

**An Introduction to Mathematical
Analysis in Economics:
Some Advanced Math from an
Elementary Point of View**

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