Homework #1, Micro III, Spring 2007

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All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. Throughout, $X, X', Y : \Omega \to \mathbb{R}$ are random variables having finite range, and $P(Y = 1) = P(Y = 2) = \frac{1}{2}$.

Time line: first, $\omega \in \Omega$ is drawn according to P; then the value of a signal $X(\omega)$ or $X'(\omega)$ is observed; then a choice $a \in A$ is made; and finally the utility $u(a, Y(\omega))$ is received.

Notation and definitions:

1. $\beta_x^X := P(Y = 1 \big| X = x)$ is the **posterior probability**, aka **belief** that $Y = 1$ given that $X = x$;
2. $V_{u,A}(X) := \max_{f:S\to A} E\, u(f(X), Y)$;
3. the **convex hull** of $C \subset \mathbb{R}^k$ is written $\mathbf{con}\,(C)$; A^S is the set of functions $f : S \to A$;
4. for a (finite) set C, $\Delta(C) = \{p \in \mathbb{R}^C_+ : \sum_{c\in C} p_c = 1\}$ is the set of probability distributions over C;
5. the range of a function $X : \Omega \to \mathbb{R}$ is $\mathcal{R}(X) = \{r \in \mathbb{R} : (\exists\omega \in \Omega)[X(\omega) = r]\}$;
6. for p, q distribution on $\mathbb{R}^k$, $k \geq 2$, p is **riskier** that q if for all continuous, concave $g : \mathbb{R}^k \to \mathbb{R}$, $\int g(x)\, dp(x) \leq \int g(x)\, dq(x)$;
7. for distributions p, q on $\mathbb{R}^1$, q is **riskier** that p if for all continuous, concave, non-decreasing $g : \mathbb{R}^1 \to \mathbb{R}$, $\int g(x)\, dp(x) \geq \int g(x)\, dq(x)$.
8. for distributions p, q on $\mathbb{R}^1$, q is a **mean preserving spread of** p if $\int x\, dq(x) = \int x\, dp(x)$ and there is an interval $|a, b| \subset \mathbb{R}$, such that
   a. for all $E \subset (a, b]$, $q(E) \leq p(E)$,
   b. for all $E \subset (b, \infty)$, $q(E) \geq p(E)$, and
   c. for all $E \subset (-\infty, a]$, $q(E) \geq p(E)$.
9. For $r \in \mathbb{R}$, δ_r is point mass on $\mathbb{R}$, that is, $\delta_r(E) = 1_E(r)$.
10. for distributions p, q on $\mathbb{R}^1$, p **first order stochastically dominates** q if for all non-decreasing $g : \mathbb{R} \to \mathbb{R}$, $\int g(x)\, dp(x) \geq \int g(x)\, dq(x)$.

1. Consider the problem (u, A) and the random variables X, X' given by

$a = 3$	0	40
$a = 2$	30	30
$a = 1$	50	0
	$Y = 1$	$Y = 2$

$X = 2$	0.3	0.8
$X = 1$	0.7	0.2
	$Y = 1$	$Y = 2$

$X' = 2$	0.1	0.7
$X' = 1$	0.9	0.3
	$Y = 1$	$Y = 2$

where, e.g. $50 = u(a = 1, Y = 1)$ and $0.3 = P(X = 2\big|Y = 1)$.

a. Let $(\beta, 1 - \beta) \in \Delta(\mathcal{R}(Y))$ be a distribution over the range of Y. Give the set of β for which $\text{argmax}_{a\in A} \int u(a, y)\, d\beta(y) = \{1\}$, $\text{argmax}_{a\in A} \int u(a, y)\, d\beta(y) = \{2\}$, and $\text{argmax}_{a\in A} \int u(a, y)\, d\beta(y) = \{3\}$.

b. Give the β^X_x and the $\beta^{X'}_{x'}$. Using the previous problem, give the solutions f^*_X and $f^*_{X'}$ to the problems $\max_{f\in A^S} E\, u(f(X), Y)$ and $\max_{f\in A^S} E\, u(f(X'), Y)$. From these calculate $V_{(u,A)}(X)$ and $V_{(u,A)}(X')$. [You should find that $X' \succ_{(u,A)} X$.]

c. Let $M = (\beta^X_x, P(X = x))_{x=1,2} \in \Delta(\Delta(\mathcal{R}(Y))$ and $M' = (\beta^{X'}_{x'}, P(X' = x'))_{x'=1,2} \in \Delta(\Delta(\mathcal{R}(Y))$. Show directly that M is not risker than M' and that M' is not riskier than M.

d. Graph, in $\mathbb{R}^2$, the sets of achievable, Y-dependent utility vectors for the random variables X and X'. That is, graph

$$F_{(u,A)}(X) = \{(E\,(u(f(X), Y)\big|Y = 1), E\,(u(f(X), Y)\big|Y = 2)) \in \mathbb{R}^2 : f \in A^S\}$$

and

$$F_{(u,A)}(X') = \{(E\,(u(f(X'), Y)\big|Y = 1), E\,(u(f(X'), Y)\big|Y = 2)) \in \mathbb{R}^2 : f \in A^S\}.$$

e. If we allow random strategies, that is, pick f according to some $q \in \Delta(A^S)$, then the sets of achievable Y-dependent utility vectors become $\textbf{con}\,(F_{(u,A)}(X))$ and $\textbf{con}\,(F_{(u,A)}(X'))$. Show that the same is true if we allow "behavioral strategies," that is, $f \in \Delta(A)^S$.

f. Show that $\textbf{con}\,(F_{(u,A)}(X)) \not\subset \textbf{con}\,(F_{(u,A)}(X'))$ and $\textbf{con}\,(F_{(u,A)}(X')) \not\subset \textbf{con}\,(F_{(u,A)}(X))$.

g. Give a different problem, (u°, A°) for which $X \succ_{(u^\circ,A^\circ)} X'$.

