Concavity and continuity, *Mathematics for Economists*
Economics 392M.8, Fall 2013

This alternative proof that a concave function is continuous on the relative interior of its domain first shows that it is bounded on small open sets, then from boundedness and concavity, derives continuity.

**Theorem 1.** If \( f : C \to \mathbb{R} \) is concave, \( C \subset \mathbb{R}^\ell \) convex with non-empty interior, then \( f \) is continuous on \( \text{int}(C) \).

Three lemmas deliver the proof. Throughout, we maintain the assumptions that \( f : C \to \mathbb{R} \) is concave, that \( C \) is convex, and that it has non-empty interior. We first show that a concave \( f \) is bounded below on small balls around any point in the interior.

**Lemma 1.** For all \( x_0 \in \text{int}(C) \), there exists \( \epsilon > 0 \) and \( r \in \mathbb{R} \) such that \( f(B_r(x_0)) \geq r \).

**Proof.** Pick \( \epsilon' > 0 \) such that \( B_{2\epsilon'}(x_0) \subset C \). Consider the \( 2 \cdot \ell \) points \( S = \{x_0 \pm \epsilon'e_i : i \in \{1, \ldots, \ell\} \} \) where \( e_i \) is the unit vector in the \( i \)th direction. By concavity, any \( y \in \text{co}(S) \) is a convex combination of the \( x \in S \), therefore \( f(y) \geq \sum \alpha_x x \geq \min\{f(x) : x \in S\} \). Now pick \( \epsilon > 0 \) such that \( B_\epsilon(x_0) \subset \text{co}(S) \). \( \square \)

We now show that \( f \) is bounded on small balls around any point in the interior.

**Lemma 2.** For \( B_\epsilon(x_0) \subset C \), if \( f(B_\epsilon(x_0)) \geq m \), then \( |f(B_\epsilon(x_0))| \leq |m| + 2f(x_0) \).

**Proof.** Translate \( B_\epsilon(x_0) \) by subtracting \( x_0 \), i.e. suppose that \( x_0 = 0 \). For any \( y \in B_\epsilon(0) \), \( f(0) \geq \frac{1}{2}f(y) + \frac{1}{2}f(-y) \), so that

\[
\frac{1}{2}f(y) \leq f(0) - \frac{1}{2}f(-y), \quad \text{or} \quad f(y) \leq 2f(0) - f(-y).
\]

Now \( 2f(0) - f(-y) \leq 2f(0) - m \leq 2|f(0)| + |m| \) and \( m \leq f(y) \), we have \( m \leq f(y) \leq 2|f(0)| + |m| \), so in particular, \( |f(y)| \leq 2|f(0)| + |m| \). \( \square \)

We now show that \( f \) is Lipschitz continuous on small neighborhoods.

**Lemma 3.** For \( B_\epsilon(x_0) \subset C \) and \( B \) its closure, if \( |f(B)| \leq M \), then for any \( \epsilon \in (0,r) \), \( f \) is \( \left( \frac{2M}{\epsilon} \right) \)-Lipschitz on \( B_{(r-\epsilon)}(x_0) \).

**Proof.** Pick \( x \neq y \in B_{(r-\epsilon)}(x_0) \). We want to show that \( |f(y) - f(x)| \leq \frac{2M}{\epsilon} \|y - x\| \).

Consider the point \( z := y + \frac{\epsilon}{\epsilon + \|y - x\|} (y - x) \) that belongs to \( B \), and note that \( y \) is between \( x \) and \( z \), specifically, \( y = \frac{\epsilon}{\epsilon + \|y - x\|} x + \frac{\|y - x\|}{\epsilon + \|y - x\|} z \). By concavity, \( f(y) \geq \frac{\epsilon}{\epsilon + \|y - x\|} f(x) + \frac{\|y - y\|}{\epsilon + \|y - x\|} f(z) \), hence \( (\epsilon + \|y - x\|) f(y) \geq \epsilon f(x) + \|y - y\| f(z) \), or

\[
\epsilon(f(x) - f(y)) \leq ||y - x|| (f(y) - f(z)).
\]

Now \( (f(y) - f(z)) \leq 2M \), hence \( (f(x) - f(y)) \leq \frac{2M}{\epsilon} \|y - x\| \). Interchanging the names/roles of \( x \) and \( y \), \( (f(y) - f(x)) \leq \frac{2M}{\epsilon} \|y - x\| \). Combining, \( |f(y) - f(x)| \leq \frac{2M}{\epsilon} \|y - x\| \). \( \square \)