Econometrics Homework #3, Spring 2007
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1. [Comparing linear prediction and best prediction] Let X_1 and X_2 be independent random variables with $P(X_i = -1) = P(X_i = +1) = \frac{1}{2}$, $i = 1, 2$ and define $Y = X_1 \cdot X_2$.

   (a) Give the distribution of Y.

   (b) Give $E(Y \mid X_1)$, $E(Y \mid X_2)$, and $E(Y \mid X_1, X_2)$.

   (c) Solve the problem $\min_{\beta_0,\beta_1,\beta_2} E\left(Y - [\beta_0 + \beta_1 X_1 + \beta_2 X_2]\right)^2$.

2. Let X_t follow an invertible MA(1) process: $X_t = \alpha X_{t-1} + u_t$, where $|\alpha| < 1$ and $u_t \sim N(0, \sigma_u^2)$. Let v_t be another white noise process such that for all s, t, v_t is uncorrelated with u_s, and $Var(v_t) = \sigma_v^2$.

   We will now see that $Z_t := X_t + v_t$ can also be represented as an invertible MA(1) process. To do this, it is sufficient to show that there exists $|\theta| < 1$ such that the process
$$\epsilon_t := Z_t - \theta Z_{t-1} + \theta^2 Z_{t-2} - \theta^3 Z_{t-3} + \cdots \tag{1}$$
   is a white noise process.

   (a) Why does the claim follow from eqn. (1)?

   (b) Find the variance and autocovariances of Z_t. Are the autocovariances consistent with the MA(1) pattern in general? Give the spectral density of Z_t.

   (c) Assuming Z_t indeed has an MA(1) representation, set up a system of equations that θ would have to satisfy by equating variances and autocovariances from this representation with those under (b).

   (d) Show that this system of equations has a solution $0 < |\theta^*| < 1$.

   (e) Using the θ^* just described under (d), define Z_t as in eqn. (1). Show that this is a white noise process. [Think spectral density.]

3. This exercise will demonstrate through a simple example how to use maximum likelihood or method of moments to estimate moving average models. Consider the MA(1) model $Y_t = \epsilon_t + \theta\epsilon_{t-1}$ where ϵ_t is iid $N(0, \sigma^2)$.

   (a) Show that the conditional distribution of Y_t given $Y_{t-1}, \ldots, Y_1, \epsilon_0$ is the same as the conditional distribution of Y_t given $\epsilon_{t-1}, \ldots, \epsilon_0$.

   (b) Show that the conditional distribution of Y_t given $Y_{t-1}, \ldots, Y_1, \epsilon_0$ is the same as the conditional distribution of Y_t given ϵ_{t-1}.

   (c) Suppose we have three observations Y_3, Y_2 and Y_1. Using the previous, Express the conditional density of (Y_3, Y_2, Y_1) given ϵ_0 in terms of the conditional densities $Y_t|\epsilon_{t-1}$, $t = 3, 2, 1$. Using the normality assumption, write down an explicit expression for this conditional density.

(d) Suppose $\epsilon_0 = 0$. Substitute out ϵ_2 and ϵ_1 in terms of Y_2, Y_1. Find the conditional MLE of θ if $Y_1 = -0.5$, $Y_2 = 0$, $Y_3 = -0.5$.

(e) Using the same sample, calculate the method of moments estimator of θ by equating the population autocorrelations of the process with their sample counterparts.

4. [For both this problem and the next one, you can also simulate to get your answers. If you do this, be explicit about your simulation process.] Suppose that $Y_t = c + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \epsilon_t$ where ϵ_t is a white noise process, $(\varphi_1, \varphi_2) = (0.5, 0.24)$, and $t \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

   (a) Find the eigen-values of the associated F matrix and show that they are inside the unit circle. [From here onwards, you can assume that Y_t is a weakly stationary sequence with $\sum_j |\gamma_j| < \infty$.]

   (b) Give $E\,Y_t$, and the γ_j, $j \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

   (c) Based on the sample $Y_1, \ldots, Y_T$, let $\hat{\mu}_T$ and $\hat{\gamma}_{j,T}$ be the population moments, $j = 0, 1, 2, 3$. Find $E\,(\hat{\mu}_T)$, $Var\,(\hat{\mu}_T)$, $E\,\hat{\gamma}_{j,T}$, and $Var\,\hat{\gamma}_{j,T}$.

5. One expects problems when the eigen-values get close to the edge of the unit circle. Repeat the previous problem for $(\varphi_1, \varphi_2) = (1.85, -0.9)$ and explain the source(s) of the difference(s).

6. Suppose that $Y_t = a + b \cdot t + \varphi_1 Y_{t-1} + \cdots + \varphi_p Y_{t-p} + \epsilon_t$, $b \neq 0$, ϵ_t a white noise process.

   (a) Is Y_t any kind of stationary? Explain.

   (b) Define $X_t = \Delta Y_t := Y_t - Y_{t-1}$. What kind of process is X_t? Is it weakly stationary? If ϵ_t is strictly stationary, is X_t also strictly stationary?

   (c) Define $Z_t = \Delta X_t = X_t - X_{t-1}$. What kind of process is Z_t? Is it weakly stationary? If ϵ_t is strictly stationary, is Z_t also strictly stationary?

7. Suppose that $Y_t = a + b \cdot t + c \cdot t^2 + \varphi_1 Y_{t-1} + \cdots + \varphi_p Y_{t-p} + \epsilon_t$, $c \neq 0$, ϵ_t a white noise process.

   (a) Is Y_t any kind of stationary? Explain.

   (b) $X_t = \Delta Y_t := Y_t - Y_{t-1}$. Is X_t any kind of stationary? Explain.

   (c) Define $Z_t = \Delta X_t = X_t - X_{t-1}$. What kind of process is Z_t? Is it weakly stationary? If ϵ_t is strictly stationary, is Z_t also strictly stationary?

8. Volatile time series are often "smoothed out" by some sort of averaging. In particular, let X_t be a covariance stationary time series and let $\tilde{X}_t$ denote its smoothed version defined by an m-period centered moving average

$$\tilde{X}_t = (X_{t-m} + ... + X_{t-1} + X_t + X_{t+1} + ... + X_{t+m})/(2m + 1).$$

(a) Let $m = 1$. Using lag operator notation, write down the linear filter that transforms X_t into $\tilde{X}_t$.

(b) Find the filter function (i.e. the function by which the spectral density of X_t has to be multiplied to obtain the spectral density of $\tilde{X}_t$).

(c) Compare the spectral density of X_t with the spectral density of $\tilde{X}_t$. Which frequencies are missing from the spectrum of $\tilde{X}_t$? Which frequencies are dampened down? Which are amplified?

(d) Let $m = 2$. Write down and graph the filter function (most easily done with a computer program). By averaging over more observations in the time domain, we should get a smoother series than before. Justify this claim in the frequency domain by comparing the graphs of the filter functions obtained for $m = 1$ and $m = 2$.