2. Let X, X' be two signals, and define $X'' = (X, X')$ to be both signals. Let S, S' and S'' be the ranges of the three signals

a. Show directly that for all (u, A), $X'' \succsim_{(u,A)} X$ (hence $X'' \succsim_{(u,A)} X'$).

b. Show directly that $\{(\beta^{X''}_{x''}, P(X'' = x''))_{x''\in S''}$ is riskier than $\{(\beta^X_x, P(X = x))_{x\in S}$.

c. Interpret X as the result of a doctor's diagnostic test and X' is the result of a possible additional test. Show that if $f^*_{(X,X')}(x, x') = f^*_X(x)$ for a problem (u, A), then there is no point in doing the extra test.

3. Most of the following results are in the Müller [1] article, which covers and extends the famous Rothschild and Stiglitz [2], [3] articles on increases in risk.

a. If q is a mean preserving spread of p, then q is riskier than p.

b. If $X \sim p$ (i.e. $P(X \in A) = p(A)$), $Y \sim q$, and there is a random variable Z such that $E\,(Z\big|X) = 0$ and $X + Z \sim q$, then q is riskier than p.

c. Let $p_0 = \delta_0$, $p_1 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. Let $p_2 = \frac{1}{2}\delta_{-1} + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_2$, and $p_3 = \frac{1}{4}\delta_{-2} + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_2 = \frac{1}{4}\delta_{-2} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2$. Continuing in this fashion, $p_4 = \frac{1}{8}\delta_{-4} + \frac{6}{8}\delta_0 + \frac{1}{8}\delta_4$, and so on.

   i. Show that p_1 is a mean preserving spread of p_0.
   ii. Show that p_{k+1} is a mean preserving spread of p_k.
   iii. Show that $p_k \to_w p_0$.

d. The previous problem showed that a sequence can become riskier and riskier and still converge to something that is strictly less risky. Show that this cannot happen if $p_k([a, b]) \equiv 1$ for some compact interval $[a, b]$. Specifically, show that if $p_k([a, b]) \equiv 1$, for all k, p_{k+1} is riskier than p_k, and $p_k \to q$, then q is riskier than all of the p_k.

4. In each time period, $t = 1, \ldots$, a random wage offer, $X_t \geq 0$, arrives. The X_t are iid with cdf F. The problem is which offer to accept. If offer $X_t = x$ is accepted, utility is $\beta^t u(x)$ where $0 < \beta < 1$, and $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly monotonic, concave, and $\int u(x)\,dF(x) < \infty$. A "reservation wage" policy is one that accepts all offers of $\underline{x}$ or above for some $\underline{x}$.

   a. Show that the optimal policy is a reservation wage policy, and give the distribution of the random time until an offer is expected.
   b. In terms of u and F, give the expected utility of following a reservation wage policy with reservation wage $\underline{x}$.
   c. If the offers are, instead, iid Y_t with cdf G and G is riskier than F, then the optimal reservation wage is higher, and the expected utility is also higher.

5. X_a is your random income depending on your action $a \geq 0$, understood as money that you spend on stochastically increasing X_a. The distribution of X_a is $R_{a,c} := cQ_a + (1 - c)\mu$, $0 \leq c \leq 1$. Here, μ does not depend on a, but, if $a > a'$, then Q_a first order stochastically dominates $Q_{a'}$. The parameter c is the amount of "control" that you have, $c = 0$ means you have no control, $c = 1$, means you have the most possible. This question asks you how a^* depends on c. Intuitively, increasing c ought to increase the optimal action, more control means that your actions have more effect.

   Let $f(a, c) = Eu(X_a - a)$ where u is an increasing, concave function. Increases in a pull down $X_a - a$, hence $u(X_a - a)$, by increasing the direct cost, but increase $X_a - a$ by stochastically increasing X_a.

   a. Show that $f(\cdot, \cdot)$ is not, in generaly, supermodular.
   b. Suppose that $f(a, c)$ is smooth, that we can interchange integration and differentiation, and that the optimum, $a^*(c)$ is differentiable. The $f_a := \partial f/\partial a$ is equal to
   $$f_a = -\int u'(x - a)d\mu(x) + cd/da[\text{messy term with } Q_a \text{ and } \mu].$$
   We let $m =$ [messy term with Q_a and μ].
      i. Show that if $f_a(a, c) = 0$, then $\partial m(a, c)/\partial a > 0$.
      ii. Show that $f_{a,c} := \partial^2 f/\partial a \partial c = \partial m/\partial a > 0$.
      iii. Show that $da^*(c)/dc > 0$.

References

1. Alfred Müller, *Comparing risks with unbounded distributions*, J. Math. Econom. **30** (1998), no. 2, 229–239. MR MR1652641 (99m:90049)
2. Michael Rothschild and Joseph E. Stiglitz, *Increasing risk. I. a definition*, J. Econom. Theory **2** (1970), 225–243. MR MR0503565 (58 #20284a)
3. ———, *Increasing risk. II. Its economic consequences*, J. Econom. Theory **3** (1971), 66–84. MR MR0503567 (58 #20284c)