** t. Three sufficient conditions in the differentiable case are: $\forall x$, $f_x(x, \cdot)$ is nondecreasing; $\forall t$, $f_t(\cdot, t)$ is nondecreasing; and $\forall x, t$, $f_{xt}(x, t) \geq 0$.

Theorem 1. *If* $f : X \times T \to \mathbb{R}$ *is supermodular and* $x^*(\tau)$ *is the largest solution to* $\max_{x \in X} f(x, \tau)$ *for all* τ, *then* $[t' \succ t] \Rightarrow [x^*(t') \succsim x^*(t)]$.

If there are unique, unequal maximizers at t' and t, then $x^*(t') \succ x^*(t)$.

Proof. Suppose that $t' \succ t$ but that $x' = x^*(t') \prec x = x^*(t)$. Because $x^*(t)$ and $x^*(t')$ are maximizers, $f(x', t') \geq f(x, t')$ and $f(x, t) \geq f(x', t)$. Since x' is the largest of the maximizers at t' and $x \succ x'$, we know a bit more, that $f(x', t') > f(x, t')$. Adding the inequalities, we get $f(x', t') + f(x, t) > f(x, t') + f(x', t)$, or

$$f(x, t) - f(x', t) > f(x, t') - f(x', t').$$

But $t' \succ t$ and $x \succ x'$ and supermodularity imply that this inequality must go the other way. □

Going back to Example 1 (p. 2), we can substitute f in the relations of Definition 3 (p. 3). Then these relations for the supermodularity of f reduce to those for supermodularity of $-c$. Thus assuming $-c$ (and hence f) is supermodular, we can use Theorem 1 for f, which implies that $x^*(t)$ is increasing. None of the second derivative conditions except $c_{xt} < 0$ are necessary, and this can be replaced by the looser condition that $-c$ is supermodular.

Clever choices of T's and f's can make some analyses criminally easy.

Example 2. *Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$ so that $CS(p) = \int_p^\infty x(r)\,dr$ is the consumer surplus when the price p is charged. Let $p(\cdot)$ be $x^{-1}(\cdot)$, the inverse demand function. From intermediate microeconomics, you should know that the function $x \mapsto CS(p(x))$ is nondecreasing.*

The monopolist's profit when they produce x is $\pi(x) = x \cdot p(x) - c(x)$ where $c(x)$ is the cost of producing x. The maximization problem for the monopolist is

$$\max_{x \geq 0} \ \pi(x) + 0 \cdot CS(p(x)).$$

Society's surplus maximization problem is

$$\max_{x \geq 0} \ \pi(x) + 1 \cdot CS(p(x)).$$

Set $f(x, t) = \pi(x) + tCS(p(x))$ where $X = \mathbb{R}_+$ and $T = \{0, 1\}$. Because $CS(p(x))$ is nondecreasing, $f(x, t)$ is supermodular.[3] *Therefore $x^*(1) \geq x^*(0)$, the monopolist always (weakly) restricts output relative to the social optimum.*

Here is the externalities intuition: increases in x increase the welfare of people the monopolist does not care about, an effect external to the monopolist; the market gives the monopolist insufficient incentives to do the right thing.

To fully appreciate how much simpler the supermodular analysis is, we need to see how complicated the differentiable analysis would be.

Example 3. ($\uparrow$*Example* 2 (p. 4)) *Suppose that for every $t \in [0, 1]$, the problem*

$$\max_{x \geq 0} \ \pi(x) + t \cdot CS(p(x))$$

has a unique solution, $x^*(t)$, and that the mapping $t \mapsto x^*(t)$ is continuously differentiable. (This can be guaranteed if we make the right kinds of assumptions on $\pi(\cdot)$ and $CS(p(\cdot))$.) To find the sign of $dx^*(t)/dt$, we assume that the first order conditions,

$$\pi'(x^*(t)) + tdCS(p(x^*(t)))/dx \equiv 0$$

[3]This is an invitation/instruction to check this last statement.

characterize the optimum. In general, this means that we need to assume that $x \mapsto \pi(x) + tCS(p(x))$ *is a smooth, concave function. We then take the derivative of both sides with respect to* t*. This involves evaluating* $d(\int_{p(x^*(t))}^{\infty} x(r)\,dr)/dt$*. In general,* $d(\int_{f(t)}^{\infty} x(r)\,dr)/dt = -f'(t)x(f(t))$*, so that when we take derivatives on both sides, we have*

$$\pi''(x^*(t))(dx^*/dt) + dCS(p(x^*(t)))/dx - p'(x^*)(dx^*/dt)x(p(x^*)) = 0.$$

Gathering terms, this yields

$$[\pi''(x^*) - p'(x^*)x(p(x^*))](dx^*/dt) + dCS(p(x^*(t)))/dx = 0.$$

Since we are assuming that we are at an optimum, we know that $\pi''(x^*) \leq 0$*, by assumption,* $p'(x^*) < 0$ *and* $x > 0$*, so the term in the square brackets is negative. As argued above,* $dCS(p(x^*(t)))/dx > 0$*. Therefore, the only way that the last displayed equation can be satisfied is if* $dx^*/dt > 0$*. Finally, by the fundamental theorem of calculus (which says that the integral of a derivative is the function itself),* $x^*(1) - x^*(0) = \int_0^1 \frac{dx^*(r)}{dt}\,dr$*. The integral of a positive function is positive, so this yields* $x^*(1) - x^*(0) > 0$*.*

