Some Game Theory

Assignment #5 for Managerial Economics, ECO 351M, Fall 2016
Due, Monday November 21.

The Basics

In economics, we use what are called *games* to analyze strategic interactions. To specify a game, we must specify

- the actors involved, be they people, firms, or other organizations,
- the choices available to the actors, and
- the payoffs/utilities received after every actor has made their choice.

The subtlety arises in the second point, to specify "choices available" includes specifying what the actors know when they are making their choices. For example, if I do not know which of my two possible routes home have delays due to construction or heavy traffic, then I do **not** have available to me the action "Take the route that is quickest today." Rather, I can pick a route, and it may or may not turn out to be the quickest today. However, knowing the general patterns of traffic on the different days of the week, I will pick to maximize the probability that I arrive home without delays. To take another example, if I am playing, say, tennis, and I do not know whether my opponent will serve to my backhand side or my forehand side, my choice of where to position myself **cannot** depend on where the serve will go.

After we have specified the game, we must figure out what will happen. This is formalized by the process of finding an **equilibrium**. Following John Nash's definition, a vector of choices, one for each of the actors, is an equilibrium if the choice of each actor, taking as given that the others are action according to the vector, maximizes their own expected utility.

More formally now, a game is a collection $\Gamma = (A_i, u_i)_{i \in I}$. This has three pieces:

(1) I is the (usually finite) set of agents/people/players,
(2) for each $i \in I$, A_i is the set of actions or strategies available to i, and, setting $A = \times_{i \in I} A_i$,
(3) for each $i \in I$, $u_i : A \to \mathbb{R}$ represents i's preferences, specifically their von Neumann-Morgenstern utility functions to represent their preferences over the possibly random actions chosen by others.

Γ is **finite** if I and each A_i is a finite set.

Having described who is involved in a strategic situation, the set I, and having described their available choices, the sets A_i, and their preferences over their own and everybody else's choices, we try to figure out what is going to happen. We have settled on a notion of equilibrium, due to John Nash, as our answer to the question of what will happen. The answer comes in two flavors: pure; and mixed.

> A **pure strategy Nash equilibrium** is a vector $a^* \in A$, ofter written $a^* = (a_i^*, a_{-i}^*)$, of actions with the property that each a_i^* is a best response to a_{-i}^*.

Δ_i or $\Delta(A_i)$ denotes the set of probability distributions on A_i, and it called the set of **mixed strategies**; $\Delta := \times_{i \in I} \Delta_i$ denotes the set of product measures on A; each u_i is extended to Δ by integration, $u_i(\sigma) := \int_A u_i(a_1, \ldots, a_I)\, d\sigma_1(a_1) \cdots \sigma_I(a_I)$.

A **mixed strategy Nash equilibrium** is a vector $\sigma^* \in \Delta$, $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$, of with the property that each σ_i^* is a best response to σ_{-i}^*.

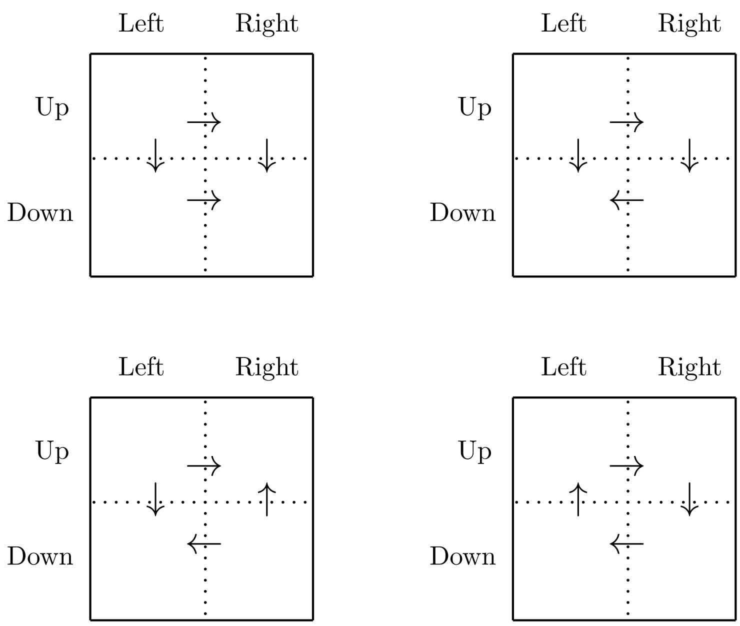
The following notation is very useful.

Notation. For $\sigma^\circ \in \Delta$, $Br_i(\sigma^\circ) = \{\sigma_i \in \Delta_i : u_i(\sigma_i, \sigma^\circ_{-i}) \geq u_i(\Delta_i, \sigma^\circ_{-i})\}$ is i's **best response to** σ.

