Assignment #6 for **Mathematics for Economists**
Economics 362M, Spring 2010

Due date: Tue. Mar. 23.
Readings: CSZ, Ch. 3.6-7, 3.9, 4.1-2.

In this part of the class, we are going to finish looking at properties of sequences in $\mathbb{R}$: the equivalence of being convergent and being Cauchy; summability; monotonicity and convergence. Related to the convergence of Cauchy sequences is the existence of the supremum and the infimum of a bounded set of numbers. Finally, combining supremum/infimum and sequences, we will consider the properties of the $\liminf_n x_n$ and $\limsup_n x_n$.

We will then turn to the topic of Chapter 4, metric spaces. The metric, or measure of distance, in $\mathbb{R}$ is given by $d(x, y) = |x - y|$. This function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ satisfies $d(x, y) = d(y, x)$, $d(x, y) = 0$ iff $x = y$, and $d(x, y) + d(y, z) \geq d(x, z)$. We are going to take these properties as the *definition* of a measure of distance. In other words, if we have a set M and a function $d : M \times M \to \mathbb{R}_+$ that satisfies these three properties, then we will call (M, d) a **metric space** and refer to $d(x, y)$ as the distance between x and y. There are two points to be made here.

First, the metric spaces that we most care about in this class are $M = \mathbb{R}^\ell$, $\ell \in \mathbb{N}$, the space of length-ℓ vectors of real numbers with the metric $d_2(\mathbf{x}, \mathbf{y}) = (\sum_{i=1}^{\ell} |x_i - y_i|^2)^{1/2}$. If $\ell = 2$, then M is the usual Euclidean plane, and the Pythagorean theorem tells you that $d_2(\mathbf{x}, \mathbf{y})$ measures the distance of the straight line joining $\mathbf{x}$ and $\mathbf{y}$. By induction, this is true for $\ell \geq 2$ as well, but after $\ell = 4$, visualization fails.

Second, for later mathematical developments, we want to make arguments that only depend on the three defining properties of a metric rather than arguments that depend on the geometry of the Euclidean plane.

From Chapter 3.6: 3.6.4, as well as parts 1 and 2 of 3.6.6.
From Chapter 3.7: 3.7.12, 3.7.16, and 3.7.17.
From Chapter 3.9: 3.9.3 and 3.9.5.

The following is a pair of uses of "a.a." and "i.o." that will be frequently useful: $[x_n \to x]$ iff $(\forall \epsilon > 0)[|x_n - x| < \epsilon]$ a.a., and $\neg[x_n \to x]$ iff $(\exists \epsilon > 0)[|x_n - x| \geq \epsilon]$ i.o.

Homework 6.A. *Show the following:*
(1) $\lim_n(\sqrt{n+1} - \sqrt{n}) = 0$.
(2) $\lim_n(\sqrt{n^2 - n} - n) = \lim_n(\sqrt{n^2 - n} - \sqrt{n^2}) = 1/2$.

Homework 6.B. *Show the following:*
(1) *If* $\sum_{t=1}^{n} x_t \to S$, *then* $x_t \to 0$.
(2) $x_t \to 0$ *need not imply that* $\sum_{t=1}^{n} x_t$ *is convergent.*

Homework 6.C. *Suppose than x_n is a sequence of non-negative numbers with $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, i.e. suppose that x_n is a non-increasing sequence on non-negative numbers.*
(1) *Show that the partial sums,* $\sum_{t=1}^{n} x_n$ *are bounded if and only if the partial sums*

$$\sum_{k=1}^{n} 2^k x_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

are bounded. [Hint: The sequence $x_n = 1/n$ is strictly positive and decreasing. We showed that the series $x_n = 1/n$ diverges by showing that the first term is greater than $1/2$, the sum of the next 2 terms is greater than $1/2$, the sum of the next 4 terms is greater than $1/2$, the sum of the next 8 terms is greater than $1/2$, and so on and so on.]

(2) *Using the previous, show that the sequence $x_n = 1/n^p$ is summable iff $p > 1$. [Don't forget $p \leq 0$ is a possibility.]*

Homework 6.D (Root test for summability)**.** *For a sequence $(x_n)_{n\in\mathbb{N}}$, define*

$$\alpha((x_n)_{n\in\mathbb{N}}) = \limsup_n (|x_n|)^{1/n}.$$

(1) *Suppose that $\alpha((x_n)_{n\in\mathbb{N}}) < 1$. Show that there exists $\beta < 1$ such that $(|x_n|)^{1/n} < \beta$ a.a.*

(2) *Show that if $\alpha((x_n)_{n\in\mathbb{N}}) < 1$, then the sequence x_n is dominated by a geometric sequence a.a., hence is summable.*

(3) *Suppose that $\alpha((x_n)_{n\in\mathbb{N}}) > 1$. Show that $|x_n| > 1$ i.o., so that x_n is not summable.*

(4) *Show that for the sequences $x_n = 1/n$ and $y_n = 1/n^2$, $\alpha((x_n)_{n\in\mathbb{N}}) = \alpha((y_n)_{n\in\mathbb{N}}) = 1$. [This means that the root test is not informative for the summability of the sequence $(x_n)_{n\in\mathbb{N}}$ if $\alpha((x_n)_{n\in\mathbb{N}}) = 1$.]*

Homework 6.E. *Consider the series $(x_1, x_2, x_3, \ldots)$ given by*

$$\left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2^2}, \tfrac{1}{3^2}, \tfrac{1}{2^3}, \tfrac{1}{3^3}, \tfrac{1}{2^4}, \tfrac{1}{3^4}, \cdots\right).$$

Show the following:

(1) $\liminf_n \frac{x_{n+1}}{x_n} = 0$ *and* $\limsup_n \frac{x_{n+1}}{x_n} = +\infty$. *[This means that the ratio test does **not** apply to this sequence.]*

(2) $\limsup_n (|x_n|)^{1/n} = \frac{1}{\sqrt{2}}$. *[This means that the root test **does** apply to this sequence.]*

Homework 6.F. *We know that the sequence $x_n = 1/2^{n-1}$, $n \in \mathbb{N}$, is summable because it is geometric. Consider the following rearrangement,*

$$(y_1, y_2, y_3, \ldots) = (1/2, 1, 1/8, 1/4, 1/32, 1/16, \ldots).$$

Show the following:

(1) $\limsup_n \frac{y_{n+1}}{y_n} = 2$ *and* $\liminf_n \frac{y_{n+1}}{y_n} = 1/2$.

(2) $\limsup_n (|y_n|)^{1/n} = \liminf_n (|y_n|)^{1/n} = \lim_n (|y_n|)^{1/n} = 1/2$.

Assignment #7 for **Mathematics for Economists**
Economics 362M, Spring 2010

Due date: Tue. Mar. 30.
Readings: CSZ, Ch. 4.3-4.
From Chapter 4.3: 4.3.6 and 4.3.11.
From Chapter 4.4: 4.4.3 and 4.4.6.