1.3. **Monotone comparative statics.** Suppose that $(X, \precsim_X)$ and $(T, \precsim_T)$ are linearly ordered sets. Define the order $\precsim_{X \times T}$ on $X \times T$ by $(x', t') \succsim_{X \times T} (x, t)$ iff $x' \succsim_X x$ and $t' \succsim_T t$. (This is the unanimity order again.)

Lemma 1. $(X \times T, \precsim_{X \times T})$ *is a lattice.*

Proof. $(x', t') \vee (x, t) = (\max\{x', x\}, \max\{t', t\}) \in X \times T$.
$(x', t') \wedge (x, t) = (\min\{x', x\}, \min\{t', t\}) \in X \times T$. □

Definition 4. *For a lattice* $(L, \precsim)$*,* $f : L \to \mathbb{R}$ *is* ***supermodular*** *if for all* $\ell, \ell' \in L$*,*

$$f(\ell \wedge \ell') + f(\ell \vee \ell') \geq f(\ell) + f(\ell'),$$

equivalently,

$$f(\ell \vee \ell') - f(\ell') \geq f(\ell) - f(\ell \wedge \ell').$$

Exercise 5. *Show that taking* $\ell' = (x', t)$ *and* $\ell = (x, t')$ *recovers Definition* 3*.*

Exercise 6. *Let* $(L, \precsim) = (\mathbb{R}^n, \leq)$*. Show that* $f : L \to \mathbb{R}$ *is supermodular iff it has increasing differences in* x_i *and* x_j *for all* $i \neq j$*. Show that a twice continuously differentiable* $f : L \to \mathbb{R}$ *is supermodular iff* $\partial^2 f/\partial x_i \partial x_j \geq 0$ *for all* $i \neq j$*.*

Sometimes different orders are useful.

Exercise 7. *Let* $A := \{a_m : m \in \{1, \ldots, M\}\}$ *be a collection of vectors in* $\mathbb{R}^n$*. Define* $x \leq_A y$ *if* $a_m \cdot x \leq a_m \cdot y$ *for all* $m \in \{1, \ldots, M\}$*. With* $A = \{e_i : i \in \{1, \ldots, n\}\}$ *being the set of unit vectors, this recovers the ordering in the previous problem.*

(1) *Show that if* A *is a spanning, linearly independent collection of vectors, then* $(\mathbb{R}^n, \leq_A)$ *is a lattice.*
(2) *More generally, we define* $C \subset \mathbb{R}^n$ *to be a* ***pointy, closed, convex cone*** *if* (a) *it is closed,* (b) *it is a convex cone, that is, for all* $x, y \in C$ *and all* $\alpha, \beta \geq 0$*,* $\alpha x + \beta y \in C$*, and* (c)*,* 0 *is the point of the cone, that is, if* $x, y \in C$ *and* $0 = \alpha x + (1 - \alpha)y$ *for some* $\alpha \in [0, 1]$*, then either* $x = 0$ *or* $y = 0$*.*

(a) *Show that if A is a linearly independent collection of vectors, then $C = \{x \in \mathbb{R}^n : \forall a \in A,\ x \cdot a \leq 0\}$ is a pointy closed convex cone.*
(b) *Show that if C is a pointy closed convex that spans $\mathbb{R}^n$ and we define $x \leq_C y$ if $(x - y) \in C$, then $(\mathbb{R}^n, \leq_C)$ is a lattice.*
(c) *Let $x \neq 0$ be a point in $\mathbb{R}^n$ and $r > 0$ such that $\|x\| > r$. Show that there is a smallest closed cone with a point containing $\{y : \|y - x\| \leq r\}$, and that it spans $\mathbb{R}^n$. [For $n \geq 3$, this is a cone that is **not** defined by finitely many linear inequalities.]*

(3) *Give conditions on A such that the ordering $\leq_A$ is trivial, that is, $x \leq_A y$ iff $x = y$.*
(4) *For A a spanning, linearly independent set and a twice continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$, give a condition on the derivatives that is equivalent to f being supermodular on the lattice $(\mathbb{R}^n, \leq_A)$. [Doing this in $\mathbb{R}^2$ is enough.]*

Exercise 8. *Let $Non \subset \mathbb{R}^n$ be the set of vectors $x' = (x_1, , x_n)$ with the property that $x_1 \leq x_2 \leq \cdots \leq x_n$, that is, the set of non-decreasing vectors. Define $x \geq_{Non} y$ if $(x - y) \in Non$, equivalently, $x \leq_{Non} y$ if $(y - x) \in Non$.*