Thus, σ^* is an equilibrium iff for all $i \in I$ we have $\sigma_i^* \in Br_i(\sigma^*)$.

An equilibrium σ^* is **strict** if $(\forall i \in I)[\#Br_i(\sigma^*) = 1]$.

A 2×2 game is one with two actors each of which has two actions. In lecture, we went through the best response arrow analysis of these games and found the following patterns.



• In the class of games represented by the top left box, both players have a dominant strategy. We will first see this under the names of the "Prisoners' Dilemma" and "Joint Investment."
• In the class of games represented by the top right box, exactly one player has a dominant strategy. We will first see this under the name of "Rational Pigs."
• In the class of games represented by the bottom left box, neither player has a dominant strategy and there are two action pairs with arrows pointing into them. Games in this class are called coordination games, the named games in this class are the "Battle of the Sexes," the "Stag Hunt," and "Hawk-Dove."
• In the class of games represented by the bottom right box, neither player has a dominant strategy and there is no action pair with arrows pointing into it. Games in this class have no pure strategy equilibrium, the named games in this class are "Matching Pennies" or "Penalty Kick," and "Monitoring/Auditing Games."

Problems

1. [The Advantage of Being Small] Here I = {Big, Little}, $A_1 = A_2$ = {Push, Wait} and the utilities are given in the following table.

Rational Pigs

	Push	Wait
Push	$(-1, 5)$	$(-1, 6)$
Wait	$(3, 2)$	$(0, 0)$

The numbers were generated from the following story: there are two pigs, one Big and one Little, and each has two actions. Little pig is player 1, Big pig player 2, the convention has 1's options being the rows, 2's the columns, payoffs (x, y) mean "x to 1, y to 2." The two pigs are in a long room. A lever at one end, when pushed, delivers food, worth 6 in utility terms, into a trough at the other end. Until the food has been emptied from the trough, the lever is non-functional, once the food has been emptied, it will again deliver food. Pushing the lever gives the pigs a shock on their sensitive snouts, causing a dis-utility of -1. The Big pig can move the Little pig out of the way and take all the food if they are both at the food trough together, the two pigs are equally fast getting across the room. During the time that it takes the Big pig to cross the room, Little can eat $\alpha = \frac{1}{2}$ of the food.

a. Give the best response arrows diagram for this game.
b. Solve this game using iterated deletion of dominated strategies (as explained in the textbook). Be explicit about what different actions are or are not dominated.
c. Give some other strategic situation, from your life or any other source, in which the weaker/smaller person ends up better off because they are weaker/smaller.

2. Consider the following two games.

Prisoners' Dilemma

	Squeal	Silent
Squeal	$(-8, -8)$	$(0, -9)$
Silent	$(-9, 0)$	$(-1, -1)$

Joint Investment

	Don't invest	Invest
Don't invest	$(2, 2)$	$(12, 0)$
Invest	$(0, 12)$	$(9, 9)$

a. Give the best response arrows diagram for these games.
b. Solve this game using iterated deletion of dominated strategies. Be explicit about what different actions are or are not dominated.

3. In this game, the CEO or CFO can Fiddle the Books (accounts) in a fashion that makes their stock options or restricted stock awards or phantom stock plans or stock appreciation rights are more valuable, or else they can prepare the quarterly report in a fashion that respects the letter and the spirit of the GAAP (Generally Accepted Accounting Practices) rules. The accounting firm that the CEO/CFO hires to audit the accounts can do an in-depth audit or they can let it slide. We are going to suppose that the auditing firm's reputation for probity is valuable enough to them that they would prefer to catch Fiddles when they occur. Putting symbols into the game, we have

	Audit	Let it slide
Fiddle	$(s - p, B + d)$	$(s + b, B)$
GAAP	$(s, B - c)$	(s, B)

Here s represents the base salary, p the penalty for being caught Fiddling the Books, c is the cost of an in-depth audit, d is the reputation benefit, net of costs, to the auditor from deterring other auditees from Fiddling, and B is the auditing firm's baseline payoff.

a. Give the best response arrows diagram for these games.
b. Explain why you cannot solve this game using iterated deletion of dominated strategies. Be explicit.
c. Solve for the unique mixed strategy equilibrium for this game.
d. What are the effects on the equilibrium strategy of the CEO/CFO and the equilibrium strategy of the Auditor if $p \uparrow$? Explain both of the results that you find.
e. What are the effects on the equilibrium strategy of the CEO/CFO and the equilibrium strategy of the Auditor if $b \uparrow$? Explain both of the results that you find.
f. What are the effects on the equilibrium strategy of the CEO/CFO and the equilibrium strategy of the Auditor if $d \uparrow$? Explain both of the results that you find.
g. What are the effects on the equilibrium strategy of the CEO/CFO and the equilibrium strategy of the Auditor if $c \uparrow$? Explain both of the results that you find.
h. What happens to the expected utility of the equilibrium payoffs for the four changes to the payoffs you have just considered?