(1) *Show that Non is a convex cone in $\mathbb{R}^n$, that is, for all $x, y \in Non$ and all $\alpha, \beta \geq 0$, $\alpha x + \beta y \in Non$.*
(2) *Show that $(\mathbb{R}^n, \leq_{Non})$ is a **not** a lattice because Non is not pointy.*
(3) *For $x, y \in \mathbb{R}^2$, draw Non and show that $\min\{x, y\}$ and $\max\{x, y\}$ are not defined.*
(4) *In $\mathbb{R}^2$, $x \leq_{Non} y$ iff $x \leq_A y$ for some set of vectors A. Give the set A. Generalize to $\mathbb{R}^n$. [Here you should see that $\leq_{Non}$ is not a lattice ordering because the set A does not span.]*

The idea of a convex cone extends to sets of functions.

Exercise 9. *Let Non be the set of non-decreasing functions from $[0, 1]$ to $\mathbb{R}$.*

(1) *Show that Non is a convex cone, that is, show that for all $f, g \in Non$ and all $\alpha, \beta \geq 0$, $\alpha f + \beta g \in Non$.*
(2) *Show that Non is not pointy.*
(3) *Define $f \geq_{Non} g$ if $(f - g) \in Non$, equivalently, $f \leq_{Non} g$ if $(g - f) \in Non$. Show that $(C[0, 1], \leq_{Non})$ is **not** a lattice.*

Definition 5. *For $A, B \subset L$, L a lattice, the **strong set order** is defined by $A \precsim_{Strong} B$ iff $\forall (a, b) \in A \times B$, $a \wedge b \in A$ and $a \vee b \in B$.*

Interval subsets of $\mathbb{R}$ are sets of the form $(-\infty, r)$, $(-\infty, r]$, (r, s), $(r, s]$, $[r, s)$, $[r, s]$, (r, ∞), or $[r, \infty)$.

Exercise 10. *For intervals $A, B \subset \mathbb{R}$, $A \precsim_{Strong} B$ iff every point in $A \setminus B$ is less than every point in $A \cap B$, and every point in $A \cap B$ is less than every point in $B \setminus A$. This is true when $\mathbb{R}$ is replaced with any linearly ordered set.*

The strong set order is not, in general, reflexive. Subsets of $(\mathbb{R}, \leq)$ are linearly ordered, hence they are a lattice. Subsets of $(\mathbb{R}^2, \leq)$ are not necessarily lattices. For any non-lattice subset, A, of $\mathbb{R}^2$, $\neg[A \precsim_{Strong} A]$.

For $t \in T$, let $M(t)$ be the set of solutions to the problem $\max_\ell f(\ell, t)$.

Theorem 2. *If $(L, \precsim_L)$ is a lattice, $(T, \precsim_T)$ is a partially ordered set, $f : L \times T \to \mathbb{R}$ is supermodular in ℓ for all t, and has increasing differences in ℓ and t, then $M(t)$ is nondecreasing in t.*

Exercise 11. *Prove Theorem 2.*

Exercise 12. *Show that if $(X, \precsim_X)$ is linearly ordered, then **any** function $f : X \to \mathbb{R}$ is supermodular (with equalities in Definition 4).*

1.4. **Another lattice ordering in $C[0,1]$.** The standard partial ordering for $f, g \in C[0,1]$ is $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in [0,1]$. This is the ordering used in the Stone-Weierstrass Theorem. Here we introduce another partial ordering, useful for a class of mechanism design problems.

Intuitively, the order is $f \succsim g$ iff f is below g and "steeper than" g. We will only apply it to non-decreasing functions.

Definition 6. *$f \succsim g$ iff $f \leq g$ and for all $0 \leq r < s \leq 1$, $f(s) - f(r) > g(s) - g(r)$.*

Exercise 13. *If $f(x) = x$ and $g(x) = x^2$, find $f \vee g$ and $f \wedge g$.*

Exercise 14. *Suppose that for each $e \in E \subset \mathbb{R}$, F_e is the cdf of a distribution on $[0,1]$, that is, $F_e(r) = 0$ for all $r < 0$, and $F_e(1) = 1$. Further suppose that for each $e < e'$, $e, e' \in E \subset \mathbb{R}$, $F_{e'}$ first order stochastically dominates F_e.*

For any non-decreasing reward/wage function $w : [0,1] \to \mathbb{R}_+$, consider the problem $\max_{e \in E} \int u(w(\pi))\, dF_e(\pi) - v(e)$ where $u(\cdot)$ is an increasing, concave function, and $v(\cdot)$ is a non-decreasing function. Let $e^*(w)$ denote the set of optimal e's for a given w.

Show that $w_1 \succsim w_2$ implies that $e^*(w_1)$ dominates $e^*(w_2)$ in the set order.