4. Two firms, a supplier and a manufacturer can invest in expensive, complementary technologies, and if they both do this, they will achieve the high quality output that will guarantee both high profits. The problem is that if one of them has invested, the other firm would be better "free riding" on their investment, it's an expensive investment for both of them, and the improvements on just one side will improve profits somewhat, at no cost to the non-investor. Putting numbers on the payoffs, let us suppose they are

	Don't invest	Invest
Don't invest	$(5, 7)$	$(32, 0)$
Invest	$(0, 22)$	$(28, 19)$

a. Suppose that a vertical merger or acquisition is arranged and that the joint firm receives the sum of the payoffs to the two firms. What is the optimal investment pattern for the joint firm?
b. Suppose that one of the firms hasn't had the advantage of your experience with the idea of solving dynamic interactions by "looking forward and solving backwards." Not knowing this cardinal principle, they decide that they will move first, invest, and tell the other firm what they have done. What will be the result?
c. Consider contracts of the form: "I will invest, and if I do not invest while you have invested, I owe you damages of x. You will invest, and if you do not invest while I have invested, you owe me damages of x. Further, this contract is not valid unless both of us have signed it." For what values of x will

the contract have the property that signing the contract and then investing becomes the dominant strategy?

5. If a chicken packing firm leaves the fire escape doors operable, they will lose c in chickens that disappear to the families and friends of the chicken packers. If they nail or bolt the doors shut, which is highly illegal, they will no longer lose the c, but, if they are inspected (by say OSHA), they will be fined f. Further, if the firedoor is locked, there is a risk, ρ, that they will face civil fines or criminal worth F if there is a fire in the plant that kills many of the workers because they cannot escape.[1] Inspecting a plant costs the inspectors k, not inspecting an unsafe plant costs B in terms of damage done to the inspectors' reputations and careers. Filling in the other terms, we get the game

		Inspectors	
		Inspect	Not inspect
Imperial	unlocked	$(\pi - c, -k)$	$(\pi - c, 0)$
	locked	$(\pi - f - \rho F, f - k)$	$(\pi - \rho F, -B)$

   a. Show that if f and ρF are too low, specifically, if $c > f + \rho F$, then Imperial has a dominant strategy, and the game is, strategically, another version of Rational Pigs.
   b. Show that if $f + \rho F > c > \rho F$ and $f - k > -B$, neither player has a dominant strategy, and there is only a mixed Nash equilibrium. In this case, we have another instance of an inspection game.
   c. Assume that $f + \rho F > c > \rho F$ and $f - k > -B$ in the inspection game. Show that the equilibrium is unchanged as π grows. How does it change as a function of c?

6. From Kreps Ch. 21, problem 21.1.
7. From Kreps Ch. 21, problem 21.2.
8. From Kreps Ch. 21, problem 21.5.
9. From Kreps Ch. 21, problem 21.7.
10. [Cournot equilibrium] Two firms compete by producing quantities $q_i \geq 0$ and $q_j \geq 0$ of a homogeneous good, and receiving profits of the form

$$\pi_i(q_i, q_j) = [p(q_i + q_j) - c]q_i,$$

where $p(\cdot)$ is the inverse market demand function for the good in question. Assume that $p(q) = 1 - q$ and that $0 \leq c \ll 1$.
   a. For each value of q_j, find i's best response, that is, find $Br_i(q_j)$.
   b. Find the unique point at which the best response curves cross. This is called the **Cournot equilibrium**.
   c. Show that there is no mixed strategy equilibrium for this game.
   d. Show that social surplus is inefficiently small in the Cournot equilibrium.

[1] White collar decisions that kill blue collar workers rarely result in criminal prosecutions, and much more rarely in criminal convictions. See Russell Mokhiber's 1988 book for some rather depressing statistics. Emmett Roe, the owner of the Imperial chicken processing plant that locked the doors killed 25 workers and injured 56 more on September 3, 1991. He plea-bargained to 25 counts of involuntary manslaughter, was sentenced to 20 years in prison, and was eligible for early release after 3 years, and was released after 4 and a half years, that is, 65 days in prison for each of the dead. The surviving injured workers and families of the dead only won the right to sue the state for failure to enforce safety codes on February 4, 1997, after a five-year battle that went to the state Court of Appeals. Damage claims will be limited to $100,000 per victim.

e. Show that, by appropriately including consumer surplus into the payoffs of the firms, one can increase the social welfare of the equilibrium.

11. From Kreps Ch. 21, problem 21.9.